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Abstract.

In this paper we study shape-preserving formations of three spacecraft, where the formation keeping forces arise from the electric charges deposed on each craft. Inspired by Lagrange's three body problem, the general conditions that guarantee preservation of the geometric shape of the electrically charged formation are derived. While the classical collinear configuration is a solution to the problem, the equilateral triangle configuration is found to only occur with unbounded relative motion. The three collinear spacecraft problem is analyzed and the possible solutions are categorized based on the spacecraft mass-charge ratio. Precise statements on the number of solutions associated with each category are provided. Finally, a methodology is proposed to study boundedness of the collinear solution that is inspired by past understanding and results for the three body problem. Given initial position and velocity vectors of each craft along with the charges, analytical solutions are provided describing the resulting relative motion.

Keywords: Coulomb thrusting, rotating equilibria, invariant shape formation

1. Introduction

Spacecraft naturally charge to negative potentials while in Earth orbit. The cause is the interaction of the spacecraft with the ambient plasma environment. For a given plasma temperature, the craft is more likely to be struck by electrons than by positively charged ions. Geostationary spacecraft assume a steady-state potential which causes a zero net-current to flow to and from the craft. A NASA NIAC study in 2002 was conducted by King et al. (2002) to study the spacecraft charge data of the SCATHA mission (Mullen et al., 1986). The results found that the naturally occurring GEO spacecraft charge can become large enough to produce micro- to milli-Newton level forces on neighboring spacecraft 20–30 meters away. Besides measuring the plasma environment and the spacecraft potential levels, the SCATHA mission also demonstrated that the spacecraft charge can be artificially controlled by emitting ions or electrons. This method of generating inter-spacecraft forces is referred to as the Coulomb propulsion concept. Spacecraft are designed to control their charge level relative to the plasma environment, and will thus produce forces onto neighboring charged satellites. The associated fuel consumption is extremely small, with estimated fuel efficiencies ranging from 10^{10}–10^{13} seconds (King et al., 2002; King et al., 2003). For this reason the Coulomb propulsion concept is referred to as being essentially propellantless. Further, the typical electrical power requirements are approximately 1 Watt or less. The spacecraft charge control technology has been also demonstrated more recently on the

CLUSTERS mission (Escoubet et al., 2001). However, in this mission, the idea to use spacecraft charge is not used for relative motion control, but to zero the spacecraft charge to avoid biases in the sensitive particle instruments (Torkar and et. al., 1999; Torkar et al., 2001).

The Coulomb propulsion concept can provide very fuel- and power-efficient means to control close relative motion between spacecraft. Applications range from flying 20–100 meter wide-field-of-view space interferometery sensor platforms (King et al., 2002; Chakravorty, 2004; Hussein, 2005), forming virtual semi-rigid structure of variable shape and size (King et al., 2002; King et al., 2003; Schaub et al., 2004; Berryman and Schaub, 2005b; Berryman and Schaub, 2005a; Schaub et al., 2005), using electrostatic forces to guide a smaller scout craft about a larger primary craft, as well as forming electrostatically tethered strings of spacecraft formations (Natarajan and Schaub, 2005). With Coulomb tether formations, physical ropes are replaced with massless Coulomb force fields to tie together a series of craft. Natarajan and Schaub (2005) introduced a nadir pointing 2-craft Coulomb tether formation of fixed length. Exploiting the small gravity gradient torque at GEO, a feedback control law is presented to stabilize both the length and in-plane attitude of the 2-craft systems. However, the Coulomb thrusting concept is only applicable at certain orbit altitudes. A charged craft within the space plasma environment will gather plasma particles of the opposite charge about itself. This causes a phenomena called charge shielding, where the electric field of the spacecraft is masked from another craft nearby. The colder or denser the plasma is, the worse the shielding will be. The conventional simplified mathematical model of this charge shielding effect adds an exponential decay term to the traditional vacuum electrostatic field equation. The exponential decay is controlled through a parameter called the Debye length (Nicholson, 1992; Garrett and DeFrost, 1979). At low Earth orbits, the Debye lengths are on the order of centimeters, making Coulomb thrusting infeasible. However, at GEO Debye lengths can range from 100-1000 meters, depending on solar activities (King et al., 2002; King et al., 2003). Within the solar system at 1 AU radius, the Debye length is about 10–30 meters due to interaction with the solar wind (King et al., 2002).

This paper investigates dynamic charge spacecraft formations. Here three craft are assumed to be freely floating in space where the dominant force acting on the craft is the Coulomb force. For example, spinning charged three-craft formations are of interest that are on heliocentric orbits far removed from the gravitational influence of planets. Thus, differential gravitational accelerations between the craft are not yet considered in this work. At 1 astronomical unit distance away from the sun the Debye length can be 20-40 meters due to the interaction with the solar wind. For missions with greater sun separation distances this Debye length would increase. Of interest is developing necessary conditions for the three craft to retain a fixed formation shape. Previous work on fixed Coulomb formation shapes assumed the formation is static relative to the orbiting Hill frame (King et al., 2002; King et al., 2003; Schaub et al., 2004; Berryman and Schaub, 2005b; Berryman and Schaub, 2005a; Schaub et al., 2005). The spacecraft were placed at very specific locations with specific charges such that the differential orbital accelerations were zero. In this study the three craft are assumed to be orbiting each other. For example, the three craft could dynamically sweep through an area while taking interferometry measurements. Instead of having a large number of craft occupy critical interferometric nodal points in the image plane, the three craft would visit these nodal points over time as a revolution is completed.

Mathematically this problem is strongly related to the general three-body problem of celestial mechanics (Szebehely, 1967; Roy, 1982; Battin, 1987; Schaub and Junkins, 2003). In 1772, Lagrange presented analytical solutions to the three-body problem where certain restrictions were applied (Lagrange, 1772). In particular, he found that equilateral and collinear solutions exist where the shape is invariant over time. Ignoring the shielding effect of the space plasma environment, the electric potential field strength drops with the square of the separation, equivalent to how the gravitational potential fields of point masses decay. However, with the electrostatic bodies, the potential field strength can be tuned by controlling the spacecraft charge. This provides interesting generalizations to Lagrange's gravitational three-body solutions. Starting from basic principles, feasible invariant rotating three-craft Coulomb formations are investigated. This study includes the charge shielding effect. Particular solutions for large and small Debye length scenarios are discussed.

The paper is organized as follows. In Section 2, we derive necessary conditions that any shape-preserving formation must satisfy. In Section

3, we study the collinear and triangular configurations as tentative solutions to the problem. Finally, in Section 4, we propose a procedure that stems from the two body problem to determine the boundedness (elliptic, parabolic or hyperbolic) of the orbits of the formation.

2. The Three Spacecraft Coulomb Tether Problem

The treatment in this section is almost identical to Lagrange's solution to the restricted three body problem except that the potential field is created by the electric charge deposited on each spacecraft in the formation. For the sake of completeness, we will briefly go through the full details. The basic assumption we make is that the formation is in free space. That is, there are no external forces acting on the system. The only force each spacecraft experiences is that generated by the electric field produced by the charges on the spacecraft.

Refer to Figure 1. The system is composed of three point mass particles of masses m_1, m_2, and m_3 and charges q_1, q_2 and q_3, possibly functions of time, respectively. Let $\mathcal{N}$ be an inertial reference frame and $\mathbf{r}_c$ be the position vector of the center of mass of the three particle system. The vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are the position vectors of the three masses relative to the system center of mass. Let $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ be the relative position vectors between the masses. The angles α_i, $i = 1, 2, 3$, are the angles between the relative vectors $\mathbf{r}_{ij}$ as shown in the figure. Finally, the unit vectors $\mathbf{e}_{r_i}$ and $\mathbf{e}_{\theta_i}$ are unit vectors centered at the

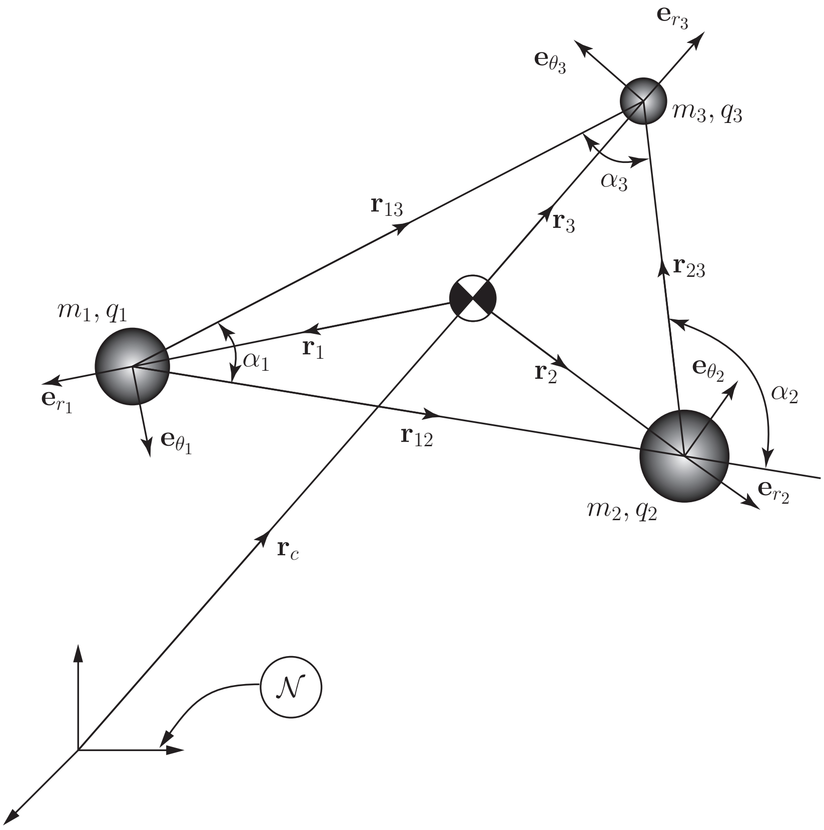


Figure 1. The Three-Spacecraft Problem.

centers of each of the three masses, where $\mathbf{e}_{r_i}$ is along $\mathbf{r}_i$ and $\mathbf{e}_{\theta_i}$ is perpendicular to $\mathbf{r}_i$ such that the triple $\mathbf{e}_{r_i}, \mathbf{e}_{\theta_i}, \mathbf{e}_3$ form a dextral set of orthonormal unit vectors and where $\mathbf{e}_3$ is perpendicular to the plane containing the three masses.

Since the system does not experience any external forces, the system center of mass satisfies

$$M\ddot{\mathbf{r}}_c = 0 \tag{1}$$

where

$$M = m_1 + m_2 + m_3 \tag{2}$$

is the total system mass. The only force acting on each craft is that due to the interaction of the electric charges generated by all spacecraft. Hence, the individual spacecraft equation of motion is given by (Schaub and Junkins, 2003)

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i = \sum_{j=1,\ \neq i}^{3} k_c q_i(t) q_j(t) \frac{\mathbf{r}_{ji}}{r_{ji}^3} \mathrm{e}^{-\frac{r_{ji}}{\lambda_d}} \tag{3}$$

where $k_c = 8.99 \times 10^9 \mathrm{Nm}^2/\mathrm{C}^2$ is Coulomb's constant and λ_d is the Debye length. The quantity $r_{ij} = r_{ji} \geq 0$ denotes the distance between spacecraft i and j. Similarly, we have $r_i = |\mathbf{r}_i| \geq 0$. Since the relative positions $\mathbf{r}_i$, $i = 1, 2, 3$, are relative to the center of mass, then

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 = 0 \tag{4}$$

This and equation (2) imply

$$\begin{aligned} M\mathbf{r}_1 &= -m_2 \mathbf{r}_{12} - m_3 \mathbf{r}_{13} \\ M\mathbf{r}_2 &= m_1 \mathbf{r}_{12} - m_3 \mathbf{r}_{23} \\ M\mathbf{r}_3 &= m_1 \mathbf{r}_{13} + m_2 \mathbf{r}_{23} \end{aligned} \tag{5}$$

Squaring these equations, we obtain

$$\begin{aligned} M^2 r_1^2 &= m_2^2 r_{12}^2 + m_3^2 r_{13}^2 + 2 m_2 m_3 \mathbf{r}_{12} \cdot \mathbf{r}_{13} \\ M^2 r_2^2 &= m_1^2 r_{12}^2 + m_3^2 r_{23}^2 - 2 m_1 m_3 \mathbf{r}_{12} \cdot \mathbf{r}_{23} \\ M^2 r_3^2 &= m_1^2 r_{13}^2 + m_2^2 r_{23}^2 + 2 m_1 m_2 \mathbf{r}_{13} \cdot \mathbf{r}_{23} \end{aligned} \tag{6}$$

Assume we seek solutions which retain the shape of the three charged particles in a rigid formation. Let $f(t)$ be a non-zero, generic, time varying function such that $f(0) = 1$. To maintain the initial configuration shape, we require

$$\frac{r_{12}(t)}{r_{12}(0)} = \frac{r_{13}(t)}{r_{13}(0)} = \frac{r_{23}(t)}{r_{23}(0)} = f(t) \tag{7}$$

or, equivalently, that the angles α_i, $i = 1, 2, 3$, be fixed. Substituting equation (7) into (6), we obtain

$$M^2 r_1^2(t) = f^2(t) \left(m_2^2 r_{12}^2(0) + m_3^2 r_{13}^2(0) + 2 m_2 m_3 r_{12}(0) r_{13}(0) \cos\alpha_1 \right) \tag{8}$$

Since $f(0) = 1$, then we have

$$r_1(0) = \frac{1}{M} \sqrt{m_2^2 r_{12}^2(0) + m_3^2 r_{13}^2(0) + 2 m_2 m_3 r_{12}(0) r_{13}(0) \cos\alpha_1} \tag{9}$$

and, therefore, we get $r_1(t) = r_1(0) f(t)$. A similar derivation for $r_2(t)$ and $r_3(t)$ gives

$$r_i(t) = r_i(0) f(t),\ i = 1, 2, 3 \tag{10}$$

Let $\boldsymbol{\omega}_i = \mathbf{r}_i \times \mathbf{v}_i$ denote the angular rate of rotation vector of particle i, $i = 1, 2, 3$, about the center of mass. To retain formation shape, we must have

$$\boldsymbol{\omega}_1(t) = \boldsymbol{\omega}_2(t) = \boldsymbol{\omega}_3(t) = \omega(t)\mathbf{e}_3 \tag{11}$$

Again, since no external forces are applied to the system, then the total angular momentum must remain constant

$$\mathbf{H} = \sum_{i=1}^{3} \mathbf{r}_i \times (m_i \dot{\mathbf{r}}_i) = \text{ constant} \tag{12}$$

By definition of the unit vectors $\mathbf{e}_{r_i}$ and $\mathbf{e}_{\theta_i}$, we have

$$\begin{aligned} \mathbf{r}_i &= r_i \mathbf{e}_{r_i} \\ \dot{\mathbf{r}}_i &= \dot{r}_i \mathbf{e}_{r_i} + r_i \omega \mathbf{e}_{\theta_i} \\ \ddot{\mathbf{r}}_i &= \left(\ddot{r}_i - r_i \omega^2\right) \mathbf{e}_{r_i} + \left(2\dot{r}_i \omega + r_i \dot{\omega}\right) \mathbf{e}_{\theta_i} \end{aligned} \tag{13}$$

This and equations (10) allow us to write the momentum as

$$\mathbf{H} = \sum_{i=1}^{3} m_i r_i^2(0) f^2 \omega \mathbf{e}_3 \tag{14}$$

Since $\mathbf{H}$ is constant, then $f^2(t)\omega(t)$ is a constant quantity. Looking at each individual momentum $\mathbf{H}_i$ we find that $\mathbf{H}_i = \mathbf{r}_i \times (m_i \dot{\mathbf{r}}_i) = m_i r_i^2(0) f^2 \omega \mathbf{e}_3$, which must be constant since $f^2\omega$ is constant. Therefore, we have

$$\dot{\mathbf{H}}_i = \mathbf{r}_i \times (m_i \ddot{\mathbf{r}}_i) = \mathbf{r}_i \times \mathbf{F}_i = 0 \tag{15}$$

where we used equation (3). This implies that *a necessary condition to maintain the shape of the formation is that* $\mathbf{F}_i$ *and* $\mathbf{r}_i$ *be aligned*. Hence, we must have

$$\mathbf{F}_i = F_i \mathbf{e}_{r_i} \tag{16}$$

Substituting this and the third equation in (13) into equation (3), we obtain

$$F_i = m_i \left(\ddot{r}_i - r_i \omega^2 \right), \ i = 1, 2, 3 \tag{17}$$

Another consequence is that we must have

$$\dot{r}_i = -\frac{r_i \dot{w}}{2\omega} \tag{18}$$

in order to achieve a rigid formation. This and the second of equations (13) imply that $\dot{\mathbf{r}}_i = -\frac{r_i \dot{\omega}}{2\omega} \mathbf{e}_{r_i} + r_i \omega \mathbf{e}_{\theta_i}$. Hence, the angle (determined from $\mathbf{r}_i \cdot \dot{\mathbf{r}}_i = \|\mathbf{r}_i\| \, \|\dot{\mathbf{r}}_i\| \cos \beta_i$) between the velocity vectors and the position vectors of each of the spacecraft relative to the system center of mass is given by

$$\cos \beta_i = \cos \beta = -\frac{\dot{\omega}}{\sqrt{\dot{\omega}^2 + 4\omega^4}} \tag{19}$$

and is therefore the same for all craft.

Substituting equation (10) into equation (17), we obtain

$$\frac{F_i}{m_i r_i} = \frac{\ddot{f}}{f} - \omega^2 =: A(t),\ i = 1, 2, 3 \tag{20}$$

which implies $F_i(t) = m_i r_i(t) A(t)$. Since $\mathbf{r}_i \times \mathbf{F}_i = 0$, then we have

$$\begin{aligned} 0 &= \mathbf{r}_1 \times \left(\frac{q_2 \mathbf{r}_2 \mathrm{e}^{-\frac{r_{21}}{\lambda_d}}}{r_{21}^3} + \frac{q_3 \mathbf{r}_3 \mathrm{e}^{-\frac{r_{31}}{\lambda_d}}}{r_{31}^3} \right) \\ 0 &= \mathbf{r}_2 \times \left(\frac{q_1 \mathbf{r}_1 \mathrm{e}^{-\frac{r_{21}}{\lambda_d}}}{r_{21}^3} + \frac{q_3 \mathbf{r}_3 \mathrm{e}^{-\frac{r_{32}}{\lambda_d}}}{r_{32}^3} \right) \\ 0 &= \mathbf{r}_3 \times \left(\frac{q_1 \mathbf{r}_1 \mathrm{e}^{-\frac{r_{13}}{\lambda_d}}}{r_{13}^3} + \frac{q_2 \mathbf{r}_2 \mathrm{e}^{-\frac{r_{23}}{\lambda_d}}}{r_{23}^3} \right) \end{aligned} \tag{21}$$

This provides general conditions for a generic fixed-shape formation motion.

To summarize, the general conditions to preserve the geometric shape of the formation are:

1. The net resultant force $\mathbf{F}_i$ on each spacecraft must pass through the system center of mass and, hence, must be along the radial vector locating each spacecraft relative to the system center of mass (equation (15)).

2. The magnitude of each spacecraft's velocity vector $\dot{\mathbf{r}}_i$ is proportional to the magnitude of the distance of the respective spacecraft to the system center of mass (from the second of equation (13) and equation (18)). Moreover, the angles β_i between the velocity vectors $\dot{\mathbf{r}}_i$, $i = 1, 2, 3$, and the respective position vector of each of the spacecraft to the system center of mass is equal for all spacecraft. That is, we have $\beta_1 = \beta_2 = \beta_3$ (equation (19)).

3. Particular Solutions

3.1. The Collinear Solution

We now study the existence of particular solutions to equation (21). One particular solution is when all spacecraft are located along a straight line. We will assume that the formation is configured as shown in Figure 2 with $r_3 > r_2 > r_1 \geq 0$. Since these distances are measured from the center of mass, the configuration as shown in Figure 2 will result whenever $m_1 > m_3$. In this case we have

$$\mathbf{r}_1 = r_1 \mathbf{e}_r, \ \mathbf{r}_2 = -r_2 \mathbf{e}_r, \ \mathbf{r}_3 = -r_3 \mathbf{e}_r \tag{22}$$

where $\mathbf{e}_r$ is as shown in the figure. This configuration will be assumed without any loss of generality. Particularly, if $m_3 > m_1$ then the system center of mass will lie between the second and third craft and, in this case, only the position vector of craft 2 will change to be $\mathbf{r}_2 = r_2 \mathbf{e}_r$. Note also that we are assuming without loss of generality that $r_2 > r_1$, which will not always hold true. If this condition does not hold, then the ensuing analysis will only need some adjustment, but the qualitative results will hold. In this and later sections we will assume the configuration given in Figure 2 with $r_2 > r_1$.

Substituting equations (22) into the equations of motion (3) and using the expression $\mathbf{F}_i = F_i \mathbf{e}_r$ (condition 1 above for preservation of

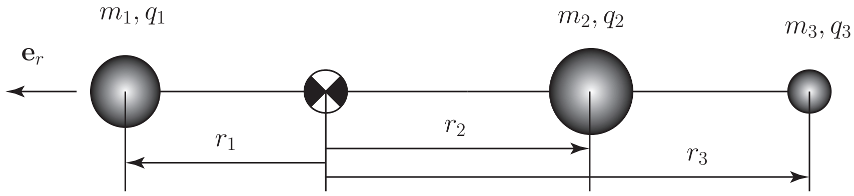


Figure 2. The Collinear Three Spacecraft Problem.

geometric shape) and equation (20), we obtain

$$\begin{aligned}
\frac{F_1}{m_1} = A(t)r_1 &= \frac{k_c q_1}{m_1}\left[q_2 \frac{r_1+r_2}{r_{21}^3} e^{-\frac{r_{21}}{\lambda_d}} + q_3 \frac{r_1+r_3}{r_{31}^3} e^{-\frac{r_{31}}{\lambda_d}}\right] \\
\frac{F_2}{m_2} = A(t)r_2 &= \frac{k_c q_2}{m_2}\left[q_1 \frac{r_1+r_2}{r_{12}^3} e^{-\frac{r_{12}}{\lambda_d}} + q_3 \frac{r_2-r_3}{r_{32}^3} e^{-\frac{r_{32}}{\lambda_d}}\right] \\
\frac{F_3}{m_3} = A(t)r_3 &= \frac{k_c q_3}{m_3}\left[q_1 \frac{r_3+r_1}{r_{13}^3} e^{-\frac{r_{13}}{\lambda_d}} + q_2 \frac{r_3-r_2}{r_{23}^3} e^{-\frac{r_{23}}{\lambda_d}}\right]
\end{aligned} \tag{23}$$

Let us define the positive non-dimensional spacecraft separation distance ratio χ as

$$\chi = \frac{r_{32}}{r_{21}} > 0 \tag{24}$$

then

$$\frac{r_{31}}{r_{21}} = 1 + \chi \tag{25}$$

Recall that $r_1 + r_2 = r_{12} = r_{21}$, $r_1 + r_3 = r_{13} = r_{31}$ and $r_3 - r_2 = r_{32} = r_{23}$. In equations (23), add the first equation to the second to obtain

$$\frac{A(t)}{k_c}(r_1 + r_2) = \frac{q_1 q_2}{r_{12}^2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) e^{-\frac{r_{32}}{\lambda_d}} + \frac{q_1 q_3}{m_1 r_{31}^2} e^{-\frac{r_{31}}{\lambda_d}} - \frac{q_2 q_3}{m_2 r_{32}^2} e^{-\frac{r_{32}}{\lambda_d}} \tag{26}$$

and subtract the third from the second to obtain

$$\frac{A(t)}{k_c}(r_2 - r_3) = \frac{q_2 q_3}{r_{23}^2}\left(\frac{1}{m_2} + \frac{1}{m_3}\right) e^{-\frac{r_{23}}{\lambda_d}} - \frac{q_1 q_2}{m_2 r_{21}^2} e^{-\frac{r_{21}}{\lambda_d}} + \frac{q_1 q_3}{m_3 r_{31}^2} e^{-\frac{r_{31}}{\lambda_d}} \tag{27}$$

Since $r_1 + r_2 = r_{12}$ and $r_3 - r_2 = r_{23}$, divide equation (26) by r_{12} and equation (27) by $-r_{23}$ and equating the two resulting equations we obtain

$$
\begin{aligned}
0 \;=\; & \frac{q_1 q_2}{m_1 m_2 r_{12}^2}\left[\frac{1}{r_{12}}\left(m_2 + m_1\right) + \frac{m_1}{r_{23}}\right] \mathrm{e}^{-\frac{r_{21}}{\lambda_d}} \\
& + \frac{q_1 q_3}{m_1 m_3 r_{13}^2}\left[\frac{m_3}{r_{12}} - \frac{m_1}{r_{23}}\right] \mathrm{e}^{-\frac{r_{31}}{\lambda_d}} \\
& - \frac{q_2 q_3}{m_2 m_3 r_{23}^2}\left[\frac{m_3}{r_{12}} + \frac{1}{r_{23}}\left(m_2 + m_3\right)\right] \mathrm{e}^{-\frac{r_{32}}{\lambda_d}}
\end{aligned} \tag{28}
$$

which, after multiplying by $r_{23} r_{12}^2$ throughout, simplifies to

$$
\begin{aligned}
0 \;=\; & c_1 c_2 \left[m_1 + \chi\left(m_1 + m_2\right)\right] \mathrm{e}^{-\frac{r_{21}}{\lambda_d}} + \frac{c_1 c_3 \left(\chi m_3 - m_1\right) \mathrm{e}^{-\frac{r_{31}}{\lambda_d}}}{\left(\chi + 1\right)^2} \\
& - \frac{c_2 c_3 \left(m_2 + m_3 + \chi m_3\right) \mathrm{e}^{-\frac{r_{32}}{\lambda_d}}}{\chi^2}
\end{aligned} \tag{29}
$$

where $c_i = q_i/m_i$. After multiplying throughout by $\chi^2 \left(1 + \chi\right)^2 \mathrm{e}^{r_{21}/\lambda_d}$ we finally obtain

$$
\begin{aligned}
0 \;=\; & c_1 c_2 \left[m_1 + \chi\left(m_1 + m_2\right)\right] \chi^2 \left(1 + \chi\right)^2 + c_1 c_3 \left(\chi m_3 - m_1\right) \chi^2 \mathrm{e}^{\frac{-r_{21}\chi}{\lambda_d}} \\
& - c_2 c_3 \left(m_2 + m_3 + \chi m_3\right) \left(1 + \chi\right)^2 \mathrm{e}^{\frac{r_{21}(1-\chi)}{\lambda_d}}
\end{aligned} \tag{30}
$$

There are two special cases that we may wish to consider next.

3.1.1. *Case I: Large Debye Lengths*

In this case we consider mission scenarios where $r_{ji} << \lambda_d$. This situation occurs in environments where the ambient plasma is of (almost) neutral charge. Interstellar (or, deep space) missions are one possibility for this to occur. Expression (30) now takes a particularly simple form that is only a function of χ and is given by

$$\begin{aligned} 0 &= -c_2c_3(m_2+m_3) - c_2c_3(2m_2+3m_3)\chi \\ &+ \left[c_1m_1(c_2-c_3) - c_2c_3\left(m_2+3m_3\right)\right]\chi^2 \\ &+ \left[c_1c_2\left(3m_1+m_2\right) + c_3m_3(c_1-c_2)\right]\chi^3 + c_1c_2(3m_1+2m_2)\chi^4 \\ &+ c_1c_2(m_1+m_2)\chi^5 \end{aligned} \tag{31}$$

This is a quintic equation in χ, which one can solve for. By the definition of χ, each solution to this equation gives a family of equilibria for the formation. Since the coefficients are of arbitrary signs, depending on the signs of $q_1,\ q_2,\ q_3$ there could be as many as five real roots to this equation, each with a family of solutions. Specifying either $r_2 + r_1$ or $r_3 + r_1$, along with a particular solution of the above equation we are able to uniquely solve for the equilibrium configuration.

Remark. Note that in equation (3), if we replace q_i with m_i and ignore the exponential (Debye) term, we obtain the equations of motion due to gravitational interaction between the three spherical bodies (the restricted three body problem), or what is known as Lagrange's problem. In this case equation (31) is now given by Lagrange's famous quintic equation in celestial mechanics (see Chapter 10 in (Schaub and Junkins, 2003)), which is precisely

$$\begin{aligned} 0 &= -(m_2+m_3) - (2m_2+3m_3)\chi - \left(m_2+3m_3\right)\chi^2 + \left(3m_1+m_2\right)\chi^3 \\ &+ (3m_1+2m_2)\chi^4 + (m_1+m_2)\chi^5 \end{aligned} \tag{32}$$

and has exactly one real root. This verifies our results so far. Also observe that if we set $c_1 = c_2 = c_3 \neq 0$, which corresponds to all craft having the same ratio of mass to charge, then again we obtain Lagrange's quintic equation (32). •

Depending on the signs and values of c_1, c_2, c_3, there will be many solution possibilities ranging from zero shape preserving solutions to a maximum of five. A special and important case is when all spacecraft have identical masses. Let $m_1 = m_2 = m_3 \neq 0$. The quintic equations now becomes

$$\begin{aligned} 0 &= -2q_2q_3 - 5q_2q_3\chi + (q_1q_2 - q_1q_3 - 4q_2q_3)\chi^2 \\ &\quad + (4q_1q_2 + q_1q_3 - q_2q_3)\chi^3 + 5q_1q_2\chi^4 + 2q_1q_2\chi^5 \end{aligned} \tag{33}$$

Dividing throughout by q_2q_3 (since we assume that all craft carry some nonzero charge) and setting $\delta = \frac{q_1}{q_3}$ and $\sigma = \frac{q_1}{q_2}$, we get

$$\begin{aligned} 0 &= -2 - 5\chi + (\delta - \sigma - 4)\chi^2 + (4\delta + \sigma - 1)\chi^3 + 5\delta\chi^4 + 2\delta\chi^5 \\ &=: f(\chi) \end{aligned} \tag{34}$$

To facilitate studying this equation, we first recall Descartes' rule of signs.

Fact 3.1 (Descartes' Rule of Signs)**.** Let $f(\chi) = a_n\chi^n + a_{n-1}\chi^{n-1} + \cdots + a_1\chi_1 + a_0$ be a polynomial where $a_n, a_{n-1}, \ldots, a_0$ are real coefficients. The number of positive real roots of f is either equal to the number of sign changes of successive terms of $f(\chi)$ or is less than that number by an even number (until 1 or 0 is reached). The number of negative real zeros of f is either equal to the number of sign changes of successive terms of $f(-\chi)$ or is less than that number by an even integer (until 1 or 0 is reached).

For equation (34), there are four general cases, which are as follows:

Case A: $\delta - \sigma - 4 > 0$ **and** $4\delta + \sigma - 1 > 0$**.** In this case, δ is guaranteed to be positive (see Figure 3). Hence, Eq. (34) has only a single sign change, implying that there is exactly one real positive solution. Recall that $\chi > 0$ by definition and negative solutions are not allowed.

Case B: $\delta - \sigma - 4 < 0$ **and** $4\delta + \sigma - 1 > 0$**.** In this case, δ is not guaranteed to be sign-definite (see Figure 3). If $\delta > 0$, $f(\chi)$ has a single sign change and, hence, exactly one positive real solution. If $\delta < 0$, there are two sign changes implying that we have either two or no positive real solutions.

Case C: $\delta - \sigma - 4 < 0$ **and** $4\delta + \sigma - 1 < 0$**.** In this case, δ again is not guaranteed to be sign-definite (see Figure 3). If $\delta > 0$, $f(\chi)$ has a single sign change for exactly one positive real solution for χ. In case $\delta < 0$, $f(\chi)$ has no sign changes and, hence, does not yield any positive real solutions. So, for Case C, δ must be positive to obtain exactly one positive real collinear formation.

Case D: $\delta - \sigma - 4 > 0$ **and** $4\delta + \sigma - 1 < 0$**.** In this case, σ is guaranteed to be negative (see Figure 3), though this does not affect the number of sign changes. If $\delta > 0$, then f has three sign changes and, hence, either three or one positive real solutions. Therefore, for $\delta > 0$, we are guaranteed at least one positive solution for χ.

If $\delta < 0$ then Eq. (34) has two sign changes and, hence, either two or zero positive real solutions.

Only Cases A and C require that δ be positive for a collinear solution to exist. In these cases, q_1 and q_3 must have the same sign. Cases B and D allow for a charge sign difference between craft 1 and 3 for a collinear solution to exist. We also note that for all four cases, there are at most three positive real solutions for a given δ and σ.

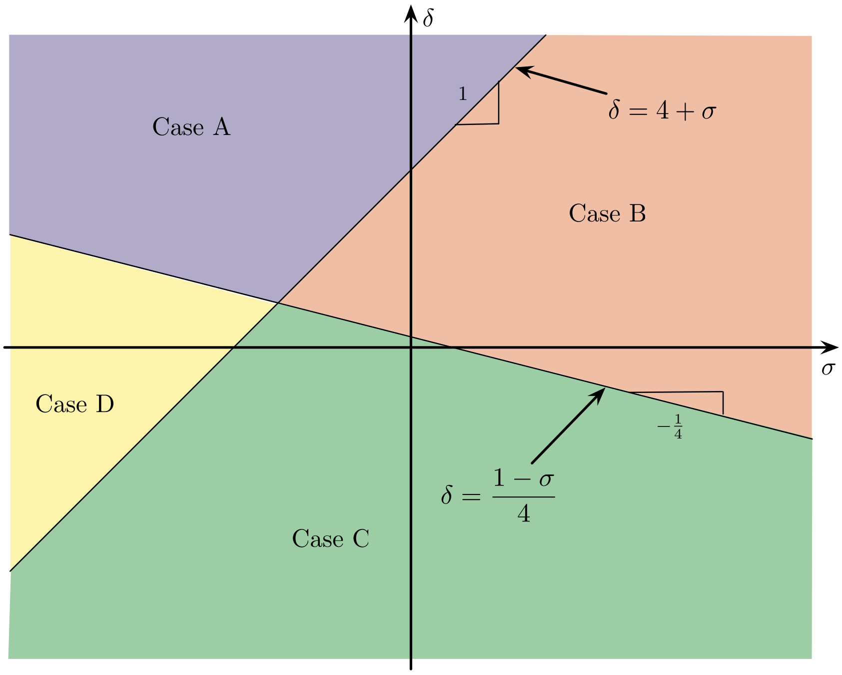


Figure 3. Different cases for collinear formation with large Debye lengths.

3.1.2. *Case II: Small Debye Lengths*

In this case we cannot ignore the exponents of $\frac{r_{ij}}{\lambda_d}$. Equation (30) is a nonlinear algebraic equation that has to be solved numerically. Note, however, that in this case we need to specify r_{21} a priori and then solve for χ (or vice-versa). If the separation distance r_{12} is time varying, then equation (30) will yield time varying χ values. This violates the condition that χ be constant, which is necessary for shape-preserving dynamic Coulomb formations to exist. As a result, if the Debye length effect is included, then dynamic invariant shape solutions are only possible if r_{21} is constant. This corresponds to having all three craft orbit on circular orbits about their common center of mass. This statement does not exclude other bounded relative motion solutions to exist, it merely states that circular solutions are the only shape-invariant collinear solutions if Debye lengths are included.

3.2. On the Equilateral Triangle Configuration

Lagrange's gravitational 3-body problem also yielded equilateral triangle solutions. This section investigates if such invariant shapes are feasible with the charged three-craft problem. To do this we revisit equations (21) and set $|\mathbf{r}_j - \mathbf{r}_i| = r_{ij} = \rho > 0$ for $i \neq j = 1, 2, 3$. Note that the exponential terms $e^{-r_{ij}/\lambda_d} = e^{-\rho/\lambda_3}$ factor out trivially. In this case the conditions (21) simplify to become

$$\begin{aligned} 0 &= \mathbf{r}_1 \times (q_2\mathbf{r}_2 + q_3\mathbf{r}_3) \\ 0 &= \mathbf{r}_2 \times (q_1\mathbf{r}_1 + q_3\mathbf{r}_3) \\ 0 &= \mathbf{r}_3 \times (q_1\mathbf{r}_1 + q_2\mathbf{r}_2) \end{aligned} \tag{35}$$

These equilateral triangle conditions are satisfied if either the terms in parentheses on the right hand side are zero or if each of the terms in parentheses is aligned with the respective $\mathbf{r}_i$ vector. Let's consider the latter case first. Assume that the spacecraft charges q_i are proportional to the mass m_i through

$$q_i = \kappa m_i, \qquad i = 1, 2, 3 \tag{36}$$

where κ is a non-zero constant. The center of mass condition in equation (4) can now be written as

$$q_1\mathbf{r}_1 + q_2\mathbf{r}_2 + q_3\mathbf{r}_3 = 0 \tag{37}$$

Using this modified center of mass condition in equation (37), the right hand side in equation (35) is always $\mathbf{r}_i \times (-q_i\mathbf{r}_i) = 0$. Thus, if the spacecraft charges are proportional to the masses, then the equilateral triangle is an invariant shape solution of the charged three-body problem. However, note that using equation (36) results in all charges having the same sign, and thus all craft will repel each other. As a result all these equilateral triangle solutions will grow infinitely large.

Next, let us consider the case where the terms in parentheses in equation (35) must be zero. For an equilateral triangle solution note that $\mathbf{r}_i$ can never be zero. Hence, equation (35) implies that we must have

$$\begin{bmatrix} 0 & q_2 & q_3 \\ q_1 & 0 & q_3 \\ q_1 & q_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} = 0 \tag{38}$$

The determinant of the 3×3 matrix on the left is given by the product $2q_1q_2q_3$. If all spacecraft charges are nonzero, then the only solution to equation (38) is $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}_3 = 0$, which is not allowed for a triangular solution. Say that only one spacecraft has a net zero charge on it. Say, without loss of generality, $q_1 = 0$. Conditions (38) imply that $q_2\mathbf{r}_2 = 0$ and $q_3\mathbf{r}_3 = 0$, which requires $q_2 = q_3 = 0$ since we can not allow $\mathbf{r}_i = 0$, $i = 1, 2, 3$. Finally, if two of the craft have zero charge, say $q_1 = q_2 = 0$, then we must have $q_3\mathbf{r}_3 = 0$ which implies that we must have $q_3 = 0$ since $\mathbf{r}_3$ is not allowed to be zero.

The above discussion shows that bounded equilateral triangle configurations are not permissible for the three spacecraft problem except for the trivial case where $q_1 = q_2 = q_3 = 0$. A physical interpretation of this result is as follows. Equilateral solutions exist for the gravitational three-body problem. However, here all three bodies are exerting attracting forces between each other. In the charged three-body problem it is impossible for all craft to mutually attract each other. For example, if craft 1 and 3 have opposite charge signs and generate an attractive force, and craft 2 and 3 also have opposite signs to cause mutual attraction, then it is impossible for craft 1 and 2 to have opposite charge signs as well.

3.3. Are there other solutions?

In the above discussion, the collinear and equilateral triangle constant charge solutions of equation (21) are studied. While these are the only two solutions for the gravitational three body problem, we do not have a conclusive statement regarding solutions for the three spacecraft Coulomb formation other than these two solutions. We believe that there are no other solutions if the charges are constant. A future task is to verify this statement.

4. Boundedness of the Charged Three-Spacecraft Solutions

Next, we would like to determine what shape the resulting invariant shaped charged spacecraft motion will describe. Depending on the charge magnitudes, charge signs, and the spacecraft initial velocity or energy levels, the resulting motion will either be bounded or grow unbounded with time. To do so, the equations of motion are rewritten for each case in a form equivalent to the gravitational two-body problem.

4.1. Collinear Solutions

Let us study the equation of motion governing the first craft in a collinear configuration. This is the first of equations (23). Because $r_{21} = |\mathbf{r}_2 - \mathbf{r}_1| = |-r_2 - r_1| = r_2 + r_1$ and $r_{31} = |\mathbf{r}_3 - \mathbf{r}_1| = |-r_3 - r_1| = r_3 + r_1$, the equation simplifies to

$$\ddot{\mathbf{r}}_1 = \frac{k_c q_1}{m_1} \left[\frac{q_2}{r_{21}^2} e^{-r_{21}/\lambda_d} + \frac{q_3}{r_{31}^2} e^{-r_{31}/\lambda_d} \right] \mathbf{e}_r \tag{39}$$

Note that for a collinear configuration we have $\alpha_1 = \alpha_3 = 0°$ and $\alpha_2 = 180°$. Using the first of equations (6) and the relationship $r_{13} = (1+\chi) r_{12}$, then we have

$$\ddot{\mathbf{r}}_1 = \frac{k_c q_1}{\mathcal{M}_1 r_1^2} \left[q_2 e^{-r_{21}/\lambda_d} + \frac{q_3}{(1+\chi)^2} e^{-r_{31}/\lambda_d} \right] \mathbf{e}_r \tag{40}$$

where

$$\mathcal{M}_1 = \frac{m_1 M^2}{m_2^2 + m_3^2(1+\chi)^2 + 2m_2 m_3(1+\chi)} \tag{41}$$

If we let

$$\mu_1 = -\frac{k_c q_1}{\mathcal{M}_1}\left[q_2 e^{-r_{21}/\lambda_d} + \frac{q_3}{(1+\chi)^2} e^{-r_{31}/\lambda_d}\right] \tag{42}$$

then we get the following equation of motion for spacecraft 1

$$\ddot{\mathbf{r}}_1 = -\frac{\mu_1}{r_1^3}\mathbf{r}_1 \tag{43}$$

where we used the fact that $\mathbf{r}_1 = r_1\mathbf{e}_r$ and $r_1 > 0$. Note that the algebraic form of equation (42) is identical to that of the gravitational 2-Body Problem (2BP). However, the equivalent gravitational constant μ_1 can be time varying here for general motions and finite Debye length values. If the Debye length is assumed to be infinite (vacuum condition), then μ_1 is constant. Unlike the gravitational problem, this μ_1 parameter can be positive or negative for the charged spacecraft scenario.

We now study the equation of motion governing the craft 2, which is the second of equations (23). In addition to $r_{21} = r_2 + r_1$ we also have $r_{32} = -(r_2 - r_3)$, the equation simplifies to

$$\ddot{\mathbf{r}}_2 = \frac{k_c q_2}{m_2}\left[-\frac{q_1}{r_{12}^2} e^{-r_{12}/\lambda_d} + \frac{q_3}{r_{32}^2} e^{-r_{32}/\lambda_d}\right]\mathbf{e}_r \tag{44}$$

Using the second of equations (6) and the relationship $r_{32} = \chi r_{21}$, then we have

$$\ddot{\mathbf{r}}_2 = \frac{k_c q_2}{\mathcal{M}_2 r_2^2}\left[-q_1 e^{-r_{12}/\lambda_d} + \frac{q_3}{\chi^2} e^{-r_{32}/\lambda_d}\right]\mathbf{e}_r \tag{45}$$

where

$$\mathcal{M}_2 = \frac{m_2 M^2}{m_1^2 + m_3^2\chi^2 - 2m_1 m_3\chi} \tag{46}$$

If we let

$$\mu_2 = \frac{k_c q_2}{\mathcal{M}_2}\left[q_1 e^{-r_{12}/\lambda_d} - \frac{q_3}{\chi^2}e^{-r_{32}/\lambda_d}\right] \tag{47}$$

then we get the following equation of motion for spacecraft 2

$$\ddot{\mathbf{r}}_2 = -\frac{\mu_2}{r_2^3}\mathbf{r}_2 \tag{48}$$

where we used the fact that $\mathbf{r}_2 = -r_2\mathbf{e}_r$ and $r_2 > 0$. Similarly for spacecraft 3, we get

$$\ddot{\mathbf{r}}_3 = -\frac{\mu_3}{r_3^3}\mathbf{r}_3 \tag{49}$$

with

$$\mu_3 = \frac{k_c q_3}{\mathcal{M}_3}\left[\frac{q_1}{(1+\chi)^2}e^{-r_{13}/\lambda_d} + \frac{q_2}{\chi^2}e^{-r_{32}/\lambda_d}\right] \tag{50}$$

and

$$\mathcal{M}_3 = \frac{m_3 M^2}{m_1^2(1+\chi)^2 + m_2^2\chi^2 + 2m_1m_2\chi(1+\chi)} \tag{51}$$

Note that each μ_i is a function of the relative distances. Moreover, note that each μ_i is of indefinite sign depending on the values of $q_1,\ q_2,\ q_3$. In the limit when we have a large Debye length, then the expressions for μ_i, $i = 1, 2, 3$, simplify to become

$$\mu_1 = -\frac{k_c q_1}{\mathcal{M}_1}\left[q_2 + \frac{q_3}{(1+\chi)^2}\right] \tag{52a}$$

$$\mu_2 = \frac{k_c q_2}{\mathcal{M}_2}\left[q_1 - \frac{q_3}{\chi^2}\right] \tag{52b}$$

$$\mu_3 = \frac{k_c q_3}{\mathcal{M}_3}\left[\frac{q_1}{(1+\chi)^2} + \frac{q_2}{\chi^2}\right] \tag{52c}$$

which are now constant functions of the masses and charges of all spacecraft in the formation.

For general Debye lengths λ_d, the only invariant shape solution is with the spacecraft performing circular orbit motions about their common center of mass. This is justified at the end of Section 3.1 under Case II. Invariant shape unbounded collinear motion is not possible. The charges q_i must be chosen such that the resulting equivalent gravitational parameters μ_i are positive constants. In this case the spacecraft motion is equivalent to that of the 2BP. Here the required spacecraft velocity vectors are given by

$$\dot{\mathbf{r}}_1 = v_1 \mathbf{e}_\theta \qquad \dot{\mathbf{r}}_2 = -v_2 \mathbf{e}_\theta \qquad \dot{\mathbf{r}}_3 = -v_3 \mathbf{e}_\theta \tag{53}$$

where

$$v_i = \sqrt{\frac{\mu_i}{r_i}} \tag{54}$$

4.2. Equilateral Triangle Solutions

The only equilateral triangle solutions found require that the charges $q_i = \kappa m_i$ are proportional to the spacecraft masses. Here $r_{ij} = \rho$ is the common separation distance between all three craft. Defining the negative constant G as

$$G = -k_c \kappa^2 e^{-\rho/\lambda_d} < 0 \tag{55}$$

the equations of motion in (3) are written for this special case as

$$m_1 \ddot{\mathbf{r}}_1 = \tfrac{G m_1}{\rho^3} \left(m_2 \mathbf{r}_{12} + m_3 \mathbf{r}_{13} \right) \tag{56}$$

$$m_2 \ddot{\mathbf{r}}_2 = \tfrac{G m_2}{\rho^3} \left(-m_1 \mathbf{r}_{12} + m_3 \mathbf{r}_{23} \right) \tag{57}$$

$$m_3 \ddot{\mathbf{r}}_3 = -\tfrac{G m_3}{\rho^3} \left(m_1 \mathbf{r}_{12} + m_2 \mathbf{r}_{23} \right) \tag{58}$$

These equations are identical to equations of motion of the equilateral triangular solution of the gravitational three-body problem (Schaub and Junkins, 2003). However, instead of G being the positive universal gravitational constant, it is a negative effective universal gravitational constant for the charged spacecraft problem. The equations of motion of each spacecraft are again written as

$$\ddot{\mathbf{r}}_i = -\frac{\mu_i}{r_i^3}\mathbf{r}_i \tag{59}$$

where the effective gravitational constant is now $\mu_i = GM_i$ with

$$M_1 = \tfrac{1}{M^2}\left(m_2^2 + m_3^2 + m_2 m_3\right)^{3/2} \tag{60}$$

$$M_2 = \tfrac{1}{M^2}\left(m_1^2 + m_3^2 + m_1 m_3\right)^{3/2} \tag{61}$$

$$M_3 = \tfrac{1}{M^2}\left(m_1^2 + m_2^2 + m_1 m_2\right)^{3/2} \tag{62}$$

Thus, for triangular invariant shape solutions, the parameters μ_i are always negative.

4.3. Relative Orbit Geometry Discussion

The equations of motion of all charged spacecraft solutions are written in the form

$$\ddot{\mathbf{r}}_i = -\frac{\mu_i}{r_i^3}\mathbf{r}_i \tag{63}$$

where μ_i is a constant regardless of the magnitude of the Debye length. This expression is equivalent to equations of motion of the gravitational 2BP, with the exception of the sign of μ_i. Recall that for the general gravitational 2BP with $\mu > 0$, the eccentricity vector $\mathbf{c}$

$$\mathbf{c} = \dot{\mathbf{r}} \times \mathbf{h} - \mu \left(\frac{\mathbf{r}}{r} \right) = \mu e \mathbf{e}_e \tag{64}$$

is found to be conserved in the absence of perturbations. Here $\mathbf{e}_e$ is the perifocal frame unit direction vector which points towards periapsis, and $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ is the mass-less angular momentum vector. Using $\mathbf{c}$ the orbit radius is shown to satisfy

$$r = \frac{p}{1 + e \cos f} \tag{65}$$

where $e \geq 0$ is the eccentricity, $p = h^2/\mu$ is the semi-latus rectum, and f is the true anomaly angle relative to periapsis. Let us investigate what happens if we allow μ to have negative values for the charged spacecraft motion case.

Following equivalent steps as with the derivation of the gravitational two-body problem eccentricity vector, we can develop an equivalent invariant eccentricity vector $\mathbf{c}_i$ of the collinear charged spacecraft motion:

$$\mathbf{c}_i = \dot{\mathbf{r}}_i \times \mathbf{h}_i - \mu_i \left(\frac{\mathbf{r}_i}{r_i} \right) \tag{66}$$

Here $\mathbf{h}_i = \mathbf{r}_i \times \dot{\mathbf{r}}_i$ is the angular momentum vector of the i^{th} craft. Next we determine the magnitude and heading of this invariant vector. Evaluating $\mathbf{c}_i \cdot \mathbf{r}_i$ leads to the condition

$$\mathbf{c}_i \cdot \mathbf{r}_i = h_i^2 - \mu_i r_i = r_i |\mathbf{c}_i| \cos(\angle \boldsymbol{r}_i, \boldsymbol{c}_i) \tag{67}$$

Equation (67) is solved for the position coordinate r_i to obtain

$$r_i = \frac{h_i^2/\mu_i}{1 + \frac{|\boldsymbol{c}_i|}{\mu_i} \cos(\angle \boldsymbol{r}_i, \boldsymbol{c}_i)} \tag{68}$$

Comparing this to the 2BP conic section radius equation $r = p/(1 + e\cos f)$, we define the semi-latus rectum p_i to be

$$p_i = h_i^2/\mu_i \tag{69}$$

Note that depending on the sign of μ_i, the value of p_i can be both positive or negative. Let $\mathbf{e}_{e_i}$ be a unit direction vector, then we can write the eccentricity vector as

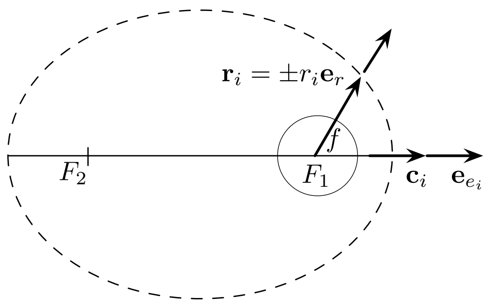
$$\mathbf{c}_i = \mu_i e_i \mathbf{e}_{e_i} \tag{70}$$

where $e_i > 0$ is the orbit eccentricity. In the conic 2BP solution in Eq. (65), the angle f is measured relative to the periapsis passage. Similarly, we define the angle f_i for our collinear charged spacecraft motion to be the angle between the position vector $\boldsymbol{r}_i$ and the eccentricity vector $\boldsymbol{c}_i$. Using Eq. (70) the radial coordinate r_i is reduced to

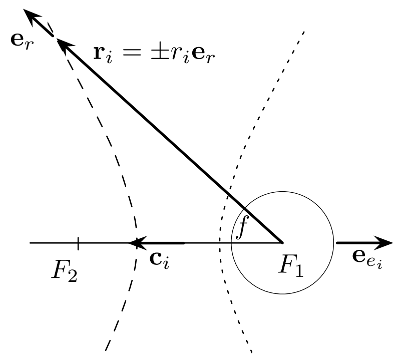
$$r_i = \frac{p_i}{1 + e_i \ \mathrm{sign}(\mu_i) \cos f_i} \tag{71}$$

To show that Eq. (71) also represents conic solutions no matter what the sign of μ_i, we need to consider the 2 cases illustrated in Figure 4. Case 1 considers the traditional $\mu_i > 0$ scenario and is shown in Figure 4(a). The result is equivalent to that of the gravitational 2BP with both $\mathbf{c}_i$ and the unit direction vector $\mathbf{e}_{e_i}$ pointing towards periapsis. The charged spacecraft radii r_i are governed by the traditional orbit radius equation of a conic section:

$$r_i = \frac{p_i}{1 + e_i \cos f_i} \tag{72}$$



(a) Case 1 with $\mu_i > 0$



(b) Case 2 with $\mu_i < 0$

Figure 4. Illustration of the Conic Solutions to the Two μ_i Sign Cases.

Note that this case will result in motions satisfying all conic solutions. Depending on the orbit energy levels, the resulting motions are either elliptic, parabolic, or hyperbolic.

Figure 4(b) illustrates Case 2 where $\mu_i < 0$. The orbit radial coordinate equation reduces to

$$r_i = \frac{p_i}{1 - e_i \cos f_i} \tag{73}$$

Studying the first equality in Eq. (67), we find that $\mathbf{c}_i \cdot \mathbf{r}_i > 0$ must hold for all times. This dictates that $\cos f_i = 0$, which shows that the true anomaly angle can only lie within $-\pi/2 \leq f_i \leq \pi/2$. Further, studying

Eq. (70) it is clear that $\mathbf{c}_i$ and $\mathbf{e}_{e_i}$ point in the opposite directions. Because $p_i = h_i^2 \mu_i < 0$, the eccentricity in Eq. (73) must be hyperbolic with $e_i > 1$. Bounded elliptic motion is not possible for this case. A physical example of such a scenario is the case where all three craft have equal charge signs. This can occur with the collinear solutions, and is required for all equilateral triangle solutions. Here the three craft repel each other regardless of the orbital speed or energy level, and travel on hyperbolic trajectories relative to the formation center of mass. Every hyperbola has two possible arcs. For the gravitational 2BP only the arc surrounding the occupied focus is the real trajectory. As illustrated in Figure 4(b), the conditions in case 2 result in the charged spacecraft using the 2[nd] hyperbolic arc which surrounds the unoccupied focus F_2. This is how the angle f is limited to $-\pi/2 \leq f \leq \pi/2$.

Table I. Orbit Types for Different μ_i Sign Cases.

	Case 1	Case 2
μ_i	> 0	< 0
elliptic a_i	> 0	N/A
hyperbolic a_i	< 0	> 0

Using the collinear position vector definition $\mathbf{r}_i = \pm r_i \mathbf{e}_r$ the energy equation can be derived for both μ sign cases in an equivalent manner to the 2BP and yields an identical expression (Schaub and Junkins, 2003):

$$\frac{v_i^2}{2} - \frac{\mu_i}{r_i} = -\frac{\mu_i}{2a_i} \tag{74}$$

where a_i is the semi-major axis of the resulting motion. Because all scenarios considered are invariant shape motions, all spacecraft trajectories about the center of mass must be the same conic solution (i.e. elliptical, parabolic, hyperbolic). Thus, it is not possible to have invariant shape solutions where the signs of μ_i differ among the craft. Considering case 1 with $\mu_i > 0$, if $a_i > 0$, then the three craft are performing elliptical or circular motion about the focus and remain bounded. If $a_i < 0$, then the craft have too much velocity for the Coulomb forces to compensate for, and the motion is hyperbolic and grows unbounded. Note that even as $r_i \to \infty$, the separation distance ratios will remain constant. Thus, by computing the semi-major axis of the collinear solutions it is possible to determine if the resulting spacecraft orbits will remain bounded, or if they will grow infinitely large. Remember that these results are only valid if $\mu_i > 0$ is a constant, which requires the Debye length to be very large compared to the spacecraft motion. Considering Case 2 with $\mu_i < 0$, then only $a_i > 0$ is possible which results in unbounded hyperbolic motion. These results are summarized in Table I.

5. Numerical Illustrations

The following numerical illustrations are generated by numerically integrating the fundamental charged particle equations of motion in equation (3). The spacecraft are assigned the masses $m_1 = 100$ kg, $m_2 = 75$ kg, and $m_3 = 50$ kg. Note that these masses satisfy the assumption that $m_1 > m_3$ used in this paper's developments. The charges are $q_1 = q_3 = 10\mu$C and $q_2 = \pm 10\mu$C. The second craft always has the same magnitude, but the charge sign can vary from case to case. The separation distance between the first and second craft is set to $r_{21} = 20$ meters for all cases considered.

First, let us consider cases where the Debye length λ_d is much larger than the formation size. Here the collinear equations of motion of each craft can be written in an equivalent form to the gravitational 2BP. The simulation parameters for the 4 cases considered are listed in Table II. Cases 1–3 have the second spacecraft charge set to a negative value. This causes an attractive force to be generated between the inner and outer spacecraft. The resulting collinear side ratio χ is the same for these cases, as well as the equivalent gravitational constant μ_1. Note that to determine if the resulting charged spacecraft motion is bounded, it is sufficient to study the motion of the first spacecraft. This numerical analysis will thus only focus on the trajectory of m_1. The resulting relative orbits are illustrated in Figure 5.

In case 1 the spacecraft velocity of m_1 is set up using the circular orbit speed $v_1 = \sqrt{\mu_1/r_1}$. The other craft velocities are then set

Table II. Set of Numerical Simulation Test Cases with Infinite Debye Length.

Parameter	Case 1	Case 2	Case 3	Case 4
q_2	-10 μC	-10 μC	-10 μC	+10 μC
χ	1.083175	1.083175	1.083175	1.131453
μ_1	0.00438648	0.00438648	0.00438648	-0.00714325
a_1	15.9252 m	15.9252 m	-79.6261 m	5.37993 m
$v_1(t_0)$	0.016596 m/s	0.016596 m/s	0.024616 m/s	0.021037 m/s
$\beta_1(t_0)$	90°	70°	90°	90°

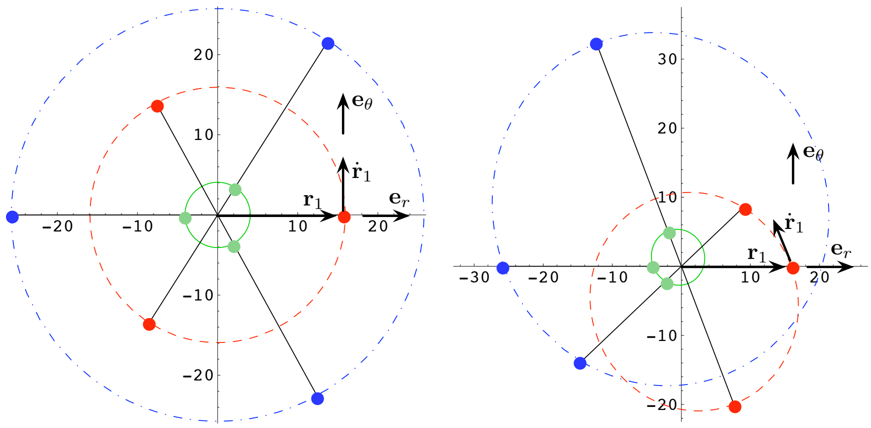
accordingly to guarantee that all craft are rotating at the same rate at the initial time. Because both μ_1 and a_1 are positive, the relative orbits are expected to be bounded. The resulting motion is seen in Figure 5(a). The spacecraft locations are shown at three different times through solid disks, while the craft trajectory is shown for one revolution. The ratio of spacecraft separation distance is maintained, yielding an invariant-shape motion.

Case 2 is set up using the same kinetic energy, and has the same semi-major axis a_1 as case 1. However, the initial velocity vector is rotated 30° relative to the $\mathbf{e}_\theta$ direction. As predicted, the resulting charged spacecraft motion are elliptical orbits with the formation center of mass being at the ellipse focus.

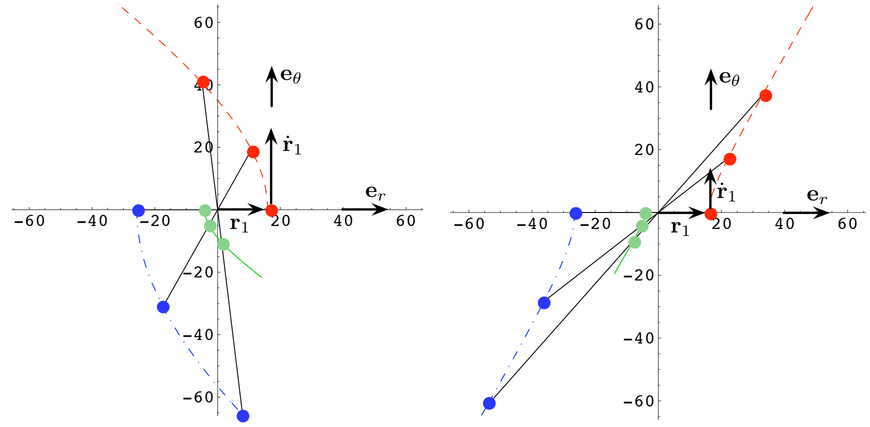
Case 3 uses the same set up as case 1, except that the initial craft velocity magnitude is increased. The resulting semi-major axis a_1 is negative, indicating hyperbolic motion. Figure 5(c) illustrates how the Coulomb forces are insufficient here to compensate for the increased formation kinematic energy level. As a result, the craft perform hyperbolic trajectories while maintaining a constant formation shape. Also, note that the formation center of mass is the focus of all three hyperbolas.

Case 4 duplicates the simulation set up of case 1, but reverses the craft 2 charge sign to $q_2 = +10\mu$C. As a result, the equivalent gravitational constant μ_1 is negative, indicating that only hyperbolic motion is possible. The resulting motion shown in Figure 5(d) shows that the spacecraft are on hyperbolic orbits where the formation center of mass is on the un-occupied focus.

The previous analysis showed that, if the Debye length is not large compared to the spacecraft separation distances, only circular collinear solutions or unbounded equal charge equilateral triangle solutions will yield shape-invariant formations. A collinear solution is simulated by using the spacecraft charges of case 1, and a Debye length of $\lambda_d = 20$ meters, we solve for a ratio $\chi = 1.06770$ and $\mu_1 = 0.0019116$. The simulation results are illustrated in Figure 6. Even with this short Debye length, an invariant-shape charged formation is produced. However, note that this solutions is more sensitive to integration or setup errors due to the higher degree of nonlinearity with the additional exponential term.



(a) Case 1: Circular Orbits, $\mu_i > 0$ (b) Case 2: Elliptical Orbits, $\mu_i > 0$



(c) Case 3: Hyperbolic Orbits, $\mu_i > 0$ (d) Case 4: Hyperbolic Orbits, $\mu_i < 0$

Figure 5. Numerical Illustration of dynamic invariant shape Coulomb formations with an infinite Debye length for spacecraft 1 (- - -), spacecraft 2 (——) and spacecraft 3 (- · -).

6. Conclusion

This paper studies shape-preserving formations of three spacecraft kept in formation via Coulomb tethers. The craft are assumed to be flying in deep space and the differential orbital accelerations are not considered

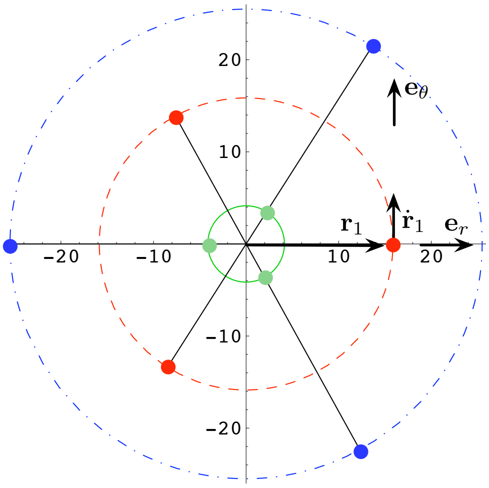


Figure 6. Numerical Illustration of circular invariant shape Coulomb formation with a small Debye length of $\lambda_d = 20$ m for spacecraft 1 (- - -), spacecraft 2 (——) and spacecraft 3 (- · -).

in this work. Inspired by Lagrange's three body problem, general conditions are derived that guarantee preservation of the geometric shape of the formation and show that the classical collinear configuration is a solution to the problem. The three collinear spacecraft problem is analyzed for the equal mass case, yielding conditions that guarantee no, unique, or multiple solutions. Finally, the boundedness of the invariant shape solutions is studied by writing the equations of motion in an equivalent form to the 2-body gravitational problem and using the energy equation to determine the semi-major axis of the charged spacecraft motion.

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