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Hanspeter Schaub, Panagiotis Tsiotras and John L. Junkins

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Principal Rotation Representations of Proper NxN Orthogonal Matrices

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Abstract

Three and four parameter representations of 3x3 orthogonal matrices are extended to the general case of proper NxN orthogonal matrices. These developments generalize the classical Rodrigues parameters, the Euler parameters, and the recently introduced modified Rodrigues parameters to higher dimensional spaces. The developments presented are motivated by, and significantly generalize and extend the classical result known as the Cayley transformation.

Introduction

It is well known in rigid body dynamics, and many other areas of Euclidean analysis, that the rotational coordinates associated with Euler's Principal Rotation Theorem [1,2,3] lead to especially attractive descriptions of rotational motion. These parameterizations of proper orthogonal 3x3 matrices include the four-parameter set known widely as the *Euler* (*quaternion*) *parameters* [1,2,3], as well as the classical three-parameter set known as the *Rodrigues parameters* or *Gibbs vector* [1,2,3,4]. Also included is a recently introduced three parameter description known as the *modified Rodrigues parameters* [4,5,6]. As we review briefly below, these parameterizations are of fundamental significance in the geometry and kinematics of three-dimensional motion. Briefly, their advantages are as follows:

Euler Parameters: This once redundant four-parameter description of three-dimensional rotational motion maps all possible motions into arcs on a four-dimensional unit sphere. This accomplishes a regularization and the representation is universally nonsingular. The kinematic differential equations contain no transcendental functions and are bi-linear without approximation.

Classical Rodrigues Parameters: This three parameter set, also referred to as the *Gibbs vector*, is proportional to Euler's principal rotation vector. The magnitude is $tan(\phi/2)$, with ϕ being the principal rotation angle. These parameters are singular at $\phi = \pm\pi$ and have elegant, quadratically nonlinear differential kinematic equations.

Modified Rodrigues Parameters: This three parameter set is also proportional to Euler's principal rotation vector, but with a magnitude of $tan(\phi/4)$. The singular orientation is at $\phi = \pm 2\pi$, doubling the principal rotation range over the classical Rodrigues parameters. They also have a quadratic nonlinearity in their differential kinematic equations.

The question naturally arises; can these elegant principal rotation parameterizations be extended to orthogonal projections in higher dimensional spaces? Cayley partially answered this question in the affirmative; his "Cayley Transform" fully extends the classical Rodrigues parameters to higher dimensional spaces [1,2,7]. A proper NxN orthogonal matrix can be generally parameterized by a vector with dimension M = ½N(N-1). Only for the 3x3 case is N equal to M. Any proper orthogonal matrix has a determinant of +1 and can be interpreted as analogous to a rigid body rotation representation. This paper extends the classical Cayley transform to parameterize a proper NxN orthogonal matrix into a set of M-dimensional modified Rodrigues parameters. Further, a method is shown to parameterize the NxN matrix into a once-redundant set of (M+1)-dimensional Euler parameters.

The first section will review the Euler, Rodrigues and the modified Rodrigues parameters for the 3x3 case, generalized later in this paper to parameterize the proper NxN orthogonal matrices. The second section will review the classical Cayley transform resulting with the representation of a proper orthogonal matrix using the Rodrigues parameters, followed by the new representation of the NxN orthogonal matrices using an M-dimensional set of modified Rodrigues parameters, and finally, a new representation of the NxN orthogonal matrices using an (M+1)-dimensional Euler parameters.

Review of Three-Dimensional Rigid Body Rotation Parameterizations

The Direction Cosine Matrix

The 3x3 direction cosine matrix C completely describes any three-dimensional rigid body rotation. The matrix elements are bounded between ±1 and possess no singularities. The famous Poisson kinematic differential equation for the direction cosine matrix is:

$$\dot{C} = -[\tilde{\omega}]C \tag{1}$$

where the tilde matrix is defined as

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

The direction cosine matrix C is orthogonal, therefore it satisfies the following constraint.

$$C^T C = CC^T = I \tag{3}$$

This constraint causes the direction cosine matrix representation to be highly redundant. Instead of considering all nine matrix elements, it usually suffices to parameterize the matrix into a set of three or four parameters. However, any minimal set of three parameters will contain singular orientations.

The constraint in equation (3) shows that besides being orthogonal, the direction cosine matrix is also normal [8]. Consequently it has the spectral decomposition

$$C = U\Lambda U^* \tag{4}$$

where U is a unitary matrix containing the orthonormal eigenvectors of C, and Λ is a diagonal matrix whose entries are the eigenvalues of C. The * symbol stands for the adjoint operator, which takes the complex conjugate transpose of a matrix. Since C represents a rigid body rotation, it always has a determinant of +1.

The Principal Rotation Vector

Euler's principal rotation theorem states that in a three-dimensional space, a rigid body (reference frame) can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single principal rotation (ϕ) about a principal line $\hat{e}$ [3].

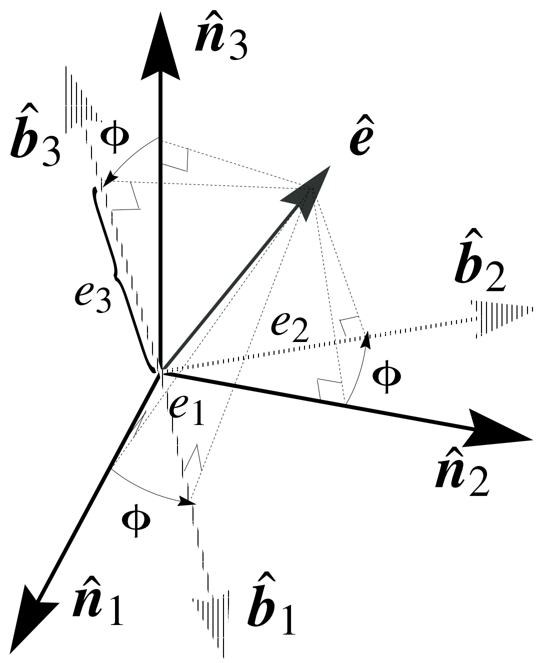

Fig. 1: Euler's Principal Rotation Theorem.

With reference to Fig. 1, the body axis $\hat{b}_i$ components of the principal line $\hat{e}$ are identical to the spatial components projected onto $\hat{n}_i$.

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \hat{e} = C \cdot \hat{e} \tag{5}$$

Therefore $\hat{e}$ must be an eigenvector of the 3x3 C matrix with a corresponding eigenvalue of +1. In this case the *det(C)* would be +1. The *principal rotation vector* $\vec{\gamma}$ is defined as:

$$\vec{\gamma} = \phi\hat{e} \tag{6}$$

Let us now consider the case where a rigid body performs a pure single-axis rotation about the fixed $\hat{e}$. This rotation axis is identical to Euler's principal line of rotation $\hat{e}$. Let the rotation angle be ϕ. The angular velocity vector for this case becomes:

$$\vec{\omega} = \dot{\phi}\hat{e} \tag{7}$$

or in matrix form:

$$[\tilde{\omega}] = \dot{\phi}[\tilde{e}] \tag{8}$$

Substituting equation (8) into (1), one obtains the following development.

$$\frac{dC}{dt} = -\frac{d\phi}{dt}[\tilde{e}]C$$

$$\frac{dC}{d\phi} = -[\tilde{e}]C$$

$$C = e^{-\phi[\tilde{e}]} \tag{9}$$

The last step follows since the $[\tilde{e}]$ matrix is constant during this single axis maneuver. Due to Euler's principal rotation theorem, however, any arbitrary rotation can always be described instantaneously by the equivalent single-axis principal rotation. Hence equation (9) will hold at any instant for an arbitrary time-varying direction cosine matrix *C*. However, ϕ and $\hat{e}$ must be considered time-varying functions. Using the following substitution

$$[\tilde{\gamma}] = \phi[\tilde{e}] \tag{10}$$

equation (9) can be rewritten as [2]

$$C = e^{-[\tilde{\gamma}]} = \sum_{n=0}^{\infty} \frac{1}{n!}(-[\tilde{\gamma}])^n \tag{11}$$

Instead of using an infinite matrix power series expansion of equation (11) to find *C*, the elegant finite transformation shown below can be used [2]. That is, the evaluation of $e^{-[\tilde{\gamma}]}$ does not require the spectral decomposition of $[\tilde{\gamma}]$, but can be written directly in term of $\vec{\gamma}$ itself. Unfortunately, this transformation only holds for the 3x3 case. A general transformation for the NxN case is unknown at this point, at least as far as the authors know.

$$\begin{aligned} &e^{-[\tilde{\gamma}]} = I\cos\phi - [\tilde{e}]\sin\phi - \hat{e}\hat{e}^T(\cos\phi - 1) \\ &\phi = \|\vec{\gamma}\|,\ \hat{e} = \vec{\gamma}/\phi \end{aligned} \tag{12}$$

To find the inverse transformation from the direction cosine matrix *C* to $[\tilde{\gamma}]$, the matrix logarithm can be taken of equation (11) to obtain

$$[\tilde{\gamma}] = -\log C = \sum_{n=1}^{\infty} \frac{1}{n}(I - C)^n \tag{13}$$

Using the spectral decomposition of *C* given in equation (4), the above equation can be rewritten as

$$[\tilde{\gamma}] = -\log(U\Lambda U^*) = -U(\log\Lambda)U^* \tag{14}$$

where calculating the matrix logarithm of a diagonal matrix becomes trivial. Since all eigenvalues of an orthogonal matrix have unit norm, the matrix logarithm in equation (14) is defined everywhere except when an eigenvalue is -1. Generally, equation (14) will return a $[\tilde{\gamma}]$ which corresponds to a principal rotation angle ϕ in (-180°,+180°). Note however, that when C has eigenvalues of -1, equation (14) does not return a skew-symmetric matrix. The transformation breaks down here for this singular event. The geometric interpretation is that a 180° rotation has been performed about one axis (leading to one positive and two negative eigenvalues of C), which is the only rotation not covered by the domain of equation (14).

The principal vector representation of C is not unique. Adding or subtracting 2π from the principal rotation angle ϕ describes the same rotation. As expected, equation (11) will always yield the same C matrix for the different principal rotation angles, since all angles correspond to the same physical orientation. However, the inverse transformation given in equation (14) yields only the principal rotation angle which lies between -180° and +180°.

As do all minimal parameter sets, the principal rotation vector parameterization has a singular orientation. The vector is not uniquely defined for a zero rotation from the reference frame. The principal rotation vector parameterization will be found convenient, however, to derive useful relationships.

The Euler (Quaternion) Parameters

The Euler parameters are a once-redundant set of rotation parameters. They are defined in terms of the principal rotation angle ϕ and the principal line components e_i as follows:

$$\beta_0 = \cos\frac{\phi}{2}\ ,\quad \beta_i = e_i \sin\frac{\phi}{2} \qquad i = 1, 2, 3 \tag{15}$$

They satisfy the holonomic constraint:

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{16}$$

Equation (16) states that all possible Euler parameter trajectories generate arcs on the surface of a four-dimensional unit hypersphere. This behavior bounds the parameters to values between ±1. However, the Euler parameters are not unique. The mirror image trajectories $\beta(t)$ and $-\beta(t)$ both describe the identical physical orientation histories. Given a 3x3 orthogonal matrix, there will be two corresponding sets of Euler parameters which differ by a sign. The Euler parameters are the only set of rotation parameters which have a bi-linear system of kinematic differential equations [1], other than the direction cosine matrix itself, as follows

$$\begin{Bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{17}$$

It is also of significance that the above 4x4 matrix is orthogonal, so “transportation" between ω_i’s and $\dot\beta_i$ ‘s is “painless". The direction cosine matrix in term of the Euler parameters is [1,3]

$$[C] = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3+\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3-\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \tag{18}$$

The Euler parameters have several advantages over all minimal sets of rotation parameters. Namely, they are bounded between ±1, never encounter a singularity, and have linear kinematic differential equations if the $\omega_i(t)$ are considered known. All of these advantages are slightly offset by the cost of having one extra parameter.

The Classical Rodrigues Parameters

The classical Rodrigues parameter vector $\vec{q}$ can be interpreted as the coordinates resulting from a stereographic projection of the four-dimensional Euler parameter hypersphere onto a three-dimensional hyperplane [6], with the projection point at the origin and the stereographic mapping hyperplane at $\beta_0 = +1$. As discussed in [6], it follows that they have their singular orientation at a principal rotation angle of $\phi = \pm 180°$ from the reference. Their transformation from the Euler parameters is

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3 \tag{19}$$

Unlike the Euler parameters, the Rodrigues parameters are unique. The q_i uniquely define a rotation on the open range of (-180°,+180°) [6]; as is evident in equation (19), reversing the sign of the Euler parameters has no effect on the q_i. Using equation (15), the classical Rodrigues parameters can also be defined directly in terms of the principal rotation angle and the principal axis components as

$$q_i = e_i \tan\frac{\phi}{2} \qquad i = 1, 2, 3 \tag{20}$$

It is apparent that $\vec{q}$ has the same direction as the principal rotation and the magnitude is $tan(\phi/2)$. The singular condition of $\phi = \pm 180°$ is evident by inspection of equation (20). The kinematic differential equation for the Rodrigues parameters contain a quadratic nonlinear dependence on the q_i. They can be verified from equations (17,20) to be [1-4]

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1+q_1^2 & q_1q_2-q_3 & q_1q_3+q_2 \\ q_2q_1+q_3 & 1+q_2^2 & q_2q_3-q_1 \\ q_3q_1-q_2 & q_3q_2+q_1 & 1+q_3^2 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{21}$$

Notice that the above coefficient matrix is not orthogonal, although the inverse is well behaved everywhere except at $\phi = \pm 180°$ where $|\vec{q}| \to \infty$. The direction cosine matrix in terms of the Rodrigues parameters is [1-4]:

$$C(\vec{q}) = \frac{1}{1+q_1^2+q_2^2+q_3^2} \begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3) & 2(q_1q_3-q_2) \\ 2(q_2q_1-q_3) & 1-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1) \\ 2(q_3q_1+q_2) & 2(q_3q_2-q_1) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{22}$$

The Modified Rodrigues Parameters

The modified Rodrigues parameter vector $\vec{\sigma}$ is also a set of stereographic parameters, closely related to the classical Rodrigues parameters [2,4-6]. The modified Rodrigues parameters have the projection point at (-1,0,0,0) and the stereographic mapping hyperplane at $\beta_0 = 0$. This projection results in a set of parameters which do not encounter a singularity until a principal rotation from the reference frame of ±360° has been performed. Therefore they are able to describe any rotation except a complete revolution ±360°. Their transformation from the Euler parameters is

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1, 2, 3 \tag{23}$$

While the classical Rodrigues parameters have a singularity at $\beta_0=0$ ($\phi = \pm 180°$), the modified Rodrigues parameters have moved the singularity out to a *single point* at $\beta_0=-1$ ($\phi = \pm 360°$). Figure 2 below illustrates these two singular conditions. Since the classical Rodrigues parameters are only defined for $-180° < \phi < +180°$, they can only describe rotations on the upper hemisphere of the four-dimensional unit hyper-sphere where $\beta_0>0$. However, the modified Rodrigues parameters can describe any rotation on this hypersphere except the point $\beta_0=-1$. Therefore the modified Rodrigues parameters have twice the nonsingular range as the classical Rodrigues parameters.

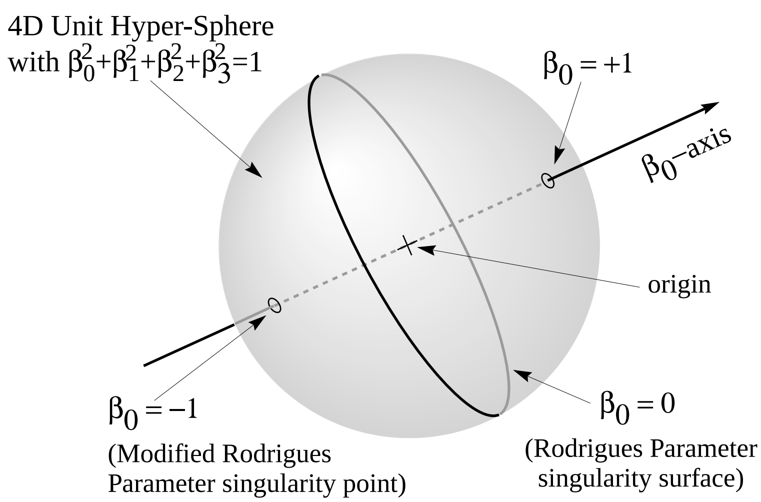

Fig. 2.: Illustration of the Singular Conditions of the Classical and the Modified Rodrigues Parameters.

Like the Euler parameters, the modified Rodrigues parameters are not unique. They have an associated "shadow" set found by using -β(t) instead of β(t) in equation (23) [5,6]. The transformation from the original set to the "shadow" set is [2,5,6]

$$\sigma_i^S = \frac{-\sigma_i}{\vec{\sigma}^T\vec{\sigma}} \qquad i = 1, 2, 3 \tag{24}$$

The "shadow" points are denoted with a superscript S merely to differentiate them from σ_i. Keep in mind that both $\vec{\sigma}$ and $\vec{\sigma}^S$ describe the same physical orientation, similar and related to the case of the two possible sets of Euler parameter and the principal rotation vector. It turns out that the modified Rodrigues "shadow" vector $\vec{\sigma}^S(t)$ has the opposite singular behavior to the original vector $\vec{\sigma}(t)$. The original parameters have differential kinematic equations which are very linear near a zero rotation and are singular at a ±360° rotation. On the other hand, the "shadow" parameters have differential kinematic equations which are linear near the ±360° rotation and singular at the zero rotation. [6] Using equation (15), the definition for the modified Rodrigues parameters in equation (23) can be rewritten as [4]

$$\sigma_i = e_i \tan\frac{\Phi}{4} \tag{25}$$

Equation (25) is very similar to equation (20), except for the scaling factor of the principal rotation angle. The singularity at ±360° is evident in equation (25), and small rotations behave like quarter angles. All three parameter representations must possess a singularity. This set maximizes the nonsingular principal rotation range to ±360°. The following differential kinematic equations display a similar degree of quadratic nonlinearity as do the corresponding equations in terms of the classical Rodrigues parameters [4-6]

$$\dot{\vec{\sigma}} = \frac{1}{4}\begin{bmatrix} 1+\sigma_1^2-\sigma_2^2-\sigma_3^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_3+\sigma_2) \\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma_1^2+\sigma_2^2-\sigma_3^2 & 2(\sigma_2\sigma_3-\sigma_1) \\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma_1^2-\sigma_2^2+\sigma_3^2 \end{bmatrix}\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix} \tag{26}$$

Note that the coefficient matrix of the differential kinematic equation is not orthogonal, but almost. Multiplying it with its transpose yields a *scalar* $\left(1+\vec{\sigma}^T\vec{\sigma}\right)^2$ /16 times the identity matrix. As far as we know, this is the *only* three parameter representation possessing this elegant property; further attesting to the uniqueness and importance of the modified Rodrigues parameterization. This almost orthogonal behavior allows for a simple transformation between the ω_i and the $\dot{\sigma}_i$

$$C(\vec{\sigma}) = \frac{1}{\left(1+\vec{\sigma}^T\vec{\sigma}\right)^2}\begin{bmatrix} 4\left(\sigma_1^2-\sigma_2^2-\sigma_3^2\right)+\Sigma^2 & 8\sigma_1\sigma_2+4\sigma_3\Sigma & 8\sigma_1\sigma_3-4\sigma_2\Sigma \\ 8\sigma_2\sigma_1-4\sigma_3\Sigma & 4\left(-\sigma_1^2+\sigma_2^2-\sigma_3^2\right)+\Sigma^2 & 8\sigma_2\sigma_3+4\sigma_1\Sigma \\ 8\sigma_3\sigma_1+4\sigma_2\Sigma & 8\sigma_3\sigma_2-4\sigma_1\Sigma & 4\left(-\sigma_1^2-\sigma_2^2+\sigma_3^2\right)+\Sigma^2 \end{bmatrix} \tag{27}$$

$$\Sigma = 1-\vec{\sigma}^T\vec{\sigma}$$

The direction cosine matrix is shown above [6,9]. It has a slightly higher degree of nonlinearity than the corresponding direction cosine matrix in terms of the classical Rodrigues parameters.

Parameterization of Proper NxN Orthogonal Matrices

A proper orthogonal matrix is an orthogonal matrix whose determinant is +1. Some aspects of parameterizing proper NxN orthogonal matrices into M-dimensional Rodrigues parameters have been studied recently by Junkins and Kim [1] and Shuster [2]. Keep in mind that M = ½N(N-1). These classical developments, generalizing the Rodrigues parameters to NxN proper rotation matrices, date from the work of Cayley [7] and are included below for comparative purposes with the new representations.

Any NxN orthogonal matrix abides by the constraint given in equation (3). This equation is an exact integral of equation (1), as can be verified by differentiation of equation (3) to obtain

$$\dot{C}^T C + C^T \dot{C} = 0 \tag{28}$$

The $\dot{C}$ matrix defined in equation (1) can be shown to satisfy this condition exactly. Substitute equation (1) into (27) and expand as follows

$$(-[\tilde{\omega}]C)^T C + C^T(-[\tilde{\omega}]C) = 0$$

$$\left(-C^T[\tilde{\omega}]^T\right)C - C^T[\tilde{\omega}]C = 0$$

$$C^T\left(-[\tilde{\omega}]^T - [\tilde{\omega}]\right)C = 0$$

The above statement is obviously satisfied if $[\tilde{\omega}]$ is a skew-symmetric matrix, e.g. $[\tilde{\omega}] = -[\tilde{\omega}]^T$. Consequently equation (1) will generate an NxN orthogonal matrix, as long as $[\tilde{\omega}]$ is skew-symmetric and the initial condition *C(t=0)* is orthogonal. This observation allows for the evolution of NxN orthogonal matrices to be viewed as higher dimensional direction cosine matrices, somewhat analogous to the motion generated by a "higher dimensional rigid body rotation," and also suggests parameterization of of higher dimensional rigid body-motivated rotation parameters.

Higher Dimensional Classical Rodrigues Parameters

Cayley's transformation [7] parameterizes a proper orthogonal matrix C as a function of a skew-symmetric matrix *Q*; these elegant transformations are

$$C = (I-Q)(I+Q)^{-1} = (I+Q)^{-1}(I-Q) \tag{29a}$$

$$Q = (I-C)(I+C)^{-1} = (I+C)^{-1}(I-C) \tag{29b}$$

The Cayley's transformation is one-to-one and onto from the set of skew-symmetric matrices to the set of proper orthogonal matrices with no eigenvalues at -1. Notice the remarkable truth that the forward and inverse transformations are identical. The transformation in equation (29b) fails if any of the eigenvalues of *C* are -1, because the *I*+*C* matrix becomes singular and is thus not invertible. The Cayley transformation in equation (29a) produces only proper orthogonal matrices *C* with *det(C)*=+1. This can be verified by examining the determinant of *C* as shown below. Using equation (29a), *det(C)* can be expressed as

$$\det(C) = \det(I-Q)\det\left((I+Q)^{-1}\right) = \frac{\det(I-Q)}{\det(I+Q)}$$

Since the *Q* matrix is skew-symmetric, it has purely imaginary complex conjugate pairs of eigenvalues of the form $\pm i\lambda_i$. Let *R* be the corresponding complex eigenvector matrix to *Q*. Multiplying and dividing the above equation by *det(R)* yields

$$\det(C) = \frac{\det(R)\det(I-Q)/\det(R)}{\det(R)\det(I+Q)/\det(R)} = \frac{\det(R)\det(I-Q)\det\left(R^{-1}\right)}{\det(R)\det(I+Q)\det\left(R^{-1}\right)}$$

$$\det(C) = \frac{\det\left(R(I-Q)R^{-1}\right)}{\det\left(R(I+Q)R^{-1}\right)} = \frac{\det\left(I-RQR^{-1}\right)}{\det\left(I+RQR^{-1}\right)}$$

where the RQR^{-1} term is a diagonal matrix containing the eigenvalues of the Q matrix. Since the determinant of a matrix is the product of all the eigenvalues, the above can be written as

$$\det(C) = \frac{\prod_{j=1}^{p}(1-i\lambda_j)(1+i\lambda_j)}{\prod_{j=1}^{p}(1+i\lambda_j)(1-i\lambda_j)} = \frac{\prod_{j=1}^{p}\left(1+\lambda_j^2\right)}{\prod_{j=1}^{p}\left(1+\lambda_j^2\right)} = +1 \quad q.e.d$$

where p is the number of nonzero (imaginary) eigenvalues of Q. The above statement proves that all C matrices formed with equation (29a) are indeed proper matrices.

To proof that the classical Cayley transform is a map from the skew-symmetric matrices *onto* the set of proper orthogonal matrices with no eigenvalue at -1,let us decompose the orthogonal C matrix into *real* matrices T and D

$$C = TDT^T$$

where T is orthogonal and D is block diagonal of the form

$$D = \begin{bmatrix} +1 & 0 & 0 & \cdots & 0 \\ 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & A_k \end{bmatrix}$$

If the NxN C matrix is of even dimension, then the +1 term is omitted. The $A_j \in \Re^{2x2}$ block entries are of the form

$$A_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$$

where, in fact, $-180° < \theta_j < +180°$. Using the above real decomposition of C in equation (29b), the Cayley transformation can be rewritten as

$$Q = T(I-D)(I+D)^{-1}T^T = T\hat{D}T^T$$

Since D is block-diagonal, the $\hat{D}$ matrix will also be block-diagonal and of the form

$$\hat{D} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & B_k \end{bmatrix}$$

where $B_j \in \Re^{2x2}$ is found to be

$$B_j = \tan\left(\frac{\theta_j}{2}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that the B_j *matrix* is not defined for $\theta_j = \pm 180°$, meaning that the orthogonal C matrix cannot contain any eigenvalues of -1 and must therefore be proper. Clearly, since $\hat{D} = -\hat{D}^T$ then Q is skew-symmetric and the mapping from skew-symmetric matrices to proper orthogonal matrices is *onto*.

For the 3x3 case, let the Q matrix be defined as the following skew-symmetric matrix:

$$Q = [\tilde{q}] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{30}$$

After substituting equation (30) into (29a), it can be verified that resulting C matrix is indeed equal to equation (22). Cayley's transformation (29) is a generalization of the classical Rodrigues parameter representation for NxN proper orthogonal matrices [1,2], while the Q matrix generalizes the Gibbs vector in higher dimensions [2,10].

Using the $[\tilde{\gamma}]$ matrix defined in equation (14) the Q matrix can be expressed as follows [2]:

$$Q = -\tanh\left(\frac{[\tilde{\gamma}]}{2}\right) = -\left(e^{\frac{[\tilde{\gamma}]}{2}} - e^{-\frac{[\tilde{\gamma}]}{2}}\right)\left(e^{\frac{[\tilde{\gamma}]}{2}} + e^{-\frac{[\tilde{\gamma}]}{2}}\right)^{-1} \tag{31}$$

The above transformation can be verified by performing a matrix power series expansion of equation (31) and substituting it into a matrix power series expansion of equation (29a). The result is a matrix power series expansion for the matrix exponential function as expected from equation (11). However, equation (12) cannot be used to calculate the matrix exponentials, since this equations only holds for the 3x3 case. Note the similarity between equation (31) and (20). Both calculate the Rodrigues parameters in terms of half the principal rotation angle!

The differential kinematic equations of the C matrix were shown in equation (1), where the skew-symmetric matrix $[\tilde{\omega}]$ is related to Q and $\dot{Q}$ via the kinematic relationship [1]

$$[\tilde{\omega}] = 2(I+Q)^{-1}\dot{Q}(I-Q)^{-1} \tag{32}$$

or conversely, $\dot{Q}$ can be written as

$$\dot{Q} = \frac{1}{2}(I+Q)[\tilde{\omega}](I-Q) \tag{33}$$

The equations (32-33) are proven to hold for the higher dimensional case in reference 1. For NxN orthogonal matrices, $[\tilde{\omega}] = -[\tilde{\omega}]^T$ represents an analogous "angular velocity" matrix.

Higher Dimensional Modified Rodrigues Parameters

As is evident above, the modified Rodrigues parameters have twice the principal rotation range as the classical Rodrigues parameters. It can be shown that the higher dimensional modified Rodrigues parameters also have twice the nonsingular domain as the higher dimensional classical Rodrigues parameters.

To find a transformation from the NxN proper orthogonal matrix C to the modified Rodrigues parameters, let us first examine what happens when taking the matrix square root of C. Let the square root matrix W be defined by the necessary, but not sufficient condition

$$WW = C \tag{34}$$

Obviously, for the general NxN case, there will be many W matrices that satisfy equation (34). Using the spectral decomposition of C given in equation (4), the spectral decomposition of W can be written as

$$W = U\sqrt{\Lambda}U^* \tag{35}$$

Since the C matrix is orthogonal, all the eigenvalues in Λ must have unit magnitude. Keep in mind that the Λ matrix in equation (35) is diagonal and that the matrix square root is trivial to calculate. Since taking the square root of an eigenvalue with unit magnitude results in another expression with unit magnitude, the W matrix itself is unitary, or orthogonal if all entries are real. It turns out that W is always real and orthogonal, as long as no eigenvalue of C is -1. If an eigenvalue of C is -1, then W has complex values and is a unitary matrix. The product of all eigenvalues of C is the determinant of C and must be +1 since C is proper. For even dimensions of C, the eigenvalues must all be complex conjugate pairs for the *det(C)* to be +1. For odd dimensions, the extra eigenvalue must be real and +1 in order for the matrix to be proper.

Each time a square root is calculated, there are two possible solutions. If the eigenvalue in question is one of the complex conjugate pairs, then the sign does not matter for W to be a proper matrix. If the matrix dimension is odd, then the root of the extra eigenvalue must be +1 for W to be proper. In the 3x3 case there is only one complex conjugate pair of eigenvalues. Hence only two W matrices satisfy the above conditions. This is to be expected, since any three-dimensional rotation can be described by two principal rotation angles which differ by 2π, one of which is positive and the other is negative. To make the choice of W unique, let us select all the roots of the complex conjugate pairs to have a positive real part.

Since the W matrix is orthogonal, with one exception, it has a principal line and angle associated with it. If the C matrix had an eigenvalue of -1, the same numerical problems arise as we encountered with finding the principal rotation vector. Multiplying W with itself in equation (34) simply doubles the principal angle, but leaves the principal line unchanged. Therefore W represents a rotation about the same principal line as C, but with half the principal angle. This provides conceptually elegant interpretations of the square root of C as defined above..

For three-dimensional rotations, the simple restriction on the square roots of the eigenvalues can be shown to restrict the principal rotation angle to satisfy $-180° < \phi < +180°$. This choice is consistent with many numerical matrix manipulation packages and their computation of a square root of a matrix. Let the j-th complex conjugate eigenvalue of C be denoted as $e^{\pm i\theta_j}$, where the the phase is $-180° \le \theta_j \le +180°$. If the dimension N is an odd number, W has the structure

$$W = U \cdot \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{+i\frac{\theta_{N-1}}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-i\frac{\theta_{N-1}}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix} \cdot U^* \tag{36}$$

If the dimension N is even, then W is

$$W = U \cdot \begin{bmatrix} e^{+i\frac{\theta_1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & e^{-i\frac{\theta_1}{2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & e^{+i\frac{\theta_N}{2}} & 0 \\ 0 & 0 & 0 & 0 & e^{-i\frac{\theta_N}{2}} \end{bmatrix} \cdot U^* \tag{37}$$

Using the parameterization given in equation (11), the matrix W can also be written directly in terms of the principal rotation matrix $[\tilde{\gamma}]$ as follows

$$W = e^{\frac{-[\gamma]}{2}} \tag{38}$$

This solution for W can be verified by substituting it into equation (34). Comparing equation (38) with equation (11) it becomes obvious that the W matrix has indeed the same principle rotation direction as C, with half the principle angle. Since, for three-dimensional rotations, there are two possible principal angles for a given attitude, there are two possible solutions for equation (38). Again, by keeping $|\phi| < 180°$, the same W matrix is obtained as with the matrix square root method discussed above.

Remember that the modified Rodrigues parameters have a nonsingular range corresponding to $|\phi| < 360°$. Since W is the direction cosine matrix corresponding to half of the principal rotation angle of C, the resulting nonsingular range of the W matrix has been reduced to $|\phi| < 180°$. This is the same nonsingular range as the classical Rodrigues parameters. Therefore the Cayley transformations, defined in equations (29a,b), can be applied to W. Let S be the skew-symmetric matrix composed of the modified Rodrigues parameters, similar to the construction of the Q matrix in equation (30). Then the transformation from W to S and its inverse are given as:

$$W = (I - S)(I + S)^{-1} = (I + S)^{-1}(I - S) \tag{39a}$$

$$S = (I - W)(I + W)^{-1} = (I + W)^{-1}(I - W) \tag{39b}$$

Using equation (39a) and (34), a direct transformation from S to C is found.

$$C = (I - S)^2(I + S)^{-2} = (I + S)^{-2}(I - S)^2 \tag{40}$$

This direct transformation is very similar to the classical Cayley transform, but no elegant direct inverse exists (i.e. we lose the elegance of equation (29b); no analogous equation can be written for S as a function of C). This is due to the overlapping principal rotation angle range of ±360° causing the transformation in equation (40) not to be injective (one-to-one). Since the classical Rodrigues parameters are for principal rotations between (-180°,+180°), they have a unique representation and the Cayley transform has the well known elegant inverse.

However, an alternate way to obtain the S matrix from the C matrix is available through the skew-symmetric matrix $[\tilde{\gamma}]$ defined in equation (14).

$$S = -\tanh\left(\frac{[\tilde{\gamma}]}{4}\right) = -\left(e^{\frac{[\tilde{\gamma}]}{4}} - e^{-\frac{[\tilde{\gamma}]}{4}}\right)\left(e^{\frac{[\tilde{\gamma}]}{4}} + e^{-\frac{[\tilde{\gamma}]}{4}}\right)^{-1} \quad (41)$$

The transformations given in equation (41) can be verified by performing a matrix power series expansion and back-substituting it into equation (40). Note again the similarity between equation (41) and equation (25). The principal rotation angle is divided by four in both cases.

Either the W or the $[\tilde{\gamma}]$ matrix can be solved from the proper NxN orthogonal C matrix to obtain the corresponding S matrix. Neither method is as elegant, however, as equation (29b) of the Cayley transformation. The method using the $[\tilde{\gamma}]$ matrix has the advantage that $[\tilde{\gamma}]$ is found by taking the matrix logarithm of the eigenvalues of the C matrix as shown in equation (14). The uniqueness questions do not arise here as in the matrix square root method because solutions are implicitly restricted to proper rotations with $|\phi| < 180°$. Both methods produce the same results using, for example, the matrix exponential and matrix square root algorithms available as MATLAB or MATHEMATICA operators. Note that both the classical and the “updated" Cayley transform have numerical problems when transforming a proper orthogonal matrix C into a skew-symmetric matrix if C has eigenvalues of -1.

Since each set of modified Rodrigues parameters has its associated “shadow” set [6], it is usually not important which S parameterization one obtains, as long as at least one valid S matrix is found. Once a parameter set is found, either the original ones or the “shadow” set, it is trivial to remain with this set during the forward integration of the differential equations governing the evolution of S.

The differential kinematic equations for S are not written directly from C as they were with the classical Cayley transform. Instead W is used to describe the kinematics of the NxN system. The relationship between W and S is the same as between C and Q. Therefore the same equations can be used. The differential kinematic equation for W is:

$$\dot{W} = -[\tilde{\Omega}]W \quad (42)$$

where the skew-symmetric matrix $[\tilde{\Omega}]$ is:

$$[\tilde{\Omega}] = 2(I+S)^{-1}\dot{S}(I-S)^{-1} \quad (43)$$

or conversely $\dot{S}$ could be defined as:

$$\dot{S} = \frac{1}{2}(I+S)[\tilde{\Omega}](I-S) \quad (44)$$

Equation (34) can be used during the forward integration to obtain *C(t)*. The time evolution of *C* in terms of *W* and $[\tilde{\Omega}]$ is:

$$\dot{C} = -[\tilde{\Omega}]WW - W[\tilde{\Omega}]W = -[\tilde{\Omega}]C - W[\tilde{\Omega}]W \tag{45}$$

Equating equation (45) and (1), the direct transformation from $[\tilde{\Omega}]$ to $[\tilde{\omega}]$ is:

$$[\tilde{\omega}] = [\tilde{\Omega}] + W[\tilde{\Omega}]W^T \tag{46}$$

To verify that equation (46) yields a skew-symmetric matrix $[\tilde{\omega}]$, the definition of a skew-symmetric matrix is used:

$$[\tilde{\omega}] = -[\tilde{\omega}]^T = -\left([\tilde{\Omega}] + W[\tilde{\Omega}]W^T\right)^T$$

$$[\tilde{\omega}] = -[\tilde{\Omega}]^T - \left(W^T\right)^T[\tilde{\Omega}]^T W^T$$

$$[\tilde{\omega}] = [\tilde{\Omega}] + W[\tilde{\Omega}]W^T \quad q.e.d.$$

Although this new parameterization is somewhat more complicated than the classical parameterization into M-dimensional Rodrigues parameters, the complications arise only when setting up the parameterization in terms of *S*. Once an *S* matrix and a corresponding *W* matrix have been found, this method is no different from the classical method. The important improvement is that the range of possible principle rotations has been doubled over the classical M-dimensional Rodrigues parameters.

A Preliminary Investigation of Higher Dimensional Euler Parameters

The classical Euler parameters stood apart from the other parameterizations, because they were bounded, universally nonsingular and had an easy-to-solve bi-linear differential kinematic equations. All of these attractive features were only slightly affected by the cost of increasing the dimension of the parameter vector by one. These classical Euler parameters are extended below to higher dimensions, where they will retain some, but not all, of the above desirable features.

The Rodrigues parameters and the Euler parameters are very closely related as seen in equation (19). They are identical except for the scaling term of β_0. The classical Rodrigues parameters have been shown to expand to the higher dimensional case where they parameterize a NxN orthogonal matrix *C* [1]. Analogous to equation (19), they can always be described as the ratio of a once-redundant set of parameters.

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3, ..., M = \frac{N(N-1)}{2} \tag{47}$$

The skew-symmetric matrix *Q* in equation (29a) can be written as:

$$Q = \frac{1}{\beta_0} B \tag{48}$$

where *B* is a NxN skew-symmetric matrix containing the numerators β_i of *Q*. For the three dimensional case, this matrix is the “vector" part of the classical Euler parameters β_1, β_2, β_3, and has the familiar structure

$$B = \begin{bmatrix} 0 & -\beta_3 & \beta_2 \\ \beta_3 & 0 & -\beta_1 \\ -\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{49}$$

Substituting the transformation relating Q to $\{\beta_0,\beta_1,...,\beta_M\}$, as given in equation (48) the Cayley transform of equation (29a) results in the following

$$C = (\beta_0 I - B)(\beta_0 I + B)^{-1}$$

$$C(\beta_0 I + B) = (\beta_0 I - B)$$

$$(I - C)\beta_0 - (I + C)B = 0 \tag{50}$$

Equation (50) represents an NxN system of linear equations in $\{\beta_0,\beta_1,...,\beta_M\}$. Let the $[N^2 x(M+1)]$ matrix A represent the linear relationship between the β_i.

$$A \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_M \end{bmatrix} = 0 \tag{51}$$

Clearly the set of all possible higher dimensional Euler parameters spans the kernel of A. We know that the M Rodrigues parameters are a minimal set to parameterize the orthogonal NxN matrix C. By adding the scaling factor β_0, a once redundant set of parameters has been generated. Even though there are N^2 linear equations in equation (50), the dimension of the range of A is only M. The problem is still under determined. The dimension of the kernel of A must be one, since only one additional term was added to a minimal set of rotation parameters. The solution space is a multi-dimensional line through the origin.

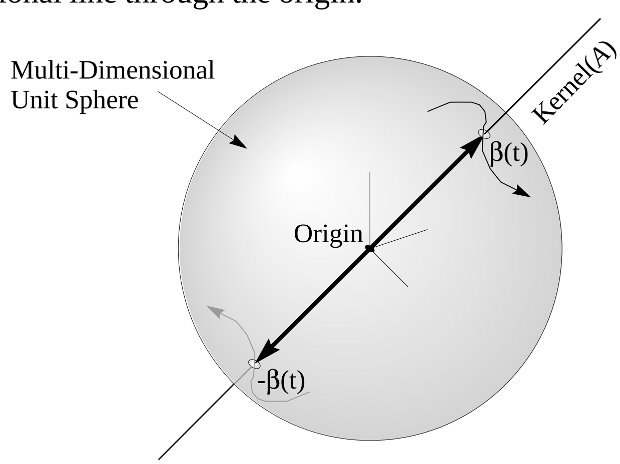

Fig. 3: Solution of the Higher Dimensional Euler Parameters.

After finding the kernel base vector, an infinite number of solutions still exist. Another constraint is needed. Let us set the norm of the higher dimensional Euler parameter vector to be unity. This concept is illustrated in Fig. 3 above.

$$\beta_0^2 + \beta_1^2 + \cdots + \beta_M^2 = 1 \tag{52}$$

Equation (52) is the higher dimensional equivalent of the holonomic constraint of the classical Euler parameters introduced in equation (16).

Two solutions are found scaling the base vector of the kernel of A to unit length. Just as with the classical Euler parameters, any point on the multi-dimensional Euler parameter unit sphere describes the same physical orientation as its antipodal pole. Therefore the higher order Euler parameters are not unique, but contain a duality. This is exactly analogous to the classical case. This duality does not pose any practical problems, except under one circumstance discussed below.

$$C = (\beta_0 I - B)(\beta_0 I + B)^{-1} = (\beta_0 I + B)^{-1}(\beta_0 I - B) \tag{53}$$

The inverse transformation from higher order Euler parameters to the orthogonal matrix C is found by using Q from equation (48) in the classical Cayley transform. The result is shown in equation (53). Using a B, as shown in equation (49) for the three-dimensional case, in equation (53) results in the same transformation as given in equation (18). Observe that the inverse transformation has a singularity when β_0 is zero. This singularity is a mathematical singularity only. Contrary to the Rodrigues parameters, the higher order Euler parameters are well defined at this orientation. After an appropriate skew-symmetric matrix B is constructed and carrying out the algebra in equation (53), a closed form algebraic transformation is found

For the 2x2 case, the B matrix is given by

$$B = \begin{bmatrix} 0 & -\beta_1 \\ \beta_1 & 0 \end{bmatrix} \tag{54}$$

Using the B defined above in equation (53), the 2x2 direction cosine matrix C is:

$$C_{2x2} = \begin{bmatrix} \beta_0^2 - \beta_1^2 & 2\beta_0\beta_0 \\ -2\beta_0\beta_0 & \beta_0^2 - \beta_1^2 \end{bmatrix} \tag{55}$$

The 2x2 C matrix contains no polynomial fractions and is easy to calculate. To find the direction cosine matrix for the 3x3 case, use the B matrix defined in equation (51) in equation (53).

$$C_{3x3} = \frac{1}{\beta_0(\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2)} \begin{bmatrix} \beta_0(\beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2) & 2\beta_0(\beta_1\beta_2 + \beta_0\beta_3) & 2\beta_0(\beta_1\beta_3 - \beta_0\beta_2) \\ 2\beta_0(\beta_1\beta_2 - \beta_0\beta_3) & \beta_0(\beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2) & 2\beta_0(\beta_2\beta_3 + \beta_0\beta_1) \\ 2\beta_0(\beta_1\beta_3 + \beta_0\beta_2) & 2\beta_0(\beta_2\beta_3 - \beta_0\beta_1) & \beta_0(\beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2) \end{bmatrix}$$

After making the obvious cancellations and enforcing the holonomic constraint equation, the well known result is found which represents the 3x3 direction cosine matrix as a function of the classical Euler parameters as given in equation (18). This classical representation contains no polynomial fractions and no singularities, just as was the case with the 2x2 system.

For dimensions greater than 3x3's, however, the algebraic transformation contains polynomial fractions. The nice cancelations that occur with a 2x2 and a 3x3 orthogonal matrices *do not* occur with the higher dimensions. This might have been anticipated, because [2] it is well-known that quaternion algebra does not generalize fully to arbitrary higher-dimensional spaces, and the elegant classical Euler parameter results are essentially manifestations of quaternion algebra. To find C_{4x4} in terms of the higher dimensional Euler parameters, we define the 4x4 *B* matrix as:

$$B_{4x4} = \begin{bmatrix} 0 & -\beta_6 & \beta_5 & -\beta_4 \\ \beta_6 & 0 & -\beta_3 & \beta_2 \\ -\beta_5 & \beta_3 & 0 & -\beta_1 \\ \beta_4 & -\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{56}$$

and substitute it into equation (53), this leads to

$$C_{4x4} = \frac{1}{\Delta}\begin{bmatrix} \beta_0^2(\beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2-\beta_4^2-\beta_5^2-\beta_6^2)-\delta^2 & 2\beta_0(\beta_0(\beta_2\beta_4+\beta_3\beta_5+\beta_0\beta_6)+\beta_1\delta) \\ 2\beta_0(\beta_0(\beta_2\beta_4+\beta_3\beta_5-\beta_0\beta_6)-\beta_1\delta) & \beta_0^2(\beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2+\beta_4^2+\beta_5^2-\beta_6^2)-\delta^2 \\ 2\beta_0(\beta_0(\beta_0\beta_5+\beta_3\beta_6-\beta_1\beta_4)-\beta_2\delta) & 2\beta_0(\beta_0(\beta_1\beta_2-\beta_0\beta_3+\beta_5\beta_6)-\beta_4\delta) \\ 2\beta_0(\beta_0(-\beta_0\beta_4-\beta_1\beta_5-\beta_2\beta_6)-\beta_3\delta) & 2\beta_0(\beta_0(\beta_1\beta_3+\beta_0\beta_2-\beta_4\beta_6)-\beta_5\delta) \end{bmatrix} \cdots$$

$$\cdots \begin{bmatrix} 2\beta_0(\beta_0(-\beta_0\beta_5+\beta_3\beta_6-\beta_1\beta_4)+\beta_2\delta) & 2\beta_0(\beta_0(\beta_0\beta_4-\beta_1\beta_5-\beta_2\beta_6)+\beta_3\delta) \\ 2\beta_0(\beta_0(\beta_1\beta_2+\beta_0\beta_3+\beta_5\beta_6)+\beta_4\delta) & 2\beta_0(\beta_0(\beta_1\beta_3-\beta_0\beta_2-\beta_4\beta_6)+\beta_5\delta) \\ \beta_0^2(\beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2+\beta_4^2-\beta_5^2+\beta_6^2)-\delta^2 & 2\beta_0(\beta_0(\beta_0\beta_1+\beta_4\beta_5+\beta_2\beta_3)+\beta_6\delta) \\ 2\beta_0(\beta_0(-\beta_0\beta_1+\beta_4\beta_5+\beta_2\beta_3)-\beta_6\delta) & \beta_0^2(\beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2-\beta_4^2+\beta_5^2+\beta_6^2)-\delta^2 \end{bmatrix} \tag{57}$$

$$\text{with} \quad \delta = \beta_3\beta_4+\beta_1\beta_6-\beta_2\beta_5$$
$$\Delta = \beta_0^2+\delta^2$$

This denominator Δ can vanish for several β_i configurations. Observe, however, that whenever Δ is zero, so is the numerator. For each singular case we can confirm that a finite limit exists, as was to be expected, since the original orthogonal *C* matrix was finite. In all cases $\beta_0 = 0$ is a prerequisite for a (0/0) condition to occur. Finding the transformations for matrices with dimensions greater than 4x4 would show the same behavior. $\beta_0 = 0$ is always a indicator that a mathematical singularity *may* occur. In none of these cases are the higher dimensional Euler parameters themselves actually singular. It is always a mathematical singularity of the transformation itself. To circumvent this problem for particular applications, the limit of the fraction can be found as $\beta_0 \to 0$. After substituting $\beta_0 = 0$ into equation (57), for example, most fractions become trivial and the matrix is reduced to

$$C_{4x4} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -I_{4x4} \tag{58}$$

Substituting $\beta_0 = 0$ into equation (55) yields the same result. Actually, as long as *C* is of *even* dimension the matrix will be *-I* if $\beta_0 = 0$. If the dimension is *odd*, as it is for the 3x3 case, the *C* matrix will be fully populated. With this observation it is easy to circumvent the singular situations if the dimension is even. If the dimension is odd a numerical limit must be found. In either case the transformation will be well behaved everywhere except the $\beta_0 = 0$ surface. The fact that the 0/0 condition can be resolved analytically to obtain finite limits should not obscure the frustrating fact that these 0/0 conditions would pose numerical difficulties in general numerical algorithms.

Let us examine the uniqueness of the transformation given in equation (53). Assuming that the transformation is not unique, two possible higher dimensional Euler parameter sets $\acute{\beta}$ and $\H{\beta}$ are chosen, these parameterize C as

$$C = \left(\acute{\beta}_0 I - \acute{B}\right)\left(\acute{\beta}_0 I + \acute{B}\right)^{-1}$$
$$C = \left(\H{\beta}_0 I + \H{B}\right)^{-1}\left(\H{\beta}_0 I - \H{B}\right)$$

Subtracting one equation from the other the following condition is obtained:

$$0 = \left(\acute{\beta}_0 I - \acute{B}\right)\left(\acute{\beta}_0 I + \acute{B}\right)^{-1} - \left(\H{\beta}_0 I + \H{B}\right)^{-1}\left(\H{\beta}_0 I - \H{B}\right)$$

$$0 = \left(\H{\beta}_0 I + \H{B}\right)\left(\acute{\beta}_0 I - \acute{B}\right) - \left(\H{\beta}_0 I - \H{B}\right)\left(\acute{\beta}_0 I + \acute{B}\right)$$

$$0 = \acute{\beta}_0 \H{B} - \H{\beta}_0 \acute{B}$$

or

$$\frac{\acute{B}}{\acute{\beta}_0} = \frac{\H{B}}{\H{\beta}_0} \tag{59}$$

Equation (59) is the necessary condition for two higher order Euler parameter sets to yield the same direction cosine matrix C. Obviously, for $\beta_0 \neq 0$ this can only occur when

$$\begin{aligned} \H{B} &= k \cdot \acute{B} \\ \H{\beta}_0 &= k \cdot \acute{\beta}_0 \end{aligned} \tag{60}$$

where k is a scalar. This condition apparently yields an infinite number of solutions. But since the higher dimensional Euler parameters must satisfy the holonomic constraint given in equation (52), only unit scaling values of k are permissible. Therefore k must be either ±1. The above uniqueness study results in exactly the same duality as is observed with the classical Euler parameters, except the restriction on $\beta_0 \neq 0$. There are always *two* possible sets of classical Euler parameters which describe an orthogonal 3x3 matrix C. It is evident that this truth extends to the more general case of NxN orthogonal matrices . This duality was seen earlier when applying the holonomic constraint to the kernel of A.

$$C_{NxN}[\beta(t)] = C_{NxN}[-\beta(t)] \tag{61}$$

Based on the above, if $\beta_0 = 0$ nothing can be said about the transformation uniqueness. As was seen with the 4x4 C matrix, the $\beta_0 = 0$ condition permits any point on the unit sphere $\sum_{i=1}^{6} \beta_i^2 = 1$.

Having established the forward and backward transformations between the NxN orthogonal matrices and the higher order Euler parameters, their kinematic equations are also of interest. To describe the orthogonal matrix C as a generalized rigid body rotation, C must satisfy a differential equation of the form given in equation (1). After substituting equation (48) into equation (33), $\dot{Q}$ is

$$\dot{Q} = \frac{1}{2}\left(I + \frac{B}{\beta_0}\right)[\tilde{\omega}]\left(I - \frac{B}{\beta_0}\right) \tag{62}$$

After differentiating equation (48) directly, $\dot{Q}$ is found to be

$$\dot{Q} = \frac{\beta_0 \dot{B} - \dot{\beta}_0 B}{\beta_0^2} \tag{63}$$

Upon substituting equation (62) into equation (63) and after making some simplifications, the following kinematic relationship is found.

$$\beta_0 \dot{B} - \dot{\beta}_0 B = \frac{1}{2}(\beta_0 I + B)[\tilde{\omega}](\beta_0 I - B) \tag{64}$$

This equation can be solved for the skew-symmetric angular velocity matrix $[\tilde{\omega}]$.

$$[\tilde{\omega}] = 2(\beta_0 I + B)^{-1}\left(\beta_0 \dot{B} - \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1} \tag{65}$$

Note that this equation contains the same *mathematical* singularity at β_0 =0 as did equation (53). Carrying out the algebra a closed form algebraic equation is found for the higher order angular velocities.

Let us verify that equation (65) for the angular velocities does indeed generate a skew-symmetric matrix. This is easily accomplished using the definition of a skew-symmetric matrix as follows

$$[\tilde{\omega}] = -[\tilde{\omega}]^T = -2\left((\beta_0 I + B)^{-1}\left(\beta_0 \dot{B} - \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1}\right)^T$$

$$[\tilde{\omega}] = -2(\beta_0 I - B)^{-1\,T}\left(\beta_0 \dot{B} - \dot{\beta}_0 B\right)^T(\beta_0 I + B)^{-1\,T}$$

$$[\tilde{\omega}] = -2\left(\beta_0 I^T - B^T\right)^{-1}\left(\beta_0 \dot{B}^T - \dot{\beta}_0 B^T\right)\left(\beta_0 I^T + B^T\right)^{-1}$$

Since the matrix B and its derivative are skew-symmetric matrices by definition, further simplifications are possible to obtain the following result

$$[\tilde{\omega}] = -2(\beta_0 I + B)^{-1}\left(-\beta_0 \dot{B} + \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1}$$

$$[\tilde{\omega}] = 2(\beta_0 I + B)^{-1}\left(\beta_0 \dot{B} - \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1} \quad q.e.d.$$

All higher order Euler parameter differentials must abide by the derivative of the constraint equation (52).

$$2\dot{\beta}_0 \beta_0 + 2\dot{\beta}_1 \beta_1 + \ldots + 2\dot{\beta}_M \beta_M = 0 \tag{66}$$

After using the B from equation (49) the linear differential kinematic equations of the classical Euler parameters are found. To verify that equation (65) generalizes correctly, known classical results let us verify two special cases. For the 2x2 case, a scalar differential kinematic equation results from equation (65) as

$$\omega_1 = 2[-\beta_1 \quad \beta_0]\begin{bmatrix}\dot{\beta}_0 \\ \dot{\beta}_1\end{bmatrix} \tag{67}$$

Adding the constraint in equation (66), equation (67) can be padded to make it full rank.

$$\begin{bmatrix}0 \\ \omega_1\end{bmatrix} = 2\begin{bmatrix}\beta_0 & \beta_1 \\ -\beta_1 & \beta_0\end{bmatrix}\begin{bmatrix}\dot{\beta}_0 \\ \dot{\beta}_1\end{bmatrix} \tag{68}$$

Note that as with the 3x3 case, the matrix transforming $\dot{\beta}$ to ω is orthogonal for the 2x2 case. Therefore the inverse transformation can be written as:

$$\begin{bmatrix}\dot{\beta}_0 \\ \dot{\beta}_1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta_0 & -\beta_1 \\ \beta_1 & \beta_0\end{bmatrix}\begin{bmatrix}0 \\ \omega_1\end{bmatrix} \tag{69}$$

It is straight forward to show that equations (65) and (66) give equation (17) for the 3x3 case. Analogous to the 3x3 case, the above differential kinematic equation for the 2x2 case is also bi-linear. As with the 4x4 and greater direction cosine matrices, for proper orthogonal matrices having dimensions greater than 3x3 the higher dimensional differential kinematic equations also contain polynomial fractions. Using the B matrix from equation (56) in equation (65) and collecting all the angular velocity term, we find the differential kinematic equations for the 4x4 case

$$\begin{Bmatrix}0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6\end{Bmatrix} = \frac{2}{\Delta}\left[\begin{matrix} \Delta\beta_0 & \Delta\beta_1 & \Delta\beta_2 \\ \beta_6(\beta_2\beta_5 - \beta_3\beta_4) - \beta_1(\beta_0^2 + \beta_6^2) & \beta_0(\beta_0^2 + \beta_6^2) & \beta_0(\beta_0\beta_3 - \beta_5\beta_6) \\ \beta_5(\beta_1\beta_6 + \beta_3\beta_4) - \beta_2(\beta_0^2 + \beta_5^2) & -\beta_0(\beta_0\beta_3 + \beta_5\beta_6) & \beta_0(\beta_0^2 + \beta_5^2) \\ \beta_4(\beta_2\beta_5 - \beta_1\beta_6) - \beta_3(\beta_0^2 + \beta_4^2) & \beta_0(\beta_4\beta_6 + \beta_0\beta_2) & -\beta_0(\beta_0\beta_1 + \beta_4\beta_5) \\ \beta_3(\beta_2\beta_5 - \beta_1\beta_6) - \beta_4(\beta_0^2 + \beta_3^2) & \beta_0(-\beta_0\beta_5 + \beta_3\beta_6) & -\beta_0(\beta_0\beta_6 + \beta_5\beta_3) \\ \beta_2(\beta_1\beta_6 + \beta_3\beta_4) - \beta_5(\beta_0^2 + \beta_2^2) & \beta_0(\beta_0\beta_4 - \beta_2\beta_6) & \beta_0(\beta_3\beta_4 + \beta_1\beta_6) \\ \beta_1(\beta_2\beta_5 - \beta_3\beta_4) - \beta_6(\beta_0^2 + \beta_1^2) & \beta_0(\beta_2\beta_5 - \beta_3\beta_4) & \beta_0(\beta_0\beta_4 - \beta_1\beta_5) \end{matrix}\right. \cdots$$

$$\cdots \left.\begin{matrix} \Delta\beta_3 & \Delta\beta_4 & \Delta\beta_5 & \Delta\beta_6 \\ \beta_0(\beta_4\beta_6 - \beta_0\beta_2) & \beta_0(\beta_0\beta_5 + \beta_3\beta_6) & -\beta_0(\beta_0\beta_4 + \beta_2\beta_6) & \beta_0(\beta_2\beta_5 - \beta_3\beta_4) \\ \beta_0(\beta_0\beta_1 - \beta_4\beta_5) & \beta_0(\beta_0\beta_6 - \beta_5\beta_3) & \beta_0(\beta_3\beta_4 + \beta_1\beta_6) & -\beta_0(\beta_0\beta_4 + \beta_1\beta_5) \\ \beta_0(\beta_0^2 + \beta_4^2) & \beta_0(\beta_2\beta_5 - \beta_1\beta_6) & \beta_0(\beta_0\beta_6 - \beta_2\beta_4) & \beta_0(\beta_1\beta_4 - \beta_0\beta_5) \\ \beta_0(\beta_2\beta_5 - \beta_1\beta_6) & \beta_0(\beta_0^2 + \beta_3^2) & \beta_0(\beta_0\beta_1 - \beta_2\beta_3) & \beta_0(\beta_0\beta_2 + \beta_1\beta_3) \\ -\beta_0(\beta_0\beta_6 + \beta_2\beta_4) & -\beta_0(\beta_0\beta_1 + \beta_2\beta_3) & \beta_0(\beta_0^2 + \beta_2^2) & \beta_0(\beta_0\beta_3 - \beta_1\beta_2) \\ \beta_0(\beta_1\beta_4 + \beta_0\beta_5) & \beta_0(-\beta_0\beta_2 + \beta_1\beta_3) & -\beta_0(\beta_0\beta_3 + \beta_1\beta_2) & \beta_0(\beta_0^2 + \beta_1^2) \end{matrix}\right]\begin{Bmatrix}\dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \\ \dot{\beta}_4 \\ \dot{\beta}_5 \\ \dot{\beta}_6\end{Bmatrix} \tag{70}$$

$$\textit{with } \Delta = \beta_0^2 + (\beta_3\beta_4 - \beta_2\beta_5 + \beta_1\beta_6)^2$$

Note that this transformation matrix is no longer orthogonal as were the corresponding matrices for both the 2x2 and 3x3 cases. The bi-linearity found for 2x2 and 3x3 cases is also lost for the higher dimensional cases. Equation (70) has the same denominator as the 4x4 direction cosine matrix. Hence it contains the identical singular situations. However, if $\beta_0 = 0$, the above transformation matrix is singular and cannot be inverted!

Thus the higher dimensional Euler parameters lose some key properties as they are generalized to parameterize higher dimensioned proper orthogonal matrices. They retain the properties of being bounded and mapping all rotations onto arcs on a unit hypersphere. However, the kinematic transformations and orthogonal matrix representations loose the elegance of their classical 3x3 counterparts. In particular, $\beta_0 = 0$ poses several unresolved issues for all dimensions higher than 3x3.

Conclusion

The principal rotation parameterizations presented show great promise as an elegant means for describing the evolution of NxN orthogonal matrices. The modified Rodrigues parameters are only slightly more complicated than their classical counterparts, but double the nonsingular rotation domain The (M+1)-dimensional Euler parameters retain some of the desirable features of their classical counterparts. However, for orthogonal matrices greater than 3x3 though, the orthogonal matrix representation formulas and the corresponding differential kinematic equations contain some mathematical singularities which require taking the limits of polynomial fractions. The computational effort for calculating the higher dimensional Euler parameters grows rapidly when increasing the dimension of the *C* matrix. For higher dimensional rotations, the modified Rodrigues parameters show the greatest promise. The gain (increased nonsingular domain in comparison to the classical Cayley transformation), significantly outweighs the extra computation.

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