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Hypersphere Stereographic Orientation Parameters

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I. Introduction

Euler Parameters (EPs), also referred to as quaternions, are a non-singular set of four attitude coordinates which are constrained to a unit norm. The first analytical mapping from EPs to Modified Rodrigues Parameters (MRPs) is performed by Wiener in his 1962 dissertation,¹ where he discovered a singularity at the 360° rotation. In Reference 2, Marandi and Modi exploit the non-uniqueness property of the MRPs by formulating a non-singular minimal attitude description. Shuster also mentions the MRPs in his well-known survey of attitude parameterizations, and gives the parameters the name Modified Rodrigues Parameters.³ Tsiotras points out that the MRPs can be viewed as the result of a stereographic projection of the EP constraint unit hypersphere onto a three-dimensional projection hyperplane.^{4,5} He also discovers that the natural logarithm function forms an elegant attitude cost (Lyapunov) function in terms of MRPs which leads to linear MRP feedback with nonlinear stability.

Schaub and Junkins⁶ further develop this work by showing that the MRP stereographic projection description discovered by Tsiotras can be expanded to describe general families of attitude parameters called the Stereographic Orientation Parameters (SOPs). In particular, Reference 6 present the sub-group of Symmetric Stereographic Orientation Parameters and shows that the MRPs and Classical Rodrigues Parameters are a sub-set of this family. Later on Southward et. al.⁷ develop the full kinematic properties of they Symmetric Stereographic Orientation Parameters by allowing the projection point to be placed anywhere on the scalar EP coordinate axis within the EP constraint hypersphere. These Symmetric Stereographic

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Orientation Parameters are expressed algebraically in terms of a scalar projection point coordinates and yield minimal set attitude coordinates where the singularity can occur at any desired orientation within $0^\circ < \Phi < 360^\circ$. In contrast, the Asymmetric Stereographic Attitude Parameters (ASOPs) of Reference 6 place the projection point at ± 1 along one of the vector EP coordinate axes. This leads to an interesting behavior where singularities are only encountered if a pure rotation about a particular principle body axis is performed. Further, a $+180^\circ$ rotation may lead to a singular attitude description, but a -270° rotation (exact same orientation) is non-singular. Only a -630° in the negative direction leads to a singular description. The non-symmetric nature of the singular rotations and their dependency of the path to a particular orientation, lead to the name asymmetric SOPs.

Other recent attitude coordinates that relate to the MRPs include the Higher Order Rodrigues Parameters.⁸ Here higher order Cayley transforms are used to develop attitude coordinates which grow infinitely large at multiples of 360° . These Higher Order Rodrigues Parameters are convenient to develop minimal sets of attitude coordinates where the differential equation can be made arbitrarily linear through the use of higher order Cayley transformations. Hurtado uses the MRPs to create inner and outer parameters for attitude representations, and presents new Cayley-like transformations.⁹

This paper investigates a sub-family of attitude coordinates called the Hypersphere Stereographic Orientation Parameters (HSOPs) which contains both the previous MRPs (particular set of Symmetric Stereographic Orientation Parameters) and the ASOP, allowing for all this work to be combined into a single, minimal attitude parameter description. HSOPs allow the projection point to lie at any point on the EP unit hypersphere constraint. Thus, depending on the choice of the project point, these attitude coordinates can display a singular behavior similar to that of the ASOP. The attitude of a spinning body can be described singularity-free with a minimal three-parameter coordinate set as long as the body is not spinning about a particular combination of principal body axes. Or, the HSOP coordinates can be chosen such that their singular behavior matches that of the MRPs where a particular 360° degree rotation about any body axis leads to a singular description. When different attitude coordinates are combined into a more general family of parameters, such as the joining of Classical Rodrigues Parameters and MRP into Symmetric Stereographic Orientation Parameters in Reference 7, the result is often a more complex set of algebraic equations. This paper investigates how a general projection point on the surface of the EP constraint hypersphere complicates the associated HSOP differential kinematic equations and their mapping to the shadow set.

This paper is organized as follows: Section II describes the geometry and algebra of a general stereographic projection. Section III provides the analytical mapping between HSOPs from EPs, the direction cosine matrix, as well as the derivation of the shadow sets,

kinematic differential equation, and singularity condition. Section IV discusses how the HSOPs can be employed in attitude control strategies.

II. Generalized Stereographic Projections

The four EP attitude coordinates are expressed as a 4x1 matrix:

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \quad (1)$$

Here each β_i represents an EP coordinate, and the magnitude of this stack is $\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1$. In terms of the principal rotation angle Φ and principal rotation axis $\hat{\mathbf{e}} = (e_1, e_2, e_3)$, the EP coordinates are expressed as: $\beta_0 = \cos(\Phi/2)$ and $\beta_i = e_i \sin(\Phi/2)$ with $i = 1, 2, 3$. Geometrically, the EP constraint $\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1$ defines the surface of a four-dimensional unit hypersphere upon which all EPs must lie.¹⁰

Generalized Stereographic Orientation Parameters (SOPs) are a minimal coordinate representation of a particular orientation. These generalized parameters are obtained by projecting the four-dimensional EP attitude description ($\boldsymbol{\beta}$) onto a three-dimensional hyperplane as illustrated in Figure 1. The coordinates of this intersection point form the SOPs. This section presents first the general stereographic mapping of the unit constraint surface onto a general projection hyperplane. This section considers the particular mapping where the projection point \mathbf{a} is constraint to the hypersphere surface. The projection point \mathbf{a} , for now, is a general location. The projection hyperplane is defined by the normal of $\mathbf{a} - \mathbf{p}_0$, where \mathbf{p}_0 is the closest hyperplane point to \mathbf{a} . Therefore, in order to solve for general SOPs, the line formed between the projection point and any EP attitude description is projected onto the hyperplane. The projection of $\boldsymbol{\beta}$ onto the hyperplane is the intersection point $\boldsymbol{\kappa}$.

The vector equation for the line between the points \mathbf{a} and $\boldsymbol{\beta}$ in n -dimensional space is:

$$\mathbf{L} = \mathbf{a} + t(\boldsymbol{\beta} - \mathbf{a}) \quad (2)$$

where t is a free scalar parameter which can describe any point on this line. The generalized equation for a plane in space with the normal vector $\mathbf{a} - \mathbf{p}_0$ that crosses through point \mathbf{p}_0 is:

$$\mathbf{P} = (\mathbf{a} - \mathbf{p}_0) \cdot \boldsymbol{\kappa} = (\mathbf{a} - \mathbf{p}_0) \cdot \mathbf{p}_0 \quad (3)$$

where $\boldsymbol{\kappa}$ is the intersection point on the hyperplane of the EP description through the

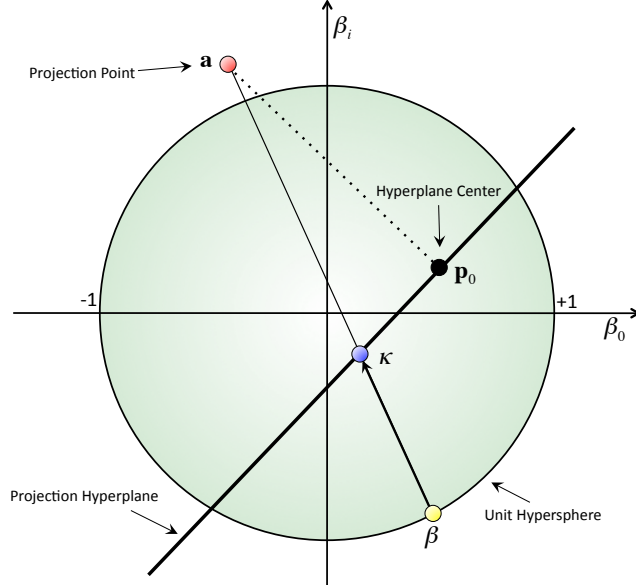


Figure 1. General Stereographic Orientation Parameter Geometry

projection point. In order to solve for $\boldsymbol{\kappa}$, the line defined in Equation 2 is intersected with the plane described by Equation 3. Solving for the scalar parameter t and substituting into Equation 2 yields:

$$\boldsymbol{\kappa} = \mathbf{a} + \frac{(\mathbf{a} - \mathbf{p}_0) \cdot (\mathbf{p}_0 - \mathbf{a})}{(\mathbf{a} - \mathbf{p}_0) \cdot (\boldsymbol{\beta} - \mathbf{a})} (\boldsymbol{\beta} - \mathbf{a}) \quad (4)$$

This four-dimensional intersection point, $\boldsymbol{\kappa}$ is to derive the HSOPs. The rest of the algebra will project this four-dimensional point to a three-dimensional hyperplane.

III. Derivation of the HSOPs

III.A. Relationship to Euler Parameters

The Hypersphere Stereographic Orientation Parameter (HSOP) attitude set is formed by two conditions: 1) The projection point \mathbf{a} is constrained to the surface of the EP constraint unit hypersphere and 2) the projection hyperplane passes through the center of this hypersphere:

With the projection point defined as $\mathbf{a} = [a_0, a_1, a_2, a_3]^T$, this means that $\mathbf{a} \cdot \mathbf{a} = 1$. Likewise, the EP stack will be defined by $\boldsymbol{\beta} = [\beta_0, \beta_1, \beta_2, \beta_3]^T$ and from the EP constraint $\boldsymbol{\beta} \cdot \boldsymbol{\beta} = 1$. Finally we set $\mathbf{p}_0 = [0, 0, 0, 0]^T$ to ensure the projection hyperplane intersects with the origin. This simplifies Equation 4 to:

$$\boldsymbol{\kappa} = \mathbf{a} - \frac{(\boldsymbol{\beta} - \mathbf{a})}{\mathbf{a} \cdot \boldsymbol{\beta} - 1} \quad (5)$$

The 4D vector $\boldsymbol{\kappa}$ represents the projection point of the attitude description $\boldsymbol{\beta}$ onto the hyperplane. In order to be a minimal set, this vector needs to be represented in terms of the 3 basis vectors that describe the projection hyperplane. The 3-D hyperplane is written in the following vector form: $\mathbf{P} = \zeta_1 \mathbf{e}_1 + \zeta_2 \mathbf{e}_2 + \zeta_3 \mathbf{e}_3$. Here $\zeta_1, \zeta_2, \zeta_3$ are the in-plane coordinates of the intersection, and thus the HSOP coordinates. The vectors $\mathbf{e}_1, \mathbf{e}_2$, and \mathbf{e}_3 form the base vectors of the hyperplane by being mutually orthogonal to the projection point vector, \mathbf{a} . The projection point can be expressed as:

$$\boldsymbol{\kappa} = [A] \begin{bmatrix} \boldsymbol{\zeta} \\ 0 \end{bmatrix} \quad (6)$$

where $\boldsymbol{\zeta}$ is the 3x1 matrix:

$$\boldsymbol{\zeta} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} \quad (7)$$

and $[A]$ is:

$$[A] = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{a} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (8)$$

Note that the base vector \mathbf{e}_i are not unique. For example, the base vector can always be rotated about \mathbf{a} to form a new set. This would result in alternate numerical HSOP descriptions which describe the same orientation. However, these alternate formulations have identical singular behaviors and do not provide any practical benefits. A goal of this paper is to determine a set of base vectors \mathbf{e}_i such that the resulting HSOPs are identical to the prior MRP or ASOP coordinates if the proper projection point is chosen.

For the developments of this paper, a unit-length orthogonal basis will be used to describe the projection plane. Therefore, this matrix is full rank and orthogonal meaning that $[A]^{-1} = [A]^T$. Solving for the matrix of HSOP coordinates $\boldsymbol{\zeta}$ results in:

$$\begin{bmatrix} \boldsymbol{\zeta} \\ 0 \end{bmatrix} = [A]^T \left[\mathbf{a} - \frac{\boldsymbol{\beta} - \mathbf{a}}{\mathbf{a} \cdot \boldsymbol{\beta} - 1} \right] \quad (9)$$

In order to generate $\mathbf{e}_1, \mathbf{e}_2$, and \mathbf{e}_3 , the composite rotation property of EPs¹⁰ is used. Because \mathbf{a} is constrained to the unit hypersphere, it can be treated as a valid EP attitude description. Therefore, an orthogonal 4-D basis of EPs (\mathbf{e}_i) is created by simply adding rotations of -180°

to \mathbf{a} . This results in the following set of unit-length basis vectors.

$$\begin{aligned}\mathbf{e}_1 &= [a_1, -a_0, -a_3, a_2]^T \\ \mathbf{e}_2 &= [a_2, a_3, -a_0, -a_1]^T \\ \mathbf{e}_3 &= [a_3, -a_2, a_1, -a_0]^T\end{aligned}\tag{10}$$

Substituting these vectors into Equation 9 and expanding results in the minimal HSOP attitude description:

$$\boldsymbol{\zeta} = \frac{1}{1 - \mathbf{a} \cdot \boldsymbol{\beta}} \begin{bmatrix} a_1\beta_0 - a_0\beta_1 - a_3\beta_2 + a_2\beta_3 \\ a_2\beta_0 + a_3\beta_1 - a_0\beta_2 - a_1\beta_3 \\ a_3\beta_0 - a_2\beta_1 + a_1\beta_2 - a_0\beta_3 \end{bmatrix}\tag{11}$$

It is important to note that the the $\boldsymbol{\zeta} = \mathbf{0}$ vector does not correspond to the zero attitude of $\beta_0 = 1$. Instead, the HSOP representation of the zero orientation is defined as:

$$\boldsymbol{\zeta}_0 = \frac{1}{1 - a_0} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\tag{12}$$

By using this definition, it is clear that the HSOPs are simply a rotation of the MRPs by the attitude described by the point \mathbf{a} , however, the zero attitude is directly dependent on the parameter \mathbf{a} . This has implications on the singularity condition as well as the attitude control which will be discussed in further detail in the following sections.

To verify these HSOP definitions are a family of attitude coordinates containing MRPs and ASOPs, we perform the following checks. Reducing Equation 11 to the MRP case where \mathbf{a} lies at $\beta_0 = -1$ results in the definition of the MRPs:¹⁰

$$\begin{bmatrix} \zeta_1 = \sigma_1 \\ \zeta_2 = \sigma_2 \\ \zeta_3 = \sigma_3 \end{bmatrix} = \begin{bmatrix} \frac{\beta_1}{1+\beta_0} \\ \frac{\beta_2}{1+\beta_0} \\ \frac{\beta_3}{1+\beta_0} \end{bmatrix}\tag{13}$$

Reducing Equation 11 to the ASOP case where \mathbf{a} lies at $\beta_1 = -1$ results in:

$$\begin{bmatrix} \zeta_1 = \eta_1 \\ \zeta_2 = \eta_2 \\ \zeta_3 = \eta_3 \end{bmatrix} = \begin{bmatrix} \frac{-\beta_0}{1+\beta_1} \\ \frac{\beta_3}{1+\beta_1} \\ \frac{-\beta_2}{1+\beta_1} \end{bmatrix}\tag{14}$$

The original ASOP case (as presented in References 6 and 10) has a sign discrepancy on the first and third parameters and the second and third parameters are switched. This is due to the uniqueness issue when defining the basis of the projection plane. The orientation of the original ASOP projection hyperplane was chosen at random without any further information. The presented ASOP hyperplane definition is preferred because it allows the presented base vector \mathbf{e}_i definition to map between EP and HSOP in a general way. Clearly there is no practical difference between the ζ_i and η_i coordinates. Further advantages of this conventions become apparent when considering the HSOP differential kinematic equations.

Solving for the inverse transformation from $\boldsymbol{\zeta}$ to $\boldsymbol{\beta}$ is similar to solving for the intersection point on the projection hyperplane. However, rather than solving for the intersection of a plane the reverse will happen and the intersection of the projection line onto the unit hypersphere will be accomplished.

Assume vectors \mathbf{a} and $\boldsymbol{\kappa}$ (the projection point and intersection point, respectively) are known. Therefore Equation 2 can be written in the equivalent form: $\mathbf{L} = \mathbf{a} + u(\boldsymbol{\kappa} - \mathbf{a})$. Where u is a scalar parameter of the line. Because \mathbf{a} and $\boldsymbol{\beta}$ lie on the unit hypersphere, intersecting this line with the unit hypersphere results in two intersection points: one at \mathbf{a} and one at $\boldsymbol{\beta}$. The equation of a n -dimensional unit hypersphere in space is given: $|\mathbf{S}|^2 = 1$. Intersecting both of these equations results in the following vector equation:

$$|\mathbf{a} + u(\boldsymbol{\kappa} - \mathbf{a})|^2 = 1 \quad (15)$$

Solving for the parameter u and applying the unit hypersphere constraints results in the following:

$$u = \frac{1 \pm 1}{1 + \zeta^2} \quad (16)$$

where $\zeta^2 = \boldsymbol{\zeta}^T \boldsymbol{\zeta} = \zeta_1^2 + \zeta_2^2 + \zeta_3^2$. Due to the unit-length constraints for projection point vector and the EP description, this means there are two solutions since both of these points lie on the unit hypersphere. Substituting the non-trivial solution back into the general equation for a line results in the desired EP description vector, $\boldsymbol{\beta}$. The inverse relation between the HSOPs and EPs is shown below:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} a_0 + \frac{2}{1+\zeta^2} (a_1\zeta_1 + a_2\zeta_2 + a_3\zeta_3 - a_0) \\ a_1 + \frac{2}{1+\zeta^2} (-a_0\zeta_1 + a_3\zeta_2 - a_2\zeta_3 - a_1) \\ a_2 + \frac{2}{1+\zeta^2} (-a_3\zeta_1 - a_0\zeta_2 + a_1\zeta_3 - a_2) \\ a_3 + \frac{2}{1+\zeta^2} (a_2\zeta_1 - a_1\zeta_2 - a_0\zeta_3 - a_3) \end{bmatrix} \quad (17)$$

In the simple compact matrix form:

$$\boldsymbol{\beta} = \mathbf{a} + \frac{2}{1 + \zeta^2}([A]\boldsymbol{\zeta} - \mathbf{a}) \quad (18)$$

To verify this general inverse mapping from EP to HSOP, let us consider the special cases of mapping from EPs to the previously developed MRPs or ASOPs. Calculating the inverse relation for the MRP case ($\mathbf{a} = [-1, 0, 0, 0]^T$) and switching coordinates from ζ to σ results in the definition of the inverse for MRPs:¹⁰

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{1 + \sigma^2} \begin{bmatrix} 1 - \sigma^2 \\ 2\sigma_1 \\ 2\sigma_2 \\ 2\sigma_3 \end{bmatrix} \quad (19)$$

Calculating the inverse relation for the ASOP case ($\mathbf{a} = [0, -1, 0, 0]^T$) results in the definition of the inverse relation for ASOPs:

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \frac{1}{1 + \eta^2} \begin{bmatrix} -2\eta_1 \\ 1 - \eta^2 \\ -2\eta_3 \\ 2\eta_2 \end{bmatrix} \quad (20)$$

III.B. Direction Cosine Matrix Derivation

Because the HSOPs were derived from the EP set, the Direction Cosine Matrix (DCM) definition for the EP is used:

$$[C] = [I_{3 \times 3}] - 2\beta_0[\tilde{\boldsymbol{\beta}}] + 2[\tilde{\boldsymbol{\beta}}][\tilde{\boldsymbol{\beta}}] \quad (21)$$

where $\tilde{\boldsymbol{\beta}}$ is the vectorial component of the EPs and $[\tilde{\boldsymbol{\beta}}]$ is the cross-product matrix defined as:

$$[\tilde{\boldsymbol{\beta}}] = \begin{bmatrix} 0 & -\beta_3 & \beta_2 \\ \beta_3 & 0 & -\beta_1 \\ -\beta_2 & \beta_1 & 0 \end{bmatrix} \quad (22)$$

Define $\bar{\mathbf{a}}$ to be:

$$\bar{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (23)$$

Substituting Equation 17 into Equation 22 results in the following expression:

$$[\tilde{\boldsymbol{\beta}}] = \frac{2}{1 + \zeta^2} \left[\frac{\zeta^2 - 1}{2} [I_{3 \times 3}] + [\tilde{\boldsymbol{\zeta}}] \right] [\tilde{\mathbf{a}}] - \frac{2}{1 + \zeta^2} (a_0 [I_{3 \times 3}] + [\tilde{\mathbf{a}}]) [\tilde{\boldsymbol{\zeta}}] \quad (24)$$

Simplifying β_0 results in:

$$\beta_0 = a_0 \frac{\zeta^2 - 1}{\zeta^2 + 1} + \frac{2}{\zeta^2 + 1} \tilde{\mathbf{a}}^T \boldsymbol{\zeta} \quad (25)$$

Reducing this DCM down to the MRP case where $\mathbf{a} = [-1, 0, 0, 0]^T$ and performing a variable switch from ζ to σ results in the same DCM as for MRPs:¹⁰

$$[C]_{\text{MRP}} = \frac{1}{(1 + \sigma^2)^2} \begin{bmatrix} 4(\sigma_1^2 - \sigma_2^2 - \sigma_3^2) + \Sigma_\sigma^2 & 8\sigma_1\sigma_2 + 4\sigma_3\Sigma_\sigma & 8\sigma_1\sigma_3 - 4\sigma_2\Sigma_\sigma \\ 8\sigma_2\sigma_1 - 4\sigma_3\Sigma_\sigma & 4(-\sigma_1^2 + \sigma_2^2 - \sigma_3^2) + \Sigma_\sigma^2 & 8\sigma_2\sigma_3 + 4\sigma_1\Sigma_\sigma \\ 8\sigma_3\sigma_1 + 4\sigma_2\Sigma_\sigma & 8\sigma_3\sigma_2 - 4\sigma_1\Sigma_\sigma & 4(-\sigma_1^2 - \sigma_2^2 + \sigma_3^2) + \Sigma_\sigma^2 \end{bmatrix} \quad (26)$$

where $\Sigma_\sigma = (1 - \sigma^2)$. Reducing the HSOP DCM to the ASOP case where $\mathbf{a} = [0, -1, 0, 0]^T$ and switching parameters from ζ to η results in a similar DCM as seen in Equation 26:

$$[C]_{\text{ASOP}} = \frac{1}{(1 + \eta^2)^2} \begin{bmatrix} 4(\eta_1^2 - \eta_2^2 - \eta_3^2) + \Sigma_\eta^2 & -8\eta_1\eta_2 - 4\eta_3\Sigma_\eta & -8\eta_1\eta_3 + 4\eta_2\Sigma_\eta \\ 8\eta_2\eta_1 - 4\eta_3\Sigma_\eta & 4(\eta_1^2 - \eta_2^2 + \eta_3^2) - \Sigma_\eta^2 & -8\eta_2\eta_3 - 4\eta_1\Sigma_\eta \\ 8\eta_3\eta_1 + 4\eta_2\Sigma_\eta & -8\eta_3\eta_2 + 4\eta_1\Sigma_\eta & 4(\eta_1^2 + \eta_2^2 - \eta_3^2) - \Sigma_\eta^2 \end{bmatrix} \quad (27)$$

with $\Sigma_\eta = (1 - \eta^2)$.

III.C. HSOP Shadow Set Derivation

Because EPs have four parameters, there is redundancy in the attitude description (a single EP is not unique). The other EPs that describes the same attitude are known as the shadow set $\boldsymbol{\beta}^s = -\boldsymbol{\beta}$. The EP shadow set simply represents another way to rotate the object to the desired attitude. For example: if the desired attitude is 45° about the \mathbf{e}_1 body axis, one could either rotate 45° (short rotation) or -315° (long rotation) about the body 1 axis. Both rotations describe the same orientation. However, there are two different EPs that describe both a short and long rotation. Because of this shadow parameter with EPs, there is also a shadow set of HSOPs. The geometry of this is illustrated in Figure 2.

Substituting the definition of the EP shadow set into Equation 6 results in the HSOP shadow set solution.

$$\boldsymbol{\zeta}^s = [A]^T \left[\mathbf{a} - \frac{\boldsymbol{\beta} + \mathbf{a}}{\mathbf{a} \cdot \boldsymbol{\beta} + 1} \right] \quad (28)$$

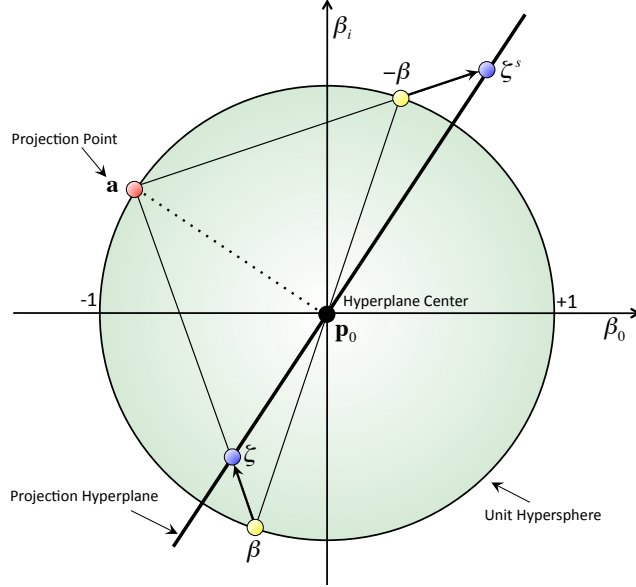


Figure 2. HSOP Shadow Set Geometry

Expanding this form results in the explicit HSOP shadow-set definition.

$$\zeta^s = \frac{1}{\mathbf{a} \cdot \boldsymbol{\beta} - 1} \begin{bmatrix} -a_1\beta_0 + a_0\beta_1 + a_3\beta_2 - a_2\beta_3 \\ -a_2\beta_0 - a_3\beta_1 + a_0\beta_2 + a_1\beta_3 \\ -a_3\beta_0 + a_2\beta_1 - a_1\beta_2 + a_0\beta_3 \end{bmatrix} \quad (29)$$

$$\zeta^s = \frac{1}{\mathbf{a} \cdot \boldsymbol{\beta} - 1} (-\beta_0 \bar{\mathbf{a}} + a_0 [I_{3 \times 3}] + [\tilde{\bar{\mathbf{a}}}] \bar{\boldsymbol{\beta}}) \quad (30)$$

Substituting in the inverse relation between EPs and HSOPs (Equation 17) and simplifying results in the elegant shadow set relation:

$$\zeta_i^s = -(\zeta_1, \zeta_2, \zeta_3) / |\boldsymbol{\zeta}|^2 \quad (31)$$

This is the same shadow set algebraic relationship as found in the MRP shadow set transformation. This result is expected as the HSOPs, geometrically, can be represented as a rotation of the MRPs about a particular attitude. The HSOP shadow set, much like the shadow set of the MRP cases, can be used to avoid singularities during integration. As shown in Figure 2, as one parameter nears the projection point, the HSOP description begins to grow infinitely large whereas the shadow parameter stays well bounded. This switching could, however, occur at any point before a singularity is reached as the shadow set describes the same attitude as the normal set. A property about the shadow sets of HSOPs is when one set magnitude is greater than one, the shadow set is always less than one. When one set has a magnitude

of 1, the other also has a magnitude of one. Switching at a magnitude of one is somewhat arbitrary and can happen at any time, however, when $|\boldsymbol{\sigma}| = 1$ always corresponds to a 180° rotation from the null attitude. Likewise, when the HSOPs are at $|\boldsymbol{\zeta}| = 1$, it corresponds to the 180° rotation from the attitude specified by the parameter \mathbf{a} . By switching both sets at a magnitude of one keeps the parameters well bounded by maintaining an attitude description that is always less than 180° , while avoiding singularities.

III.D. Kinematic Differential Equation

The derivation of the HSOP differential kinematic equation requires the kinematic differential equation of EPs:¹⁰

$$\dot{\boldsymbol{\beta}} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \quad (32)$$

Differentiating Equation 11 results in:

$$\dot{\zeta}_1 = \frac{-a_1\dot{\beta}_0 + a_0\dot{\beta}_1 + a_3\dot{\beta}_2 - a_2\dot{\beta}_3}{a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - 1} + \frac{(a_1\beta_0 - a_0\beta_1 - a_3\beta_2 + a_2\beta_3)(a_0\dot{\beta}_0 + a_1\dot{\beta}_1 + a_2\dot{\beta}_2 + a_3\dot{\beta}_3)}{(a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - 1)^2} \quad (33a)$$

$$\dot{\zeta}_2 = \frac{-a_2\dot{\beta}_0 - a_3\dot{\beta}_1 + a_0\dot{\beta}_2 + a_1\dot{\beta}_3}{a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - 1} + \frac{(a_2\beta_0 + a_3\beta_1 - a_0\beta_2 - a_1\beta_3)(a_0\dot{\beta}_0 + a_1\dot{\beta}_1 + a_2\dot{\beta}_2 + a_3\dot{\beta}_3)}{(a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - 1)^2} \quad (33b)$$

$$\dot{\zeta}_3 = \frac{-a_3\dot{\beta}_0 + a_2\dot{\beta}_1 - a_1\dot{\beta}_2 + a_0\dot{\beta}_3}{a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - 1} + \frac{(a_3\beta_0 - a_2\beta_1 + a_1\beta_2 - a_0\beta_3)(a_0\dot{\beta}_0 + a_1\dot{\beta}_1 + a_2\dot{\beta}_2 + a_3\dot{\beta}_3)}{(a_0\beta_0 + a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - 1)^2} \quad (33c)$$

Substituting in Equations 17 and 32, and applying the unit hypersphere constraint, results in the kinematic differential equation for HSOPs which are independent of the projection point \mathbf{a} . Despite the generalities with this attitude parameter set, the result is extremely elegant and given in the following matrix form:

$$\dot{\boldsymbol{\zeta}} = \frac{1}{4} \begin{bmatrix} 1 - \zeta^2 + 2\zeta_1^2 & 2(\zeta_1\zeta_2 - \zeta_3) & 2(\zeta_1\zeta_3 + \zeta_2) \\ 2(\zeta_2\zeta_1 + \zeta_3) & 1 - \zeta^2 + 2\zeta_2^2 & 2(\zeta_2\zeta_3 - \zeta_1) \\ 2(\zeta_3\zeta_1 - \zeta_2) & 2(\zeta_3\zeta_2 + \zeta_1) & 1 - \zeta^2 + 2\zeta_3^2 \end{bmatrix} \boldsymbol{\omega} \quad (34)$$

Due to the lack of information about \mathbf{a} in these equations, we find that all HSOP differential kinematic equations will have the exact same algebraic form! These equations are written in

compact vector notation as:

$$\dot{\boldsymbol{\zeta}} = \frac{1}{4} \left[(1 - \zeta^2) [\mathbf{I}_{3 \times 3}] + 2[\tilde{\boldsymbol{\zeta}}] + 2\boldsymbol{\zeta}\boldsymbol{\zeta}^T \right] \boldsymbol{\omega} \quad (35)$$

After applying the coordinate switch from ζ to σ , this kinematic differential equation is the same as for MRPs.¹⁰ The only difference between HSOPs and MRPs is that the HSOPs are simply a rotation of the MRPs by the attitude defined by \mathbf{a} . Therefore, the kinematic differential equations will be the same. With this new ASOP definition, the ASOP differential kinematic equations assume the identical algebraic form as the MRPs. This elegant result justifies why the ASOPs should be redefined as shown in this paper.

III.E. Singularity Condition

Because HSOPs is a minimal attitude coordinate set, there is a singular description that can be arbitrarily placed. This condition arises when the denominator in Equation 11 equals zero:

$$1 - a_0\beta_0 - a_1\beta_1 - a_2\beta_2 - a_3\beta_3 = 0 \quad (36)$$

Geometrically, this arises at one point on the unit hypersphere when $\mathbf{a} = \boldsymbol{\beta}$. Writing this in terms of principal rotation angle Φ and principal axis components e_i results in:

$$\begin{aligned} a_0 &= \cos(\Phi/2) = \beta_0 \\ a_1 &= e_1 \sin(\Phi/2) = \beta_1 \\ a_2 &= e_2 \sin(\Phi/2) = \beta_2 \\ a_3 &= e_3 \sin(\Phi/2) = \beta_3 \end{aligned} \quad (37)$$

Therefore, from Equation 37, the singularity can be placed in any desired direction with any desired rotation angle. There are two rotations that can be performed in order to reach a singularity. For example, if the singularity is placed in a certain direction at a $\Phi = 135^\circ$, a rotation of $+135^\circ$ or -585° about the appropriate direction could be performed in order to reach the singular point. This means that with HSOPs and its sub-sets, the rotational path to a particular orientation determines if the attitude description will go singular.

It is important to note that because this singular point can be placed anywhere on the unit hypersphere, there is only one attitude description that it can go singular at, and in order for this to be a singularity it must lie *exactly* on \mathbf{a} . However, attitudes around this point will grow infinitely large and depending on the control technique applied, these large values can be used to avoid a neighborhood of points in space.

IV. Spacecraft Control

Because the HSOP differential kinematic equations are algebraically equivalent to the MRP differential kinematic equations, any control development that exploits algebraic properties of the MRP differential equations,^{5,6,11,12} can also be directly applied to the HSOP. This process is illustrated in the following control development example based on the MRP-based attitude control developed in Reference 6. For the purposes of this discussion, the underlying rotational dynamics will be Euler's equations of rotational motion of a rigid body:

$$[I]\dot{\boldsymbol{\omega}} = -[\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + \mathbf{u} + \mathbf{L} \quad (38)$$

where $[I]$ is the inertia tensor, \mathbf{u} is the control torque, and \mathbf{L} is any external torque acting on the body. As developed for the MRP set, a logarithmic Lyapunov function will be used to design a stabilizing control law.^{5,6}

$$V(\delta\boldsymbol{\omega}, \boldsymbol{\zeta}_e) = \frac{1}{2}\delta\boldsymbol{\omega}^T [I]\delta\boldsymbol{\omega} + 2K \ln(1 + \boldsymbol{\zeta}_e^2) \quad (39)$$

where $\boldsymbol{\zeta}_e$ is defined as the error attitude from the zero attitude of the body so $\boldsymbol{\zeta}_e = \boldsymbol{\zeta} - \boldsymbol{\zeta}_0$ where $\boldsymbol{\zeta}_0$ is defined in Equation 12. This is required because a zero HSOP attitude does not correspond to the null attitude of a body, by adding in this offset factor, as $\boldsymbol{\zeta}_e$ is driven to zero, the attitude of the body is driven to the null attitude. To ensure global stability, the Lyapunov rate is set equal to the negative semi-definite $\dot{V} = -\delta\boldsymbol{\omega}^T [P]\delta\boldsymbol{\omega}$. Because $\delta\boldsymbol{\omega} = \boldsymbol{\omega} - \boldsymbol{\omega}_r$, we can plug in the equations of motion in order to generate the closed loop dynamics and solve for the control variable. This leads to the stabilizing control law \mathbf{u} .⁶

$$\mathbf{u} = [\tilde{\boldsymbol{\omega}}][I]\boldsymbol{\omega} + [I](\dot{\boldsymbol{\omega}}_r - [\tilde{\boldsymbol{\omega}}]\boldsymbol{\omega}_r) - K\boldsymbol{\zeta}_e - [P]\delta\boldsymbol{\omega} - \mathbf{L} \quad (40)$$

While this attitude control law has the exact same algebraic form as the MRP-based control law, the closed-loop response will be different because the HSOP represent a different attitude description in general, and has a completely different singular behavior.

V. Conclusion

The newly developed hypersphere stereographic orientation parameters are a generalization of the MRPs and ASOPs. Direct analytical mappings are present from the HSOP to the Euler parameters, as well as the direction cosine matrix. The key result of this paper is that the HSOP have the same differential kinematic equation as the MRPs, and thus can be applied to any control law which uses the algebraic form of MRP kinematic equation.

HSOPs are different than MRPs because of the different singular behaviors of each at-

titude coordinate set. This offers great flexibility as the singular orientation can be placed at a full revolution or at particular rotations about particular body axes. In all cases the kinematic differential equation only has quadratic nonlinear terms equivalent to those of the MRPs. This is highly beneficial in relation to non-linear spacecraft attitude control as HSOPs will have the same stability guarantees as a controller designed using MRPs.

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