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# Dynamics and Control of Spacecraft Formations: Challenges and Some Solutions

**Kyle T. Alfriend * and Hanspeter Schaub †**

**Abstract**

Presented is an analytic method to establish $J_2$ invariant relative orbits, that is, relative orbit motions that do not drift apart. Working with mean orbit elements, the secular relative drift of the longitude of the ascending node and the argument of latitude are set equal between two neighboring orbits. Two first order conditions constrain the differences between the chief and deputy momenta elements (semi-major axis, eccentricity and inclination angle), while the other three angular differences (ascending node, argument of perigee and mean anomaly), can be chosen at will. Several challenges in designing such relative orbits are discussed. For near polar orbits or near circular orbits enforcing the equal nodal rate condition may result in impractically large relative orbits if a difference in inclination angle is prescribed. In the latter case, compensating for a difference in inclination angle becomes exceedingly difficult as the eccentricity approaches zero. The third issue discussed is the relative argument of perigee and mean anomaly drift. While this drift has little or no effect on the relative orbit geometry for small or near-zero eccentricities, for larger eccentricities it causes the relative orbit to enlarge and contract over time. A simple control solution to this issue is presented. Further, convenient expressions are presented which allow for quick annual fuel budget estimations. For given initial orbit element differences, these formulas estimate what $\Delta v$ is required to compensate for the $J_2$ induced relative drift.

## Introduction

In recent years, the concept of spacecraft formations has been considered for various missions. One class of spacecraft formations has the satellite constellation composed of spacecraft of equal type and build which form a rotating sparse aperture. These types of formations are typically considered in remote sensing missions where each satellite is an individual element of a large, virtual antenna formed by the formation. By sharing the individual measurements, the resolution of the spacecraft cluster is potentially much higher then the resolution of any individual craft. To minimize secular relative drift among the spacecraft, these missions typically are comprised of identical spacecraft to reduce the

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differential atmospheric drag. The gravitational perturbations should be the dominant factor producing the secular drift in this case. Ignoring these perturbations leads to relative orbit designs which require more frequent corrections, and thus use more fuel. The $J_2$ perturbations cause secular drift in the longitude of the ascending node, argument of perigee and mean anomaly. The magnitude of the secular drifts are determined by the semi-major axis, eccentricity and the inclination angle.[1] If these quantities aren't carefully selected, then the relative drift rates will cause secular drift among the various spacecraft in the formation.[2]

Another class of spacecraft formations are composed of craft of different design and built. These formations are often of a lead-follower type where one spacecraft is intended to follow in the trail of another spacecraft. Since these craft may have different ballistic coefficients, the dominant cause for secular drift among these craft is the differential atmospheric drag effect. It causes the spacecraft with the higher ballistics coefficient to first fall back and slow down. This causes it to drop to a slightly lower orbit with a shorter period, which in return makes it catch up again to the other craft at a lower altitude. For these types of missions it is not possible to compensate for the secular drift by carefully picking appropriate orbit elements. Due to the differential drag, the individual orbits are losing energy at unequal rates and no choice in orbit elements will cancel this. Thus, these types of spacecraft formations cannot avoid the periodic orbit corrections to keep the relative orbit bounded.

This paper investigates spacecraft formation flying issues for the first mentioned class of formations where all the craft are of equal type and design. Thus, differential drag is a minor cause of relative drift compared to the $J_2$ gravitational perturbation. Previous studies on the relative spacecraft motion in low Earth orbit have typically used the Clohessy-Wiltshire (CW) equations[3–6] to describe the relative equations of motion. With these linearized equations periodic orbits in the relative motion reference frame have been identified. These periodic orbits include in-plane, out-of-plane, and combinations of these two motion types. Of these types of relative motion, having an out-of-plane motion at the maximum latitude (polar crossing) poses the greatest challenge when designing relative orbits which do not have secular drift. An inclination difference between the chief and deputy satellites is necessary to cause this type of out-of-plane motion. This difference in inclination angles also results in a differential nodal precession rate between the two satellites. However, the linear CW equations do not show this motion; they indicate an out-of-plane oscillatory motion with a constant amplitude. To maintain a relative orbit designed with the CW equations, periodic orbit corrections are necessary to cancel deviations caused by the $J_2$ perturbations. Further, a reference motion and the accompanying state transition matrix might result in an out-of-plane control that changes inclination because the state transition matrix does not indicate the increasing amplitude caused by the inclination difference. For these reasons, it is necessary for the reference motion to include at least the secular $J_2$ gravitational perturbation effect.

This paper will outline and discuss a method to generate $J_2$ invariant relative orbits which was first presented in Reference 2. Benefits and challenges of composing such relative orbits are discussed and illustrated in detail. Further, a method is presented to estimate the fuel required to compensate for the relative secular drift due to initial orbit element differences. These fuel cost estimates are critical when designing relative orbits in that they allow the orbit designer to quickly assess what effect relaxing any of the $J_2$ invariant orbit conditions will bring.

## Problem Statement

Adding the $J_2$ perturbation to the classical Keplerian orbit motion causes three types of changes in the osculating orbit elements, short period and long period oscillations, and secular growth. The long period term is the period of the apsidal rotation. Over a short time this looks like a secular growth of $\mathcal{O}(J_2^2)$. The short period growth manifests itself as oscillations of the orbit elements, but doesn't cause the orbits to drift apart. The relative secular growth is the motion that needs to be avoided for relative orbits to be $J_2$ invariant. This growth is best described through *mean orbit elements* rather than the osculating elements. By studying the relative motion through the use of mean orbit elements,[2,7,8] we are able to ignore the orbit period specific oscillations and address the secular drift directly. It is not possible to set all of the individual orbit drifts equal to zero. However, instead we choose to set the *relative* mean orbit element drifts to zero to avoid *relative secular growth.*

The following algebra is greatly simplified if we work with dimensionless variables. Therefore distances will be measured in Earth radii $R_e$ and time is normalized by the mean motion of a satellite at one Earth radius (i.e. $\mu = 1$). The $J_2$ gravitational perturbation causes only secular drift in the longitude of the ascending node $h$, the argument of perigee $g$ and the mean anomaly $l$. The individual drift rates are solely determined through the semi-major axis $a$, eccentricity $e$ and inclination angle $i$. Instead of using $a$ and $e$ directly, we use the equivalent dimensionless variables $L$ and $\eta$ instead which are defined through

$$L = \sqrt{\mu a} \tag{1}$$

$$\eta = \sqrt{1 - e^2} \tag{2}$$

The mean dimensionless drift rates of $l$, $g$ and $h$ are given by[1,2]

$$\dot{l} = \frac{1}{L^3} + \epsilon \frac{3}{4L^7\eta^3}\left(1 - 3\cos^2 i\right) \tag{3}$$

$$\dot{g} = \epsilon \frac{3}{4L^7\eta^4}\left(1 - 5\cos^2 i\right) \tag{4}$$

$$\dot{h} = \epsilon \frac{3}{2L^7\eta^4}\cos i \tag{5}$$

where $\epsilon = -J_2$. The momenta variables $a$, $e$ and $i$ do not exhibit secular drift due to $J_2$, only short and long period oscillations.

## $J_2$ Invariant Relative Orbit Constraints

### *Orbit Element Constraints*

The following section outlines the development of the $J_2$ invariant relative orbit constraints presented in Reference 2. Since the mean angle quantities $l$, $g$ and $h$ do not directly contribute to the secular growth caused by $J_2$, their values can be chosen at will. However, the mean momenta values $L$, $\eta$ and $i$ (and therefore implicitly $a$, $e$ and $i$) must be carefully chosen to match the secular drift rates shown in Eqs. (3) through (5). To keep the satellites from drifting apart over time, it would be desirable to match all three rates $(\dot{l}, \dot{g}, \dot{h})$ between the various satellites in a given formation. However, this can only be achieved by having the momenta equal, which in return severely restricts the possible relative orbits. Therefore, the condition that the relative average drift rate of the angle between the radius vectors be zero. This results in

$$\dot{h}_i = \dot{h}_j \qquad \forall i \neq j \tag{6}$$

$$\dot{\theta}_i = \dot{l}_i + \dot{g}_i = \dot{\theta}_j \qquad \forall i \neq j \tag{7}$$

where $\theta$ is the argument of latitude. Thus, the arguments of perigee and the mean anomalies are allowed to drift apart. In fact, they end up drifting apart at equal and opposite rates.[2] Imposing equal latitude rates instead of forcing equal argument of perigee and mean anomaly drift has little consequence on the general spacecraft formation geometry if the eccentricity is small. For the case of having a circular orbit (i.e. $e = 0$), then having the relative $g$ and $l$ drift apart has no consequence at all. However, for relative orbits with a larger eccentricity, having the $g$ and $h$ drift apart at equal and opposite rates causes the relative orbit to "balloon" out and in again as the argument of perigees drift apart from their nominal values. Combining Eqs. (3) and (4), the mean latitude rate $\dot{\theta}$ is expressed as

$$\dot{\theta} = \frac{1}{L^3} + \epsilon \frac{3}{4L^7\eta^4}\left[\eta\left(1 - 3\cos^2 i\right) + \left(1 - 5\cos^2 i\right)\right] \tag{8}$$

Let the chief mean orbit elements be denoted with the subscript "0". The drift rate $\dot{\theta}_i$ of a neighboring orbit can be written as a series expansion about the chief orbit element as

$$\dot{\theta}_i = \dot{\theta}_0 + \frac{\partial\dot{\theta}_0}{\partial L}\delta L + \frac{\partial\dot{\theta}_0}{\partial\eta}\delta\eta + \frac{\partial\dot{\theta}_0}{\partial i}\delta i + H.O.T. \tag{9}$$

where we make use of the fact that $\dot{\theta} = \dot{\theta}(L, \eta, i)$ only. Let the difference in latitude rates be $\delta\dot{\theta}$, then a first order approximation of Eq. (9) is written as

$$\delta\dot{\theta} = \dot{\theta}_i - \dot{\theta}_0 = \frac{\partial\dot{\theta}_0}{\partial L}\delta L + \frac{\partial\dot{\theta}_0}{\partial\eta}\delta\eta + \frac{\partial\dot{\theta}_0}{\partial i}\delta i \tag{10}$$

Similarly, the nodal rate $\dot{h}$ is expressed as

$$\delta\dot{h} = \frac{\partial\dot{h}_0}{\partial L}\delta L + \frac{\partial\dot{h}_0}{\partial\eta}\delta\eta + \frac{\partial\dot{h}_0}{\partial i}\delta i \tag{11}$$

To enforce equal drift rates $\dot{\theta}_i$ and $\dot{h}_i$ between neighboring orbits, we must set $\delta\dot{\theta}$ and $\delta\dot{h}$ equal to zero in Eqs. (10) and (11). Taking the appropriate partial derivatives of Eq. (8) and substituting them into Eq. (10), the condition that enforces equal latitude rates is rewritten as:

$$\begin{aligned} -\frac{3}{{L_0}^4}\delta L - \epsilon\frac{21}{4L_0^8\eta_0^4}\left[\eta_0(1 - 3\cos^2 i_0) + (1 - 5\cos^2 i_0)\right]\delta L \\ - \epsilon\frac{3}{4L_0^7\eta_0^5}\left[3\eta_0(1 - 3\cos^2 i_0) + 4(1 - 5\cos^2 i_0)\right]\delta\eta \\ + \epsilon\frac{3}{2L_0^7\eta_0^4}(3\eta_0 + 5)\cos i_0 \sin i_0 \delta i = 0 \end{aligned} \tag{12}$$

Note that only the term $\delta L$ appears without being multiplied by the small parameter $\epsilon$. Thus $\delta L$ must be itself of $\mathcal{O}(\epsilon)$ and the term involving $\epsilon\delta L$ is dropped as a higher order term. The first orbit element constraint is then simplified to

$$-\delta L - \epsilon\frac{1}{4L_0^3\eta_0^5}\left[3\eta_0(1-3\cos^2 i_0)+4(1-5\cos^2 i_0)\right]\delta\eta + \epsilon\frac{1}{2L_0^3\eta_0^4}(3\eta_0+5)\cos i_0 \sin i_0 \delta i = 0 \quad (13)$$

Taking the partial derivatives of Eq. (5), the second condition for $J_2$ invariant orbits is written as

$$\epsilon\frac{3}{2L_0^7\eta_0^5}\left[-\frac{7}{L_0}\cos i_0\delta L - 4\cos i_0\delta\eta - \eta\sin i_0\delta i\right] = 0 \quad (14)$$

Since $\delta L = \mathcal{O}(\epsilon)$ the $\delta L$ term is dropped. Thus, the condition that results in equual nodal precession rates for two neighboring orbits is:

$$\delta\eta = -\frac{\eta_0}{4}\tan i_0 \delta i \quad (15)$$

Observe that as the chief satellite approaches a polar orbit (i.e. $i$=90 degrees), the necessary change in eccentricity results in an eccentricity greater than unity (hyperbolic orbit) or less than zero. This issue will be discussed in more detail in the following section. Using the $\delta i$ defined in Eq. (15), we are able to simplify the equal relative latitude rate condition in Eq. (13) to

$$\delta L = \underbrace{-\frac{\epsilon}{4L_0^4\eta_0^5}(4+3\eta_0)\left(1+5\cos^2 i_0\right)L_0}_{D}\delta\eta \quad (16)$$

Combined, Eqs. (15) and (16) provide the two necessary conditions on the mean momenta differences between two neighboring orbits to yield a $J_2$ invariant relative orbit. When designing a relative orbit using the mean orbit element differences, either $\delta i$, $\delta e$ or $\delta a$ is chosen, and the other two momenta element differences are then prescribed through the two constraints. The remaining mean orbit element differences $\delta h$, $\delta g$ and $\delta l$ can be chosen at will without affecting the $J_2$ invariant conditions. Further, note that these two conditions are not precise answers to the nonlinear problem, but are only valid up to a first order approximation. Thus, relative orbits designed with these two conditions will still exhibit some small relative drift.

### *Effect of Dropping the* $\boldsymbol{\epsilon\delta L}$ *Term*

To achieve the two orbit element constraint equations in Eq. (15) and (16), the terms containing $\epsilon\delta L$ were dropped as a second order term. The reason for this is that in Eq. (12) $\delta L$ is the only term appearing without being multiplied by $\epsilon$ and thus must itself be of order $\epsilon$. However, as the inclination angle approaches either 0 or 90 degrees, then the term in Eq. (12) which contains $\delta i$ would also become very small. Thus, ignoring the $\epsilon\delta L$ terms in these cases could potentially contribute lead to significant numerical errors.

The following development will show that the error introduced by neglecting the $\epsilon\delta L$ terms in minimal. If the $\epsilon\delta L$ terms are retained, then the two $J_2$ invariant relative orbit conditions take on a more complicated form:

$$\delta\eta = \frac{\eta\left(7\epsilon(\eta-2)(5+3\eta)\cos i\sin i - \eta(4L^4\eta^4+7\epsilon(1+\eta))\tan i\right)}{16L^4\eta^5+7\epsilon(4\eta^2+\eta-4)-7\epsilon(\eta(11+12\eta)-20)\cos^2 i}\delta i \tag{17}$$

$$\delta L = -\frac{\epsilon(4+3\eta)(1+5\cos^2 i)}{\eta(4L^4\eta^4+7\epsilon(1+\eta))-14\epsilon(\eta-2)(5+3\eta)\cos^2 i}L\delta\eta \tag{18}$$

Note that Eqs. (17) and (18) perfectly satisfy the first order conditions in Eqs. (12) and (14). If the higher order terms are dropped, then the previously presented $J_2$ invariant orbit constraints are retrieved. However, these more precise conditions on the mean orbit element are also more complex and analytically less trackable than their simplified cousins.

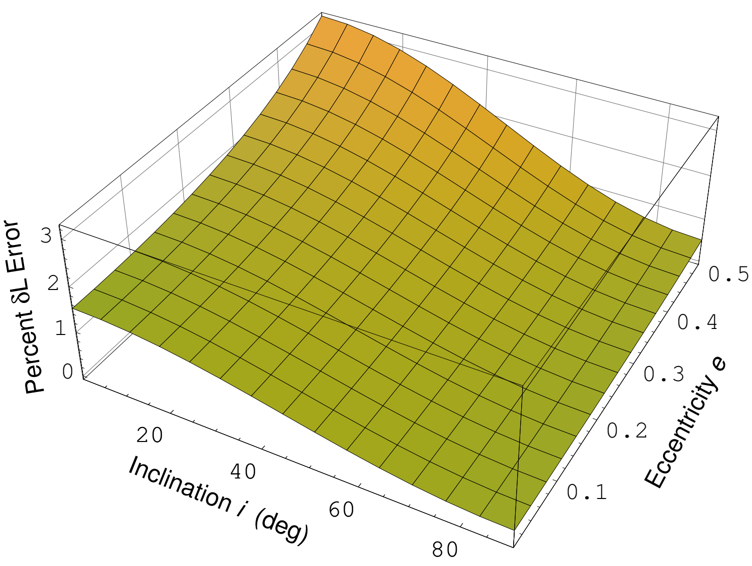



**Figure 1: Percent Error in Computing $\delta L$ by dropping the $\epsilon\delta L$ Terms**

Figure 1 illustrates the percent error in computing the $\delta L$ correction for a range of eccentricities and inclination angles. Here $L$ is set to be 1.054. As the figure shows, the numerical errors involved in using the simplified orbit constraint conditions are typically less than or equal to 1.5 percent. Only as the eccentricities grow larger do the numerical errors start to grow larger. It is interesting to note that dropping the $\epsilon\delta L$ term causes the largest numerical errors for near-zero inclination angles, while near-polar orbits show the least numerical errors. The numerical errors in computing $\delta\eta$ and $\delta i$ are essentially equivalent. Thus, using the simplified $J_2$ orbit element constraints results in minimal numerical errors. For cases where the numerical errors are too large, the more precise expressions in Eqs. (17) and (18) can be used.

*Orbit Element Constraints for $a$ and $e$*

For more physical insight into the $J_2$ invariant relative orbit constraints, it is convenient to map the differences in $L$ into differences in the semi-major axis $a$. Recalling that $L=\sqrt{a}$

($L$ is a non-dimensional variable), the variations in $L$ and $a$ are related through

$$\delta L = \frac{1}{2L}\delta a = \frac{\delta a}{2\sqrt{a}} \tag{19}$$

Substituting Eqs. (19) into Eq. (16), the constraint enforcing equal latitude rates between two orbits is rewritten as

$$\delta a = 2Da_0\delta\eta \tag{20}$$

Note that this $a$ is the non-dimensional semi-major axis and must be multiplied by the Earth radius $R_e$ to obtain proper physical units. Combined, Eqs. (15) and (20) form the two necessary momenta constraints expressed in terms of a difference in semi-major axis, eccentricity and inclination angle.

It is preferable to express the differences in eccentricity in terms of $\delta\eta$, and not in terms of the eccentricity $\delta e$ itself. The reason for this is clear when we observe the variation of $\eta = \sqrt{1-e^2}$.

$$\delta e = -\frac{\eta}{e}\delta\eta \tag{21}$$

Using Eq. (21) clearly poses numerical difficulties whenever the orbits become circular. A finite change in $\eta$ would erroneously appear as an infinite change in $e$. Thus, it is preferable to deal with $\delta\eta$ quantities and then use the precise mapping $\eta = \sqrt{(1-e^2)}$ to map these differences into corresponding $\delta e$ quantities.

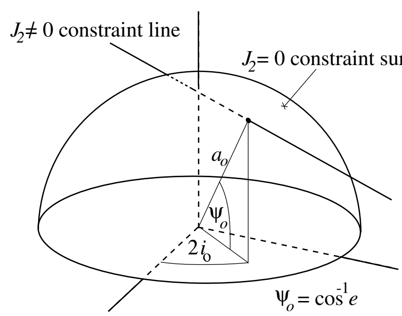



**Figure 2: Drift Free Constraint Illustration In Momenta Space**

If $J_2$ is set to zero (i.e. pure Keplerian motion), then we are only left with the constraint that $\delta a = 0$. This makes sense intuitively, since the semi-major axis $a$ determines the orbit period. For Keplerian motion, if the orbit periods are not equal, then the two spacecraft will drift apart. Thus, for Keplerian motion the initial conditions that result in relative motion orbits that do not drift apart are constrained to a five dimensional manifold , or in the momenta space, $(a, e, i)$, a two dimensional manifold, the surface of the sphere as illustrated in Figure 2. For a particular chief orbit with $a_0$, $e_0$ and $i_0$, the neighboring orbit momenta elements must lie on this surface. However, once the $J_2$ perturbation is included, the geometric constraint on the momenta elements to achieve drift free relative motion is a straight line which is not tangent to the sphere surface. Thus, the presence of gravitational perturbations changes the dimension of the constraint manifold from two to one.

*Energy Levels between two $J_2$ Invariant Relative Orbits*

It is interesting to study the energy levels of two neighboring orbits that are $J_2$ invariant using the necessary first order conditions established in Eqs. (15) and (20). For the system studied, the Hamiltonian $M$ is the total energy. Including the $J_2$ term, the averaged energy in terms of normalized orbit elements is given by

$$M = -\frac{1}{2a} + \epsilon \frac{1}{4a^3\eta^3}(-1 + 3\cos^2 i) \tag{22}$$

Where for Keplerian motion the energy level of an orbit only depends on the semi-major axis $a$, including the $J_2$ effect makes the energy expression depend on all three momenta elements $a$, $e$ and $i$. The difference in energy $\delta M$ of a neighboring orbit and a reference orbit is approximated as

$$\delta M = M - M_0 \approx \frac{\partial M_0}{\partial a}\delta a + \frac{\partial M_0}{\partial \eta}\delta\eta + \frac{\partial M_0}{\partial i}\delta i \tag{23}$$

Computing the partial derivatives in Eq. (22) while keeping in mind that $\delta a$ is of order $\mathcal{O}(\epsilon)$ we find that

$$\delta M = \frac{1}{2a_0^2}\delta a + \epsilon \frac{3}{4a_0^3\eta_0^4}\left[(1 - 3\cos^2 i_0)\delta\eta - 2\eta_0 \sin i_0 \cos i_0 \delta i\right] \tag{24}$$

For two neighboring orbits to be $J_2$ invariant, the differences in $a$, $\eta$ and $i$ must satisfy the two conditions in Eqs. (15) and (20). Substituting these variational constraint, the energy difference between two $J_2$ invariant orbits is given by

$$\delta M = \epsilon \frac{\tan i_0}{4a_0^3\eta_0^4}(1 + 5\cos^2 i_0)\delta i \tag{25}$$

Eq. (25) states that if the two orbits have a non-zero difference in inclination angle $\delta i$ (or implicitly a difference in $\eta$ or $a$), then the two orbits must have different energies. Only if all three momenta elements $a$, $\eta$ and $i$ between two orbits are equal will the orbit energies themselves be equal. Note that this condition still allows the two orbits to have different mean $l$, $g$ and $h$.

## Spacecraft Formation Flying Challenges

*Near-Polar Orbits*

For near-polar orbits, where the inclination $i$ approaches 90 degrees, the equal relative nodal rate condition in Eq. (15), given by

$$\delta\eta = -\frac{\eta_0}{4}\tan i_0 \delta i$$

may pose some practical problems in designing $J_2$ invariant relative orbits. The issue here is that as $i$ approaches 90 degrees, and the relative orbit design commands out-of-plane motion at the maximum latitude (i.e. $\delta i$ is non-zero), then the corresponding change in eccentricity becomes unpractically large. The result is that the relative orbit becomes excessively large. Note that this near-polar issue only arises if a specific mean inclination angle difference is prescribed and the two $J_2$ constraints are then used to compute the necessary mean

$\delta a$ and $\delta i$. If a change in mean semi-major axis or eccentricity were required for a near-polar orbit, then the equal nodal rate condition in Eq. (15) would require a very small corresponding mean inclination angle difference. Thus, achieving out-of-plane motion the maximum latitude poses the greatest challenge in designing $J_2$ invariant relative orbits. If the out-of-plane motion should occur during the equator crossing, then this can be achieved by describing a difference in ascending nodes $\delta h$. Since the three angular quantities $\delta h$, $\delta g$ and $\delta l$ can be chosen at will, no practical issues would arise here.

That the relative orbits become excessively large for near-polar orbits if a $\delta i$ is prescribed was also shown in the relative energy discussion in the previous section. Studying Eq. (25) is it clear that if the chief orbit is a polar orbit, a finite $\delta i$ requires an infinite difference in orbit energy, an unrealistic condition. Thus, as the inclination approaches 90 degrees the size of the relative motion orbits increases.

The problem posed by attempting to design a $J_2$ invariant relative orbit for a near-polar chief orbit is illustrated in the following numerical simulation. The chief mean orbit elements used are shown in Figure 1.

**Table 1: Chief Satellite Orbit Elements for Near-Polar Case Study.**

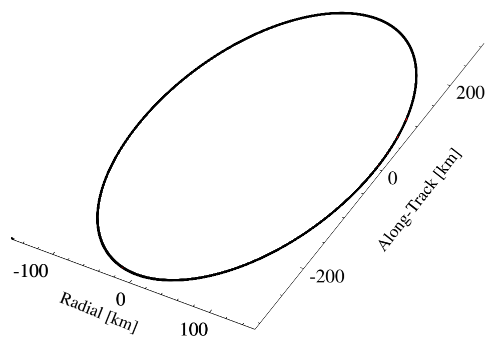
| Desired Average Orbit Elements | Value | Units |
|---|---|---|
| $a$ | 7153 | km |
| $e$ | 0.05 | |
| $i$ | 88 | deg |
| $h$ | 0.0 | deg |
| $g$ | 30.0 | deg |
| $l$ | 0.0 | deg |

The numerical simulations are performed by integrating the nonlinear orbit equation

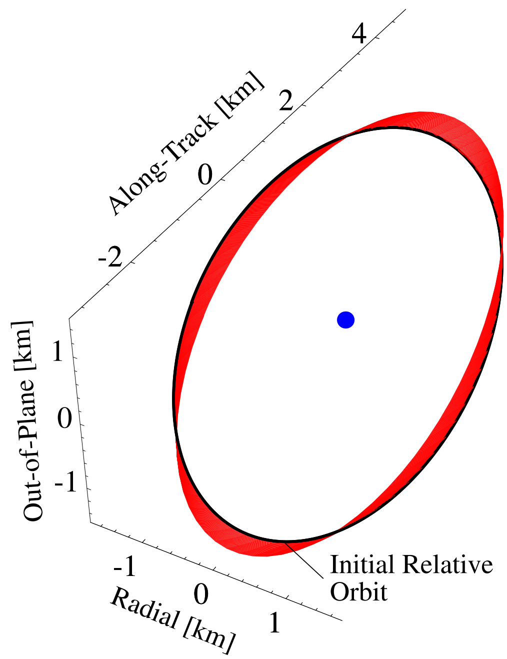
$$\ddot{\boldsymbol{r}} = -\frac{\mu}{|\boldsymbol{r}|^3}\boldsymbol{r} + \boldsymbol{f}(\boldsymbol{r}) \tag{26}$$

where the perturbative acceleration $\boldsymbol{f}(\boldsymbol{r})$ includes the zonal $J_2$ through $J_5$ effects. The relative orbit is described by choosing the mean orbit element differences $\delta h = 0.0$ degrees (all out-of-plane motion produced through $\delta i$), $\delta g = 0.1$ degrees and $\delta l =$ -0.1 degrees. Case 1 assumes the relative orbit geometry requires a $\delta i$ of 0.01 degrees to achieve roughly 1 km of out-of-plane motion at maximum latitude. To achieve a desired $\delta i$ of 0.01 degrees without inducing relative drift in the other orbit elements, the remaining two momenta elements differences must be $\delta e = 0.020648$ degrees and $\delta a =$ -27.2122 meters. The resulting relative orbit is shown in Figure 3(a). Note that the necessary difference in eccentricity is very large, causing the relative orbit to become very large in the along track and radial direction. However, no apparent drift is visible for the 45 orbits plotted on the scale shown.

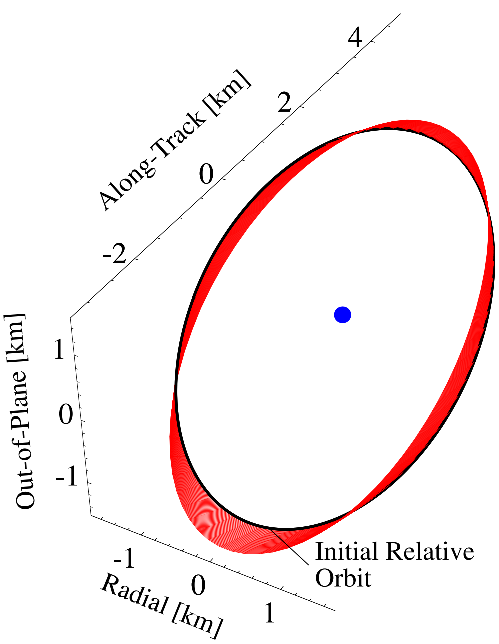
One method suggested in Reference 2 is to drop the equal relative nodal rate condition in Eq. (15) when a $\delta i$ is prescribed for a near-polar chief orbit. The $\delta i$ of 0.01 degrees is retained in case 2 shown in Figure 3(b), but it is not used to prescribe a corresponding difference in eccentricity. Instead, a $\delta e$ of 0.0001 is chosen and the corresponding $\delta a$ of -0.24157 meters



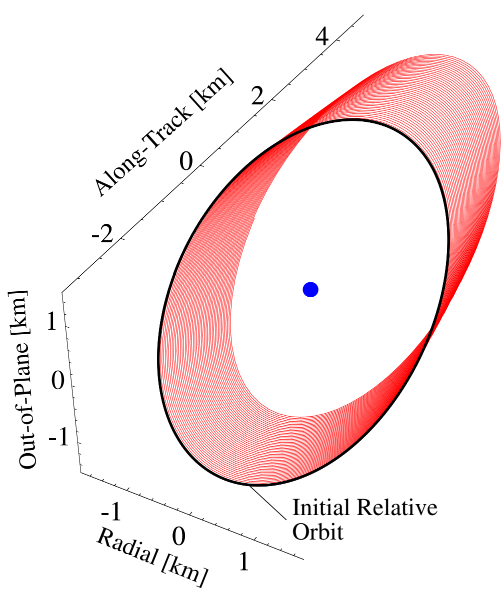

(a) Relative Orbit Enforcing Both $J_2$ Invariant Conditions




(b) Relative Orbit Enforcing Only $\delta a = 2Da\delta\eta$

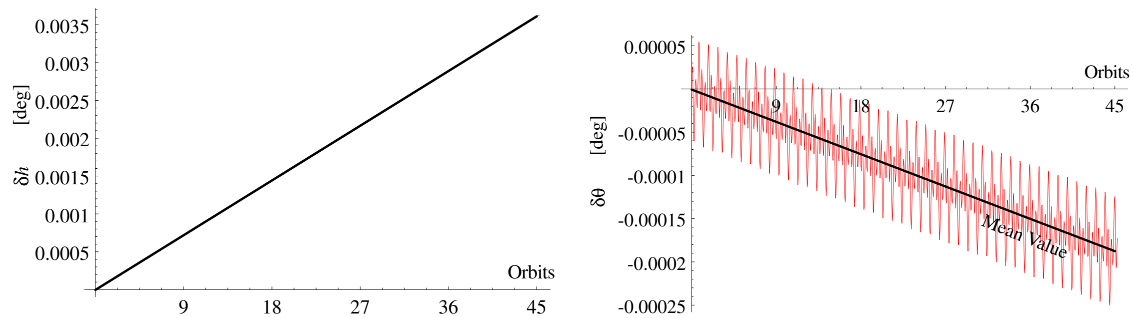



(c) Inclination Angle Error $\delta i$ $(deg)$




(d) Relative Orbit Setup Performed in Osculating Orbit Elements

**Figure 3: Relative Orbit Drift for a Near-Polar Chief Orbit**




(a) Difference in Ascending Node $h$

(b) Difference in Latitude Angle $\theta$

**Figure 4: Mean and Osculating Orbit Element Differences for Case 2**

is computed through the equal relative latitude rate condition in Eq. (20). The resulting relative orbit does exhibit some drift since the ascending nodes are drifting apart. Over a year, the $\Delta v$ required to compensate for this drift is roughly 56.8 m/s. However, for case 3 the equal latitude rate condition is also dropped (i.e. $\delta a = 0$ meters for the same $\delta e$), then the resulting orbit shown in Figure 3(c) has some clear along-track drift. Case 4 has the same initial orbit element differences as the ones used in case e, but here the orbits were established using osculating orbit elements instead of mean orbit elements. The resulting relative orbit is shown in Figure 3(d). This would be analogous to setting up the relative orbit initial conditions using the CW or Hills equations. Over the 45 orbits shown, clearly substantial drift would result. This emphasized the point that one should be working with mean orbit elements when design the relative orbits.

Figure 4 illustrates the relative nodal and latitude rate drifts for case 2. By dropping the equal nodal rate condition, the nodes clearly drift apart over time. The corresponding osculating relative ascending node variations are not visible due to the large drift. While the relative latitude drift is not perfectly zero, it is kept very small. The fuel estimate to compensate for the $\delta\theta$ drift over one year is only 1.45 m/s, while it would be about 14.1 m/s if the equal latitude rate condition is dropped. However, as a comparison, to compensate for the relative ascending node drift it would take about a fuel cost of 56 m/s over year to compensate.

Thus, it is possible to design relative orbits with out-of-plane motion created by an inclination change and a chief orbit that is near-polar. However, the equal ascending node rate condition must be dropped here to obtain a relative orbit of practical value. Periodic maneuvers will be required to compensate for the $\delta h$ drift. References 7 and 8 present continuous feedback and impulsive control schemes respectively in terms of the mean orbit elements. For an orbit such as is presented in Case 2, it would make sense to use the impulsive control scheme where the ascending node is correct during the polar region crossings using:

$$\Delta v_{h_\Omega} = \frac{h \sin i}{r \sin \theta} \Delta\Omega \qquad \text{for } \theta = \pm 90 \text{ degrees} \tag{27}$$

*Near-Circular Orbits*

As the chief's orbit eccentricity becomes small, the eccentricity differences commanded by the equal nodal rate condition may cause the relative orbit to become very large in the along-track direction. This is clear from the linear mapping of differences in $e$ to differences in $\eta$ shown in Eq. (21) to be:

$$\delta e = \frac{\eta}{e}\delta\eta$$

However, the change in $e$ does not become infinitely large as $e \to 0$. The equal nodal rate condition in Eq. (15) shows a finite required difference in $\eta$ as $e$ goes to zero and $\eta$ goes to one. Using the nonlinear relationship between $\eta$ and $e$ shown in Eq. (2), this finite $\delta\eta$ corresponds to a finite $\delta e$ for a circular orbit. However, these eccentricity changes may still result in a relative orbit which is too large for practical use. Again, as was the case with near-polar chief orbits, if the out-of-plane motion can be produced by a change in node instead of a change in inclination angle, then having a chief orbit with a small eccentricity would not pose any practical difficulties.

A numerical simulation is performed to illustrate this behavior. The chief orbit elements are shown in Table 2. The relative orbit is established using the mean orbit element differences of $\delta h = 0.01$ degrees, $\delta g = 0.01$ degrees and $\delta l =$ -0.01 degrees. An inclination angle difference $\delta i$ of 0.01 degrees is requested. The relative orbits were computed for the three mean chief eccentricities 0.04, 0.05 and 0.06.

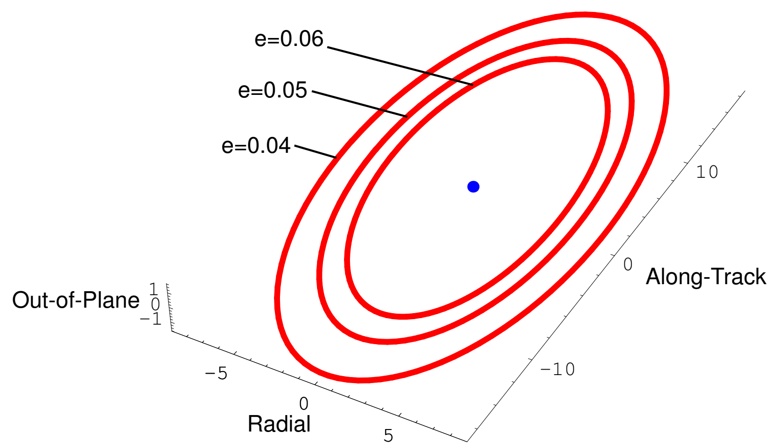
**Table 2: Chief Satellite Orbit Elements for Small Eccentricity Case Study.**

| Desired Average Orbit Elements | Value | Units |
|---|---|---|
| $a$ | 7153 | km |
| $e$ | 0.04, 0.05 or 0.06 | |
| $i$ | 48 | deg |
| $h$ | 0.0 | deg |
| $g$ | 30.0 | deg |
| $l$ | 0.0 | deg |

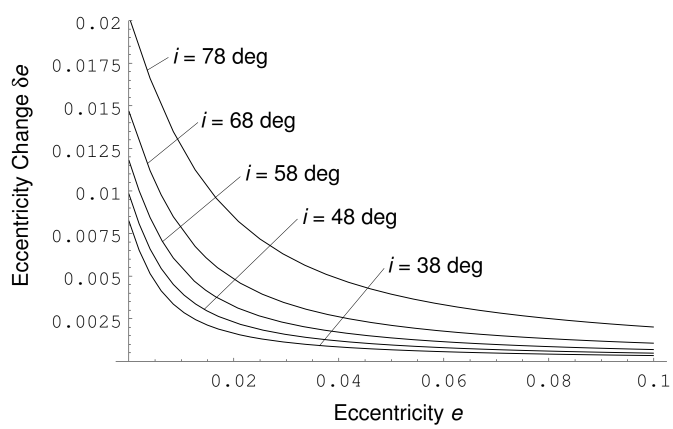
Figure 5(a) compares the resulting three relative orbits. For the case where $e = 0.06$, the requested $\delta i$ required a $\delta e$ of 0.000799. The case where $e = 0.05$ resulted in a $\delta e$ of 0.000957 and the case with $e = 0.04$ resulted in $\delta e = 0.001191$. Clearly the smaller chief eccentricities result in a larger relative orbit in the along track direction.

This general behavior is also illustrated in Figure 5(b) where the required $\delta e$ for a $\delta i$ of 0.01 degrees are displayed for various chief eccentricities $e$ and inclination angles $i$. Due to the $\tan i$ term in the equal nodal rate condition, the effect of having small eccentricities is enhances for larger inclination angles. The $\delta e$ here were computed using the nonlinear mapping between $\eta$ and $e$ in Eq. (2). While the required eccentricity for the relative motion orbit does grow large as $e$ approaches zero, it reaches a finite limit for a circular chief orbit case and does not become infinite.

This result is interesting in that it states that it is easier to compensate for out-of-plane motion induced by $\delta i$ if the chief orbit has a larger eccentricity. The richer dynamics of




(a) Relative Orbit in LVLH Frame for various Eccentricities




(b) Eccentricity Differences for Small Eccentricities

**Figure 5: Small Eccentricity Case Study**

having a more eccentric orbit makes it easier to compensate for the relative nodal drift condition.

*Relative Argument of Perigee and Mean Anomaly Drift*

To establish the $J_2$ invariant orbits, conditions are established which set the relative ascending node rate $\delta\dot{h}$ and latitude rate $\delta\dot{\theta}$ equal to zero. While this guarantees that the angle between the chief and deputy position vector remains constant, it is possible that the argument of perigee and mean anomaly differences drift apart. The effect of this is that for chief orbits with non-zero eccentricity, the relative orbit geometry swells larger as $\delta g$ and $\delta l$ drift apart and then shrinks again as they eventually approach each other. Since the relative latitude rate is equal to zero when the two presented $J_2$ constraint conditions are satisfied, then we know that

$$\delta\dot{g} = -\delta\dot{l} \tag{28}$$

To compute the relative drift in the argument of perigee, we take the partial derivative of Eq. (4).

$$\delta\dot{g} = \frac{\partial\dot{g}}{\partial L}\delta L + \frac{\partial g}{\partial\eta}\delta\eta + \frac{\partial\dot{g}}{\partial i}\delta i \tag{29}$$

After substituting the $J_2$ invariant conditions in Eqs. (20) and (15), the relative perigee drift rate is found to be

$$\delta\dot{g} = -\epsilon\frac{3}{4L^7\eta^4}\left(\tan i(5\cos^2 i - 1) - 5\sin(2i)\right)\delta i \tag{30}$$

The following numerical simulation illustrates the effect of the perigee/mean anomaly drift has on the relative orbit geometry. The chief orbit elements are the same as are shown in Table 2 with an eccentricity set to be 0.05. A mean $\delta i$ of 0.01 degrees is prescribed and the mean $\delta h$ is set equal to 0.01 degrees. The argument of perigee and mean anomaly differences are set equal to

$$\delta g = -\delta l = 0.0, \quad 0.5 \text{ or } \quad 1.0 \quad \text{degrees}$$

The resulting three relative orbits are illustrated in the rotating LVLH frame in Figure 6. As the argument of perigee and mean anomaly differences drift apart, the overall relative orbit geometry is expanded without changing the shape itself appreciably. Not that the presented orbit has a relatively large eccentricity of 0.05. If the eccentricity were closer to zero, then the effect of the perigee/mean anomaly drift on the relative orbit geometry would be even less. At the limiting case where the chief orbit becomes circular, the perigee/mean anomaly drift would have no effect on the relative geometry.

While this drift in $\delta g$ and $\delta l$ is an effect that may have to be periodically compensated for, the argument of perigee and mean anomaly drift occurs very slowly. For the presented numerical simulation, the $\delta g$ had only drifted 0.05 degrees after 45 revolutions (roughly three days). Thus, for $\delta g$ to drift the 1.0 degrees shown in Figure 6, it would take at approximately 60 days.

To correct such specific orbit element differences, Reference 8 developed an impulsive feedback control scheme with the mean orbit element errors as the feedback quantity. While

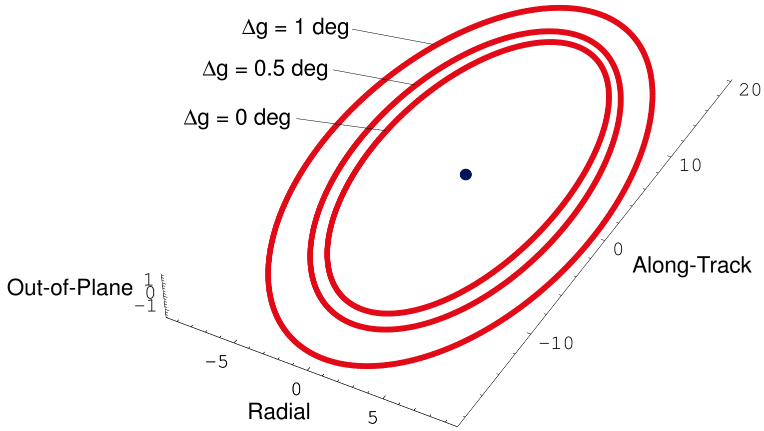



**Figure 6: Relative Orbits in LVLH Frames for Three Different Argument of Perigee and Mean Anomaly Differences**

this scheme is able to correct any types of orbit element errors, the $g$ and $l$ correction are of interest to the present problem. Let $\Delta v_{r_p}$ a orbit radial thrust performed at perigee, and $\Delta v_{r_a}$ the orbit radial thrust performed at apogee. In order to correct a specific $\Delta g = -\Delta l$ error, the following control is used.

$$\Delta v_{r_p} = -\frac{na}{4}\left(\frac{(1+e)^2}{\eta} - 1\right)\Delta g \tag{31}$$

$$\Delta v_{r_a} = \frac{na}{4}\left(\frac{(1-e)^2}{\eta} - 1\right)\Delta g \tag{32}$$

The advantage of this impulsive firing scheme is that only the osculating $g$ and $l$ are adjusted in a near-optimal manner. Reference 8 goes into further details describing how this scheme can also be used to correct for mean orbit element errors.

## Fuel Cost Estimation

As has been shown in the previous sections, at times it may be beneficial to relax the two constraints on the mean orbit elements in order to obtain a relative orbit solution which is of practical value. This section presents convenient formulas which allow us to predict the fuel cost in terms of $\Delta v$'s that must be applied to cancel any $J_2$ induced drift if the orbit elements $a$, $e$ and $i$ do not perfectly match the conditions in Eqs. (15) and (20). To perform this analysis, it is convenient to use the dimensional orbit element drift equations, as opposed to their non-dimensional versions in Eqs. (3) through (5).

$$\dot{h} = -\frac{3}{2}J_2\frac{r_e^2}{a^2}\frac{n}{\eta^4}\cos i \tag{33}$$

$$\dot{g} = \frac{3}{4}J_2\frac{r_e^2}{a^2}\frac{n}{\eta^4}(5\cos^2 i - 1) \tag{34}$$

$$\dot{l} = n - \frac{3}{4}J_2\frac{r_e^2}{a^2}\frac{n}{\eta^3}(1 - 3\cos^2 i) \tag{35}$$

Note that $n = \sqrt{\mu/a^3}$ is the mean orbit motion of the chief satellite and $r_e$ is the Earth's equatorial radius. Further, any orbit elements used in Eqs. (33) through (35) are the mean orbit elements of the chief satellite.

The methodology to compute the fuel cost to combat the $J_2$ induced drift will be the same for all the cases. First, we will compute how much drift the momenta orbit element differences $\delta a$, $\delta e$ and $\delta i$ will cause over one orbit. Then, using impulsive control, we are able to provide an estimate of what $\Delta v$ would be required to cancel the $J_2$ induced drift. Note that these fuel estimates will not be precise predictions, but rather they provide a convenient method to quickly assess how roughly much fuel would be required to combat the $J_2$ perturbation if the momenta orbit element differences are not set at their ideal $J_2$ invariant values.

*Ascending Node Relative Drift Correction Cost Estimate*

First, we find an estimate of the fuel required to control the $J_2$ induced ascending node drift. The derivative of Eq. (33) is used to compute the relative nodal drift $\delta\dot{h}$. Note that advantage is taken here of the fact that the semi-major axis differences $\delta a$ are assumed to be of order $J_2$ and are thus ignored here as higher order terms.

$$\delta\dot{h} = \frac{3}{2} J_2 \frac{r_e^2}{a^2} \frac{n}{\eta^5} \left(\eta \sin i \ \delta i + 4 \cos i \ \delta\eta\right) \tag{36}$$

The orbit period $T$ of the chief satellite is given by

$$T = \frac{2\pi}{n} \tag{37}$$

The $J_2$ induced drift in the ascending node over one orbit period is then given by

$$\Delta h_{orbit} = \delta\dot{h} \cdot T = 3 J_2 \pi \frac{r_e^2}{a^2 \eta^5} \left(\eta \sin i \ \delta i + 4 \cos i \ \delta\eta\right) \tag{38}$$

Eq. (38) provides an estimate of the amount of ascending node correction that would be required per orbit. To compute what $\Delta v$ would be required to perform these corrections, the impulsive control scheme developed in Reference 8 is used here. The impulses developed in this control law to correct specific orbit element errors are based on Gauss' variational equations.[1] The ideal time to perform a node correction is during the polar crossings where $\theta = \pm 90$ degrees. Firing an impulse $\Delta v_h$ in the orbit normal direction, the following node correction is achieved:

$$\Delta v_h = \frac{h \sin i}{r_h} \Delta h \tag{39}$$

Note that $r_h$ is the orbit radius at $\theta = \pm 90$ degrees. After substituting Eq. (38) into Eq. (39), and performing several simplifications, the following fuel estimate is found to counter a $J_2$ induced nodal drift.

$$\Delta v_h = 3 J_2 \pi \frac{r_e^2}{r_h} \frac{n}{\eta^4} \sin i \left(\eta \sin i \ \delta i + 4 \cos i \ \delta\eta\right) \tag{40}$$

Note that this $\Delta v$ estimate is the fuel required per orbit. To find a yearly fuel budget estimate, this figure needs to be multiplied by the number of orbits that occur in one year.

As expected, if the mean orbit element differences $\delta i$ and $\delta\eta$ satisfy the equal nodal rate condition in Eq. (15), then the predicted fuel budget is zero. Note that the actual fuel budget would not be zero though. This is because several first order approximations were made in developing the two constraints in Eqs. (15) and (20).

Eq. (40) does provide a very convenient method to quickly estimate the fuel budget if the $J_2$ invariant conditions are not setup perfectly. Assume the relative orbit is designed using the linear CW equations. Here the chief orbit is circular and we set the inclination angle equal to 70 degrees and the semi-major axis equal to 7000 km. To obtain an out-of-plane motion of roughly one kilometer, a $\delta i$ of 0.01 degrees is required. Using Eq. (40), this leads to an annual fuel budget estimate of 43.6 m/s solely to correct for the relative ascending node drift. A cost which could be avoided if the $J_2$ perturbation is taking into account when designing the relative orbit.

*Argument of Perigee and Mean Anomaly Relative Drift Correction Cost Estimate*

After having found a fuel budget estimate to correct the relative nodal drift, fuel budget estimates are now developed to correct for both the relative argument of perigee drift and mean anomaly drift. Taking the derivative of Eq. (34) and making use again of the fact that $\delta a$ is of the order of $J_2$, the relative argument of perigee drift rate is expressed as

$$\delta\dot{g} = -\frac{3}{4}J_2\frac{r_e^2}{a^2}\frac{n}{\eta^5}\left(5\eta\sin(2i)\delta i + 4\left(5\cos^2 i - 1\right)\delta\eta\right) \tag{41}$$

Using Eq. (37), the perigee drift over one orbit is estimated to be

$$\Delta g_{orbit} = \delta\dot{g}\cdot T = -J_2\frac{3\pi}{2\eta^4}\frac{r_e^2}{a^2}\left(5\eta\sin(2i)\delta i + 4\left(5\cos^2 i - 1\right)\delta\eta\right) \tag{42}$$

The mean anomaly drift over orbit is computed in an analogous manner. Note, however, that here $\delta a$ appears without being multiplied by $J_2$ and is thus retained.

$$\Delta l_{orbit} = \delta\dot{l}\cdot T = -\frac{3\pi}{a}\delta a - \frac{9\pi}{2}J_2\frac{r_e^2}{a^2}\frac{n}{\eta^4}\left(\eta\sin(2i)\delta i - \left(1 - 3\cos^2 i\right)\delta\eta\right) \tag{43}$$

Again, note that Eqs. (42) and (43) provide angular drift estimates for one orbit period. To compute the annual drift, these figures would be multiplied by the number of orbit period in a year.

To compute $\Delta v$'s necessary to perform the required $\Delta g$ and $\Delta l$ corrections, the two impulse technique presented in Reference 8 is used. Here an orbit radial thrust is applied at both perigee and apogee to achieve the desired orbit element corrections in a near-optimal manner and without affecting the remaining orbit elements. Using this method, the two $\Delta v$ are then computed through

$$\Delta v_{r_p} = -\frac{na}{4}\left(\frac{(1+e)^2}{\eta}\Delta g + \Delta l\right) \tag{44}$$

$$\Delta v_{r_a} = \frac{na}{4}\left(\frac{(1-e)^2}{\eta}\Delta g + \Delta l\right) \tag{45}$$

where $\Delta g$ and $\Delta l$ are computed through Eqs. (42) and (43) respectively. The total fuel estimate required to control either relative argument of perigee drift, relative mean anomaly drift or both is then computed as

$$\Delta v_{g,l} = |\Delta v_{r_p}| + |\Delta v_{r_a}| \tag{46}$$

*Latitude Relative Drift Correction Cost Estimate*

While Eq. (46) is convenient to estimate the fuel budget to correct for $g$ or $l$ relative drifts, for the formation flying problem this is of lesser importance. What is more critical is what is the fuel budget to combat the latitude drift, i.e. the sum of both the relative perigee and mean anomaly drift. For nearly circular orbits the argument of perigee and mean anomaly can drift apart with negligible effect on the relative orbit geometry, as long the sum of their drifts is zero. In this section we will provide a fuel budget estimate to control the relative latitude drift. The amount of latitude drift rate is computed through

$$\delta\dot{\theta} = \delta\dot{g} + \delta\dot{l} \tag{47}$$

To estimate how much fuel is required to correct a latitude error, it is assumed that a $\Delta v$ is applied to change the semi-major axis $a$ (and thus the orbit period) which will speed up or slow down the satellite such that it correct the $\delta\theta$ error over one orbit. At the end of the correction, a second such $\delta a$ adjustment must be made to reinsert the satellite in the previous orbit. ¿From Gauss' variational equations, the required $\Delta v$ for a given $\Delta a$ is

$$\Delta v = \frac{h(1-e)}{2a^2(1-e^2)}\delta a = \frac{n}{2}\sqrt{\frac{1-e}{1+e}}\delta a \tag{48}$$

To relative the change in semi-major axis $\delta a$ to the corresponding change in orbit period $\delta T$, we differentiate Eq. (37) and make use of $n = \sqrt{\mu/a^3}$.

$$\delta a = \frac{2a}{3}\frac{\delta T}{T} \tag{49}$$

The final step is to relate the latitude drift amount $\delta\theta$ per orbit to the required orbit period change $\delta T$ which will accomplish this correction. This is found through

$$\delta\theta_{orbit} = \delta\dot{\theta}\cdot T = n\cdot\delta T \tag{50}$$

Substituting Eqs. (49) and (50) into Eq. (48), a fuel budget estimate to correct the per orbit latitude drift is

$$\Delta v_\theta = \frac{a}{3}\sqrt{\frac{1-e}{1+e}}\delta\dot{\theta} \tag{51}$$

If the $\delta a$, $\delta e$ and $\delta i$ differences satisfy the conditions in Eqs. (15) and (20), then the latitude drift $\delta\dot{\theta}$ becomes zero, resulting in a zero fuel budget estimate. This is easily achievable.

## Conclusion

A survey of a method to establish $J_2$ invariant relative orbits for spacecraft formation flying applications is presented. The desired relative orbit geometry is designed using differences in mean orbit elements. Two constraints on the three momenta element differences $\delta a$, $\delta e$ and $\delta i$ are required to maintain zero relative drift in the mean ascending node rate and the latitude rate. Three specific challenges in designing such $J_2$ invariant relative orbits are discussed. As the inclination angle $i$ approaches a polar orbit, the corrections required in eccentricity and semi-major axis to compensate for the $J_2$ effect become too large to be of practical value. Working with near-polar orbits, setting up the relative orbit geometry in mean elements and canceling the latitude rate difference still provides a substantial drift and associated fuel savings. Another similar limitation occurs when the chief eccentricity becomes small for non-equatorial orbits. The smaller the eccentricity is, the more difficult it is to compensate for a difference in mean inclination angle. Third, the issue of of the argument of perigee and mean anomaly drift is discussed. These two quantities will not remain constant, but drift apart at equal and opposite rates. While this drift does has an enlarging effect on the relative orbit, it occurs very slowly and is easily controlled with the presented impulsive control scheme. Further, the fuel budget estimates are very valuable when designing relative orbit since they provide a cost figure for any deviations which may be made from the ideal orbit element differences which would have resulted in the relative orbit being $J_2$ invariant. Due to mission constraints or when designing near-polar orbit with out-of-plane relative motion, it may be necessary to relax one or both of the $J_2$ invariant conditions.

## Acknowledgment

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