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STEREOGRAPHIC ORIENTATION PARAMETERS FOR ATTITUDE DYNAMICS: A GENERALIZATION OF THE RODRIGUES PARAMETERS

Hanspeter Schaub* and John L. Junkins†

A new family of orientation parameters derived from the Euler parameters is presented. They are found by a general stereographic projection of the Euler parameter constraint surface, a four-dimensional unit sphere, onto a three-dimensional hyperplane. The resulting set of three stereographic parameters have a low degree polynomial non-linearity in the corresponding kinematic equations and direction cosine matrix parameterization. The stereographic parameters are not unique, but have a corresponding set of "shadow" parameters. These "shadow" parameters are distinct, yet represent the same physical orientation. Using the original stereographic parameters combined with judicious switching to their shadow set, it is possible to describe any rotation without encountering a singularity. The symmetric stereographic parameters are nonsingular for up to a principal rotation of ±360°. The asymmetric stereographic parameters are well suited for describing the kinematics of spinning bodies, since they only go singular when oriented at a specific angle about a specific axis. A globally regular and stable control law using symmetric stereographic parameters is presented which can bring a spinning body to rest in any desired orientation without backtracking the motion.

INTRODUCTION

While the Euler parameters (quaternions) describe an arbitrary orientation without a singularity, they form a once-redundant set. The following development studies a method to stereographically project the Euler parameters onto a three-dimensional hyperplane and form a family of sets of three parameters called the *stereographic parameters*. This study is motivated by earlier work done by Marandi and Modi[1], Tsiotras[2], Shuster[3] and Wiener[4]. In particular, Wiener, Marandi and Modi introduce a set of three parameters similar to the Rodrigues parameters (singular at a principal rotation of $\Phi = \pm 180°$), which move the singularity out to a principal rotation Φ of ±360°! Marandi, Modi and Tsiotras describe this modified set of Rodrigues parameters as the result of a stereographic projection of a four-dimensional unit sphere onto a three-dimensional hyperplane. This paper will explore the stereographic projection idea further and in a more generalized way, and show that both the classical Rodrigues parameters and the Modi/Wiener

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modified Rodrigues parameters can be considered a special case of the general symmetric stereographic parameters. Indeed, the method presented can be used to construct a set of three symmetric stereographic parameters which have their singular point anywhere between a principal rotation of 0° and 360°, or to construct a set of three asymmetric stereographic parameters which have their singular point determined by both a principal angle and an axis of rotation. Analogous to the Euler parameters, the stereographic parameters are generally not unique. The Euler parameters time variation, for any physical motion, generate a trajectory on the surface of the unit constraint sphere surface. The reflection of the Euler parameters (reversing all parameters signs) generates a second trajectory on the opposite side of the sphere which corresponds to the same physical rotation. Each set of stereographic parameters has a set of "shadow parameters" which correspond to the reflection set of Euler parameters. These "shadow" stereographic parameters are generally numerically different from the original parameters, yet physically parameterize the same rotation. The developments presented herein are of significant academic importance; using stereographic projections it is easy to visualize the singularities of this infinite family of three parameter sets which include the classical and modified Rodrigues parameters.

The modified Rodrigues parameters, as introduced by Wiener, Marandi and Modi, are studied in further detail, since they present the largest range of non-singular rotations for the symmetric stereographic parameters. In combination with the corresponding set of "shadow parameters," a globally regular and non-singular Lyapunov attitude control is established in feedback form.

THE EULER PARAMETER UNIT SPHERE

The four Euler parameters are well known and well studied in the literature. They can be developed directly from Euler's principal rotation theorem[3,6]. The angle Φ is the principal rotation angle and the unit vector $\underline{e}$ is the principal line of rotation, the Euler parameters are defined as

$$\beta_0 = \cos\frac{\Phi}{2} \qquad \beta_i = e_i \cdot \sin\frac{\Phi}{2} \qquad i = 1, 2, 3 \tag{1}$$

$$\underline{\beta}^T\underline{\beta} = \beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{2}$$

The four Euler parameters β_i abide by the holonomic constraint given in Eq. (2). This equation defines a four-dimensional unit sphere. The Euler parameter trajectories on this sphere completely describe any possible rotational motion without any singularities or discontinuities. However, note that the Euler parameters are not unique. The mirror image trajectory $-\underline{\beta}(t)$ describes the identical rotational motion as $\underline{\beta}(t)$. The negative sign means the rotation is accomplished about a principal axis of the opposite direction through the negative principal angle. Usually this non-uniqueness does not pose any difficulties since both sets have identical properties, correspond to the same physical orientation, and can be solved uniquely once initial conditions are prescribed.

Because the Euler parameters satisfy one holonomic constraint, they form a once redundant set of equations. Three parameters are sufficient to describe a general rotation. However, the problem with any set of three parameters is that, as is well known, singularities will occur at certain orientations. Different three-parameter sets distinguish themselves by having different geometric interpretations and, especially, having their singular behavior at different orientations. Also of significance, most three-parameter sets introduce transcendental nonlinearities into the parameterization of the direction cosine matrix and related kinematical relationships. However, the classical Rodrigues parameters and other sets discussed herein involve low degree polynomial nonlinearities in both the direction cosine matrix and associated kinematical differential equation, without approximation. The Euler parameter description represents an attractive *regularization* which has no singularity, at the cost of having one extra variable.

STEREOGRAPHIC PROJECTION OF THE 4D UNIT SPHERE

If a minimum parameter representation is desired, the four Euler parameters can be reduced to any parameter set of three by an appropriate transformation. For example, the 3-1-3 Euler angles or the Rodrigues parameters are very commonly used sets that are easily transformed from the Euler parameters[3,5]. Marandi, Modi and Tsiotras found a set of modified Rodrigues parameters by means of a stereographic projection of the four-dimensional unit sphere onto a three-dimensional hyperplane. To describe the stereographic projection, imagine a three-dimensional sphere being projected onto a two-dimensional plane (analogous to the Earth map projection problem). A certain point is chosen in the 3D space called a projection point. Next a 2D mapping plane is chosen. Every point on the unit sphere is then projected onto the mapping plane by drawing a line from the projection point through the point on the unit sphere and intersected with the mapping plane.

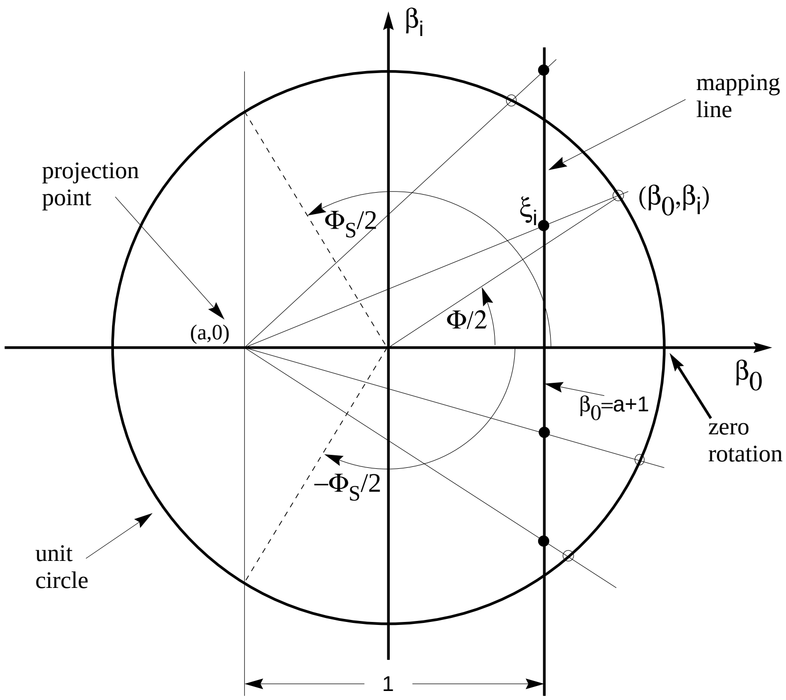

Figure 1: Illustration of a Symmetric Stereographic Projection onto Hyperplane Orthogonal to β_0 axis.

Figure 1 shows only a 2D to 1D stereographic projection to keep the illustration simple. The results though can easily be expanded to a four-dimensional sphere since the axes are orthogonal to each other. With all these projections the Euler parameter β_0 is eliminated since the mapping hyperplane normal is the β_0 axis. They are called *symmetric* projections since the non-singular principal angle range is symmetric about the zero rotation angle. *Asymmetric* stereographic projections are projections onto a hyperplane with a normal other than the β_0 axis, which do not have a symmetric principal angle range. The case where the Euler parameter β_1, β_2 or β_3 is eliminated is discussed later in this paper.

Placing the projection point on the β_0 axis yields a symmetric situation wherein the zero rotation is in the center of the nonsingular principal angle range. The mapping line is placed a distance of +1 from the projection point. The parameters are scaled by this arbitrary distance, so having a distance of 2 between the projection point and the mapping plane would simply scale all the parameters by a factor of 2.

Keep in mind that the Euler parameters are defined in terms of *half* of the principal rotation angle Φ. The point (1,0) on the circle corresponds to a zero rotation. The point (0,1) corresponds to a +180° rotation. Studying Figure 1 it becomes evident that the reduced parameters go off to infinity when a point on the circle is projected which lies directly in the plane perpendicular to the β_0 axis through the projection point. The two lines that need to be intersected are parallel to each other, causing the intersection point to move to infinity. The corresponding principal rotation obviously yields the angle at which the reduced set of parameters will go singular! By placing the projection point at different locations on the β_0 axis, the maximum principal rotation angle is varied. If the projection point is outside the unit circle, no singularity will occur, but the projection is no longer one-to-one. Clearly this is not a desirable feature because of the ambiguity this lack of uniqueness would introduce (given the projected coordinates, we cannot uniquely orient the reference frame).

The angle Φ_S is the principal angle of rotation where the stereographic parameter vector $\underset{\sim}{\zeta}$ encounters a singularity. This angle Φ_S determines the placement of the projection point a.

$$a = \cos\frac{\Phi_S}{2} \tag{3}$$

The transformation from the Euler parameters to a general set of three symmetric stereographic parameters $\underset{\sim}{\zeta}$ is defined as:

$$\zeta_i = \frac{\beta_i}{\beta_0 - a} \qquad i = 1, 2, 3 \tag{4}$$

The condition for a symmetric stereographic parameter singularity, evident in Eq. (4), is

$$a = \beta_0 = \cos\frac{\Phi}{2} \tag{5}$$

If $a < 1$ this condition is satisfied at an infinite set of orientations. If the projection point is on the unit sphere surface, then $a = -1$ and a singularity is only achieved at $\Phi = \pm 360°$. The inverse transformation from the general stereographic parameters $\underset{\sim}{\zeta}$ to the Euler parameters β_i is

$$\begin{aligned} \beta_0 &= \frac{a\underset{\sim}{\zeta}^T\underset{\sim}{\zeta} + \sqrt{1 + \underset{\sim}{\zeta}^T\underset{\sim}{\zeta}(1 - a^2)}}{1 + \underset{\sim}{\zeta}^T\underset{\sim}{\zeta}} \\ \beta_i &= \zeta_i\left[\frac{-a + \sqrt{1 + \underset{\sim}{\zeta}^T\underset{\sim}{\zeta}(1 - a^2)}}{1 + \underset{\sim}{\zeta}^T\underset{\sim}{\zeta}}\right] \qquad i = 1, 2, 3 \end{aligned} \tag{6}$$

This equation holds for both the symmetric and asymmetric stereographic projections. Since the Euler parameters are not unique, it is valid to rewrite Eq. (4) in terms of $-\beta_i$. For the general case these new stereographic parameters $\underset{\sim}{\zeta}^S$ correspond to the mirror image of the Euler parameters and are generally not numerically equal to $\underline{\zeta}$ of Eq. (4). However, the resulting vector $\underline{\zeta}^S$ will describe the same orientation as the original parameters and are herein referred to as the “shadow points” of $\underline{\zeta}$ and are denoted with a superscript S:

$$\zeta_i^S = \frac{-\beta_i}{-\beta_0 - a} = \frac{\beta_i}{\beta_0 + a} \tag{7}$$

Using Eq. (6) the shadow point $\underline{\zeta}^S$ can be expressed directly as a transformation of the original parameters $\underline{\zeta}$ and the projection point a as:

$$\zeta_i^S = \zeta_i \left[\frac{-a + \sqrt{1 + \underline{\zeta}^T \underline{\zeta}(1 - a^2)}}{a + 2a\underline{\zeta}^T \underline{\zeta} + \sqrt{1 + \underline{\zeta}^T \underline{\zeta}(1 - a^2)}} \right] \tag{8}$$

The fact that the shadow point vector $\underline{\zeta}^S$ generally has a different behavior than the original $\underline{\zeta}$ will be useful when describing a rotation. The differential kinematic equations for $\underline{\zeta}$, by differentiating Eq. (4), are found to be

$$\dot{\underline{\zeta}} = \frac{\dot{\beta}_i}{\beta_0 - a} - \frac{\beta_i \dot{\beta}_0}{(\beta_0 - a)^2} \tag{9}$$

By making use of the differential kinematic equations of the Euler parameters[6]

$$\begin{bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{10}$$

and the definition of the stereographic parameters given in Eq. (4), the differential kinematic equations for the stereographic parameters can be found. Their general form is very lengthy and not shown here due to space limitations. The most important special cases are discussed below.

Viewing Figure 1, it becomes evident that a set of three symmetric stereographic parameters cannot have the singularity point moved beyond a principal rotation of ±360°. Going beyond ±360° would mean finding a projection point that would map the entire unit sphere more than once, i.e. not a one-to-one map onto the projection plane. Therefore the symmetric parameters are better suited for regulator or “moderately large” departure motion problems, than for spinning body or large angle maneuver cases.

Note that for the zero principal rotation, the asymmetric stereographic parameters are not equal to zero. The projection of the β_0 parameter onto $\beta_i = a + 1$ is not zero because β_0 is one at the zero principal rotation.

Asymmetric stereographic parameters have a qualitatively different singular behavior from the symmetric stereographic parameters. The Euler parameter β_0 contains information about the principal rotation angle only (i.e., the direction of $\underline{e}$ does not affect β_0). Eliminating β_0 during a symmetric projection causes the singularity to appear at a certain principal rotation angle only, independent from the principal axis of rotation $\underline{e}$. Consequently symmetric projections have a symmetric range of nonsingular principal rotations $\{-\Phi_S < \Phi < +\Phi_S\}$ about the zero rotation, regardless of the direction of $\underset{\sim}{e}$.

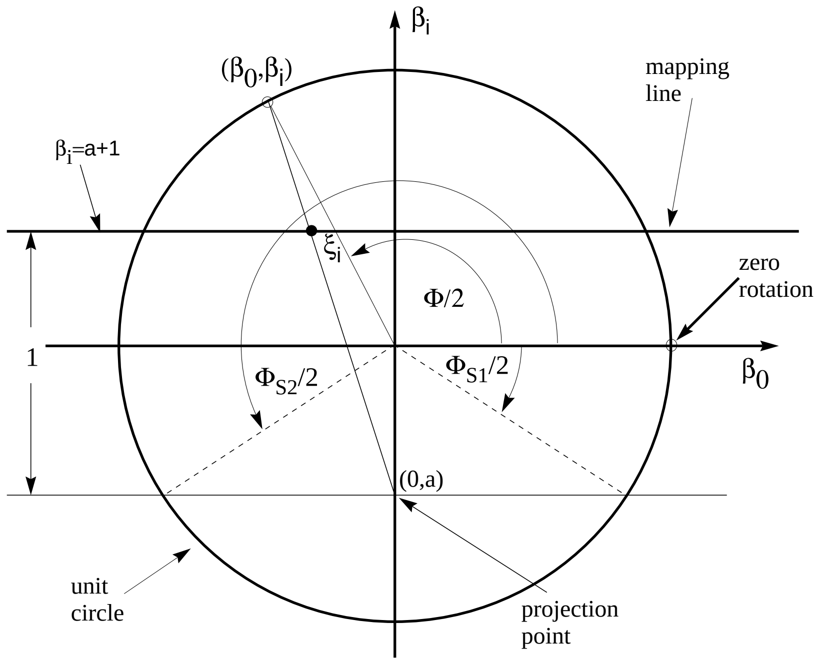

Figure 2: Illustration of a Asymmetric Stereographic Projection onto Hyperplane Orthogonal to β_i axis.

For an asymmetric projection, one of the Euler parameters β_1, β_2, or β_3 is eliminated. Each one of these parameters contains information about both the principal rotation angle and the direction of $\underset{\sim}{e}$. Therefore singularities will only occur at certain angles about the i-th axis (corresponding to β_i). Figure 2 illustrates an asymmetric stereographic projection where β_i is eliminated. All possible projections points *a* now lie on the β_i axis, and the mapping hyperplane perpendicular to β_i is defined at $\beta_i = a+1$. Since the zero rotation is no longer in the center of the nonsingular principal angle range, the valid range of principal angles is non-symmetric. A singularity will occur at Φ_{S1} or Φ_{S2}, where these two principal angles are unequal in magnitude. Given a singular principal rotation angle Φ_{S1} which lies between ±180°, the corresponding projection point *a* is defined as:

$$a = \cos\frac{\Phi_{S1}}{2} \tag{11}$$

The second singular principal rotation angle Φ_{S2} is then found as:

$$\Phi_{S2} = 2\pi - \Phi_{S1} \tag{12}$$

The transformation from Euler parameters to the corresponding asymmetric stereographic parameters is the same as given in Eq. (4), with β_0 and β_i switched. A singularity now occurs when β_i equals *a*. If the projection point *a* lies inside the four-dimensional unit sphere, this may occur at several orientations.

$$e_i \cdot \sin\frac{\Phi}{2} = a \qquad (13)$$

Using Eq. (1), the condition for a singularity becomes Eq. (13), where the index *i* stands for the β_i parameter which was eliminated. Since the *sin* function is bounded between ±1, a singularity will *never* occur if $|e_i| < a$. If the projection point *a* is moved to the sphere surface, namely to ±1, then a singularity may occur with a rotation about the *i-th* body axis only! The reason for this is evident in Eq. (12). Since *a* is ±1 and the *sin* function is bounded within ±1, the only way Eq. (13) is satisfied is if $|e_i| = 1$. Because $\underline{e}$ is a unit vector, the other two direction components must be zero if $|e_i| = 1$. Thus if the body is spinning about an axis other than the i-th body axis, a singularity will never occur. Therefore these asymmetric stereographic parameters are attractive for spinning body problems, where an object is rotating mainly about a certain axis. The principal rotation angle is now *not bounded* as with the symmetric stereographic parameters, but can grow beyond ±360°. Simply choose the normal of the projection hyperplane to be far removed from the rotation axis and place the projection point *a* on the four-dimensional unit sphere surface and the probability of encountering a singularity is virtually nil.

For both the symmetric and asymmetric stereographic parameters, having the projection point on the sphere surface means the singularity can only occur at two distinct orientations. If the projection point lies inside the sphere, there generally exists an infinite set of possible singular orientations.

The inverse transformation from asymmetric stereographic parameters to Euler parameters is the same as given in Eq. (6). These asymmetric parameters also exhibit the same shadow point behavior as the symmetric parameters do with the same transformation given in Eq. (8). Therefore, if a singular orientation is approached with the asymmetric stereographic parameters, one can switch to the shadow point to avoid the singularity.

CLASSICAL RODRIGUES PARAMETERS

The Rodrigues parameters $\underline{q}$ have a singularity at $\Phi = \pm 180°$. This corresponds to a point on the two-dimensional unit circle in Figure 1 of (0,±1). The corresponding symmetric stereographic projection has the projection point *a* at the origin and the mapping line at $\beta_0 = 1$. It becomes evident why the classical Rodrigues parameters must go singular at $\Phi = \pm 180°$ when describing them as a special case of the symmetric stereographic parameters. The transformation from the Euler parameters to the Rodrigues parameters $\underline{q}$ is found by setting $\Phi_S = \pm 180°$ in Eqs. (3-4). The well known result is shown in Eq. (14) below.

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3 \qquad (14)$$

The inverse transformation from the Rodrigues to the Euler parameters is found by using the same Φ_S in Eq. (6) and is given as:

$$\beta_0 = \frac{1}{\sqrt{1 + \underline{q}^T \underline{q}}} \qquad \beta_i = \frac{q_i}{\sqrt{1 + \underline{q}^T \underline{q}}} \qquad i = 1, 2, 3 \qquad (15)$$

The differential kinematic equation in terms of the classical Rodrigues parameters is given in vector form as:

$$\dot{\underline{q}} = \frac{1}{2}[I + [\tilde{q}] + \underline{q}\underline{q}^T]\,\underline{\omega} \tag{16}$$

An explicit matrix form of Eq. (16) is given below[5].

$$\dot{\underline{q}} = \frac{1}{2}\begin{bmatrix} 1+q_1^2 & q_1q_2 - q_3 & q_1q_3 + q_2 \\ q_2q_1 + q_3 & 1+q_2^2 & q_2q_3 - q_1 \\ q_3q_1 - q_2 & q_3q_2 + q_1 & 1+q_3^2 \end{bmatrix}\begin{bmatrix}\omega_1 \\ \omega_2 \\ \omega_3\end{bmatrix} \tag{17}$$

Using the definitions of the Euler parameters in Eq. (1), the Rodrigues parameters can also be expressed directly in terms of the principal rotation angle Φ and the principal line of rotation $\underline{e}$.

$$\underline{q} = \underline{e}\tan\frac{\Phi}{2} \tag{18}$$

From Eq. (18), it is obvious why the classical Rodrigues parameters go singular at ±180°. For completeness the direction cosine matrix C is given in explicit matrix form[5]:

$$C(q_1, q_2, q_3) = \frac{1}{1+q_1^2+q_2^2+q_3^2}\begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3) & 2(q_1q_3-q_2) \\ 2(q_2q_1-q_3) & 1-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1) \\ 2(q_3q_1+q_2) & 2(q_3q_2-q_1) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{19}$$

and in vector form[3]:

$$C(\underline{q}) = \frac{1}{1+\underline{q}^T\underline{q}}\left((1-\underline{q}^T\underline{q})I + 2\underline{q}\underline{q}^T - 2[\tilde{q}]\right) \tag{20}$$

Eq. (20) and its inverse can also be written as the Cayley Transform[3,5,7]:

$$C(\underline{q}) = (I - [\tilde{q}])(I + [\tilde{q}])^{-1} \tag{21a}$$

$$[\tilde{q}] = (I - C)(I + C)^{-1} \tag{21b}$$

and the kinematic differential equation shown in Eqs. (16-17) has the "Cayley" form[5]:

$$\frac{d}{dt}[\tilde{q}] = \frac{1}{2}(I - [\tilde{q}])[\tilde{\omega}](I - [\tilde{q}]) \tag{22}$$

The tilde matrix $[\tilde{q}]$ is defined by $-[\underline{q}\times\ldots]$ as given in Eq. (23).

$$[\tilde{q}] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{23}$$

Let the vector $\underline{q}^S$ (defined with -$\underline{\beta}$) denote the shadow point of the classical Rodrigues parameters. Solving Eq. (6), or starting with Eq. (14), the following definition for the $\underline{q}^S$ is found.

$$q_i^S = \frac{-\beta_i}{-\beta_0} = \frac{\beta_i}{\beta_0} = q_i \qquad i = 1, 2, 3 \tag{24}$$

Eq. (24) shows that for the Rodrigues parameters, the shadow point vector components are identical to the original Rodrigues parameters, with identical values and properties. Therefore the shadow point concept is of no practical consequence in this case; the classical Rodrigues parameters are unique!

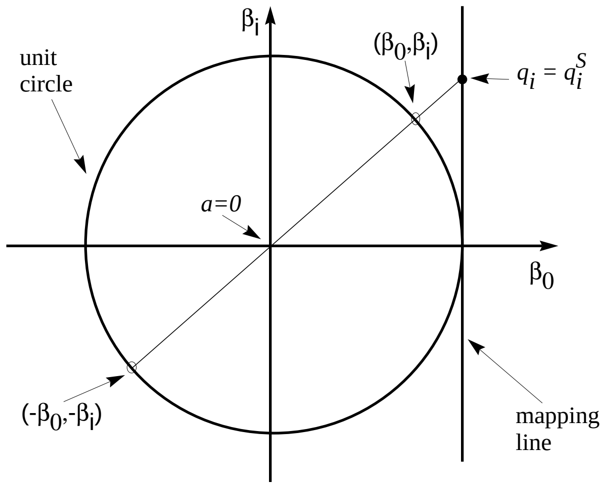

Figure 3: Original and “Shadow Point” Projection of the Classical Rodrigues Parameters.

Having the projection point *a* at the origin causes this elegant, degenerate phenomenon. Figure 3 illustrates why both sets of Rodrigues parameters are identical. The classical Rodrigues parameters are the only symmetric stereographic parameters which exhibit this lack of distinction between the original parameters and their shadow point counterparts. This proves simultaneously to be an advantage and a disadvantage.

MODIFIED RODRIGUES PARAMETERS

The modified Rodrigues parameters presented by Marandi and Modi, and Tsiotras move the projection point to the far left of the unit sphere at (-1,0,0,0) and project the Euler parameters onto the hyperplane at $\beta_0 = 0$. This pushes the singularity as far away from the zero-rotation as possible. The parameters will now go singular at $\Phi = \pm 360°$. This set of parameters is able to describe any type of rotation except a complete revolution back to its original orientation. Carrying out the stereographic projection with $\Phi_S = \pm 360°$, the transformation from Euler parameters to the modified Rodrigues parameter vector $\underline{\sigma}$ and the inverse transformation are given as[2]:

$$\sigma_i = \frac{\beta_i}{1 + \beta_0} \qquad i = 1, 2, 3 \tag{25}$$

$$\beta_0 = \frac{1 - \underline{\sigma}^T\underline{\sigma}}{1 + \underline{\sigma}^T\underline{\sigma}} \qquad \beta_i = \frac{2\sigma_i}{1 + \underline{\sigma}^T\underline{\sigma}} \qquad i = 1, 2, 3 \tag{26}$$

Using Eq. (1) again, the modified Rodrigues parameters can be written as[2]:

$$\underline{\sigma} = \underline{e}\tan\frac{\Phi}{4} \tag{27}$$

This formula immediately reveals the singularity at a principal rotation of ±360°, double the range of the classical Rodrigues parameters. It is interesting that Φ = 0° and Φ = ±360° correspond physically to the same body orientation. This fact has both theoretical and practical consequences in "avoiding" the singularity.

$$\dot{\underline{\sigma}} = \frac{1}{2}\left[I\left(\frac{1-\underline{\sigma}^T\underline{\sigma}}{2}\right) + [\tilde{\sigma}] + \underline{\sigma}\underline{\sigma}^T\right]\underline{\omega} \tag{28}$$

The kinematic differential equations in terms of $\underline{\sigma}$ are given in Eq. (28). They are very similar to Eq. (16) except for one extra term. This terms makes the equations only slightly more complicated, but not any more non-linear.

The explicit matrix form for the elements of Eq. (28) is given as[2]:

$$\dot{\underline{\sigma}} = \frac{1}{4}\begin{bmatrix} (1+\sigma_1^2-\sigma_2^2-\sigma_3^2) & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_3+\sigma_2) \\ 2(\sigma_2\sigma_1+\sigma_3) & (1-\sigma_1^2+\sigma_2^2-\sigma_3^2) & 2(\sigma_2\sigma_3-\sigma_1) \\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & (1-\sigma_1^2-\sigma_2^2+\sigma_3^2) \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{29}$$

The direction cosine matrix in terms of the modified Rodrigues parameters[2] can be shown to be:

$$C(\underline{\sigma}) = \frac{1}{(1+\underline{\sigma}^T\underline{\sigma})^2}\begin{bmatrix} 4(\sigma_1^2-\sigma_2^2-\sigma_3^2)+\Sigma^2 & 8\sigma_1\sigma_2+4\sigma_3\Sigma & 8\sigma_1\sigma_3-4\sigma_2\Sigma \\ 8\sigma_2\sigma_1-4\sigma_3\Sigma & 4(-\sigma_1^2+\sigma_2^2-\sigma_3^2)+\Sigma^2 & 8\sigma_2\sigma_3+4\sigma_1\Sigma \\ 8\sigma_3\sigma_1+4\sigma_2\Sigma & 8\sigma_3\sigma_2-4\sigma_1\Sigma & 4(-\sigma_1^2-\sigma_2^2+\sigma_3^2)+\Sigma^2 \end{bmatrix} \tag{30}$$

$$\Sigma = 1-\underline{\sigma}^T\underline{\sigma}$$

or more compactly in vector form as[3]:

$$C(\underline{\sigma}) = I - \frac{4(1-\underline{\sigma}^T\underline{\sigma})}{(1+\underline{\sigma}^T\underline{\sigma})^2}[\tilde{\sigma}] + \frac{8}{(1+\underline{\sigma}^T\underline{\sigma})^2}[\tilde{\sigma}]^2 \tag{31}$$

The modified Rodrigues parameter vector $\underline{\sigma}$ is transformed into classical Rodrigues parameters as:

$$\underline{q} = \left(\frac{2}{1-\underline{\sigma}^T\underline{\sigma}}\right)\underline{\sigma} \tag{32}$$

Naturally, this transformation goes singular at a principal rotation of ±180°, because $\|\underline{\sigma}\| \to 1$ and $\|\underline{q}\| \to \infty$ as $\Phi \to \pm 180°$.

Comparing Eq. (27) and Eq. (18) it is immediately evident that both the classical and the modified Rodrigues parameter vectors have the direction of the principal rotation vector $\underline{e}$, but a different magnitude. The transformation from modified to classical Rodrigues parameters shown in Eq. (32) can be rewritten in terms of the principal angle of rotation Φ.

$$\underline{q} = \frac{\tan\frac{\Phi}{2}}{\tan\frac{\Phi}{4}}\underline{\sigma} \tag{33}$$

Using the image set $-\underline{\beta}(t)$ of Euler parameters, the shadow point of the modified Rodrigues parameter vector $\underline{\sigma}$ is found[1,4].

$$\sigma_i^S = \frac{-\beta_i}{1-\beta_0} = \frac{-\sigma_i}{\underline{\sigma}^T\underline{\sigma}} \tag{34}$$

Contrary to the classical Rodrigues parameters, these modified Rodrigues parameter shadow points are not numerically equal to the original parameters. While they generate exactly the same direction cosine matrix, they are not generally a mirror image of one another. While generally $\underline{\sigma}^S \neq -\underline{\sigma}$, note that everywhere on the unit sphere $\underline{\sigma}^T\underline{\sigma} = 1$ that, in fact, $\underline{\sigma}^S = -\underline{\sigma} = -\beta_i$. This simple observation has significant practical consequences.

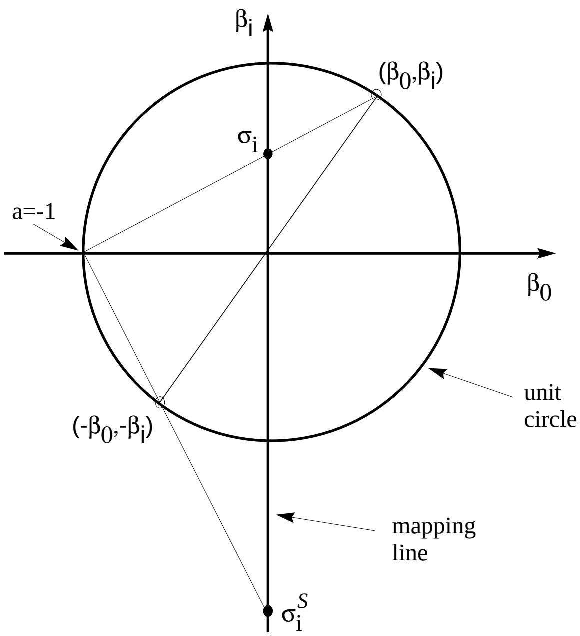

Figure 4: Original and "Shadow Point" Projection of the Modified Rodrigues Parameters.

The shadow points $\underline{\sigma}^S$ have some interesting properties. They go singular at the zero rotation and go to zero at a ±360° principal rotation! This is the exact opposite of the qualitative behavior of $\underline{\sigma}$. The reason for this behavior becomes evident in Figure 4. At a zero rotation, the shadow point will intersect the mapping line at infinity. At a rotation of ±180° the shadow points will be the negative of their original values. We note that $\underline{\sigma}^S$ is distinguished from $\underline{\sigma}$ merely for book-keeping purposes. Transforming initial conditions (from $[C]$ or $\underline{\beta}$) for any given case, could initiate motion on either $\underline{\sigma}(t)$ or $\underline{\sigma}^S(t)$.

When the attitude parameters switch to the "shadow" set, their derivatives naturally switch to. Let $\sigma^2 = (\underline{\sigma}^T\underline{\sigma})$, then their relationship is given by

$$\frac{d}{dt}\underline{\sigma}^S = \frac{1}{\sigma^4}\left(\frac{d}{dt}\underline{\sigma} - \frac{1}{2}(1+\sigma^2)\,[\tilde{\sigma}]\,\underline{\omega} - (1-\sigma^4)\frac{\underline{\omega}}{4}\right) \tag{35}$$

Using $\underline{\sigma}$ together with the shadow vector $\underline{\sigma}^S$, it is possible to describe *any* rotation without singularities and with only three parameters, but with one discontinuity at the switching point. If the original $\underline{\sigma}(t)$ trajectory approaches the singularity at $\Phi = \pm 360°$, the vector $\underline{\sigma}(t)$ can be switched to the shadow trajectory $\underline{\sigma}^S(t)$. This transformation is very simple as is seen in Eq. (34). Rather than waiting until $|\underline{\sigma}(t)| \to \infty$ or $|\underline{\sigma}^S(t)| \to \infty$ to switch, however, the most convenient switching surface is the $\underline{\sigma}^T\underline{\sigma} = 1$ sphere; the unit sphere which corresponds to a principal rotation of ±180°. The Euler parameter β_0 is zero everywhere on this sphere. This causes the shadow point to have the same unit magnitude as the original with the transformation being $\underline{\sigma}^S = -\underline{\sigma}$. Thus whenever $\underline{\sigma}(t)$ exits (enters) the unit sphere, $\underline{\sigma}^S(t)$ enters (exits) at the opposite side of the sphere.

Switching at the $\underline{\sigma}^T\underline{\sigma} = 1$ surface can be very elegantly accomplished when finding $\underline{\sigma}$ by extracting the Euler parameters from the direction cosine matrix. Simply keep $\beta_0 \geq 0$ and the resulting set of parameters will always have $\underline{\sigma}^T\underline{\sigma} \leq 1$ (Ref. 1). Switching on the $\beta_0 = 0$ sphere (where $\underline{\sigma}^T\underline{\sigma} = \underline{\sigma}^{S^T}\underline{\sigma}^S = 1$) keeps the combined set of original and shadow points bounded within the unit sphere.

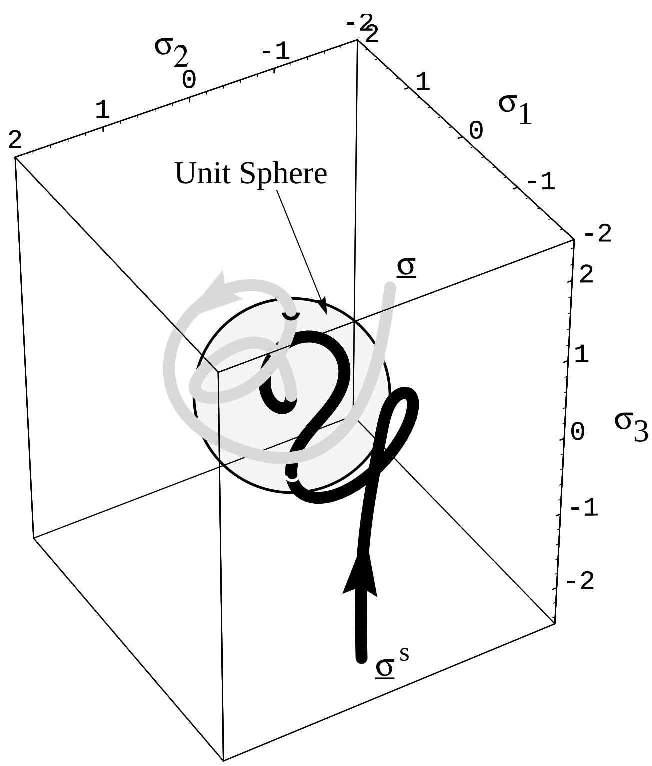

Figure 5: Illustration of the Original and Shadow Modified Rodrigues Parameter.

This bounded behavior of the combined set is illustrated in Figure 5 above. The grey line represents the $\underline{\sigma}(t)$ trajectory and the black line the corresponding shadow trajectory of $\underline{\sigma}^S(t)$. The motion starts out at a zero rotation with the grey line at the origin and the black line at infinity. After a while the principal angle of the object grows beyond 180° and the grey line exits the unit sphere. At the same time the shadow parameters (black line) enter the sphere at the opposite position. If the body rotates back to the original orientation, the shadow parameters approach zero as the original parameters go off to infinity. Any tumbling motion would give rise to a qualitatively identical discussion of $\underline{\sigma}(t)$ and $\underline{\sigma}^S(t)$.

EXAMPLE OF ASYMMETRIC STEREOGRAPHIC PARAMETERS

A sample set of asymmetric stereographic parameter vector $\underline{\eta}$ is constructed by eliminating the Euler parameter β_1 and setting *a* equal to -1. Adjusting Eq. (4), the vector $\underline{\eta}$ is defined as:

$$\eta_1 = \frac{\beta_0}{\beta_1 + 1} \qquad \eta_2 = \frac{\beta_2}{\beta_1 + 1} \qquad \eta_3 = \frac{\beta_3}{\beta_1 + 1} \tag{36}$$

Using Eqs. (11) and (12) the singular principal rotations about the positive β_1 axis become Φ_{S1} = -180° and Φ_{S1} =+540°. As mentioned earlier, the direction at which a singular orientation is approached is important with asymmetric stereographic parameters. Here a negative principal rotation of 180° about the first body axis causes a singularity. A positive principal rotation of 180° would yield an identical physical position, yet causes no singularity. Only after a +540° does this representation go singular, even though this position is the same as +180°. This non-symmetric principal angle range is due to the fact that the zero rotation point (±1,0,0,0) does not lie on the β_1 axis. Naturally the singularities could always be avoided by switching the $\underline{\eta}$ vector to its shadow set through

$$\underline{\eta}^S = -\frac{\underline{\eta}}{\underline{\eta}^T\underline{\eta}} \tag{37}$$

Differentiating Eq. (36) and using Eq. (10), the differential kinematic equation for vector $\underline{\eta}$ is found to be:

$$\dot{\underline{\eta}} = \frac{1}{4}\begin{bmatrix} (-1-\eta_1^2+\eta_2^2+\eta_3^2) & 2(\eta_1\eta_3-\eta_2) & -2(\eta_1\eta_2+\eta_3) \\ 2(\eta_3-\eta_1\eta_2) & 2(\eta_2\eta_3+\eta_1) & (-1+\eta_1^2-\eta_2^2+\eta_3^2) \\ -2(\eta_1\eta_3+\eta_2) & (1-\eta_1^2-\eta_2^2+\eta_3^2) & 2(\eta_1-\eta_2\eta_3) \end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} \tag{38}$$

Note that Eq. (38) contains no transcendental functions in it and is similar qualitatively to Eq. (29). Because $\underline{\eta}$ is an asymmetric stereographic parameter vector, however, there is less symmetry in the matrix. This lack of symmetry is linked with the absence of a symmetric principal rotation angle range. Therefore, Eq. (38) cannot be written in a more compact vector as was the case with the symmetric stereographic parameters.

The direction cosine matrix in terms of $\underline{\eta}$ can be found to be:

$$C(\underline{\eta}) = \frac{1}{(1+\underline{\eta}^T\underline{\eta})^2}\begin{bmatrix} 4(\eta_1^2-\eta_2^2-\eta_3^2)+\Sigma^2 & 8\eta_1\eta_3+4\eta_2\Sigma & -8\eta_1\eta_2+4\eta_3\Sigma \\ -8\eta_1\eta_3+4\eta_2\Sigma & 4(\eta_1^2+\eta_2^2-\eta_3^2)-\Sigma^2 & 8\eta_2\eta_3+4\eta_1\Sigma \\ 8\eta_1\eta_2+4\eta_3\Sigma & 8\eta_2\eta_3-4\eta_1\Sigma & 4(\eta_1^2-\eta_2^2+\eta_3^2)-\Sigma^2 \end{bmatrix} \tag{39}$$

$$\Sigma = 1-\underline{\eta}^T\underline{\eta}$$

Analogously, asymmetric stereographic parameters could be derived by projecting onto a hyperplane orthogonal to the β_2 or β_3 axis, or actually any non-β_0 axis. All these parameters would have a similar singular behavior.

To illustrate the use of the asymmetric stereographic parameters $\underline{\eta}$ for describing a spinning body, a sample motion was generated. The motion was achieved by forcing the following 3-1-3 Euler angle time history upon the body.

$$\theta_1(t) = t \qquad \theta_2(t) = (1-\cos 2t)\frac{\pi}{2} \qquad \theta_3(t) = (\sin 2t)\frac{\pi}{4} \tag{40}$$

The body is mainly spinning about the third body axis while oscillating about the other two. Therefore the stereographic parameter vector $\underline{\eta}$ will never go singular, since a singularity can only occur with a pure rotation about the first body axis.

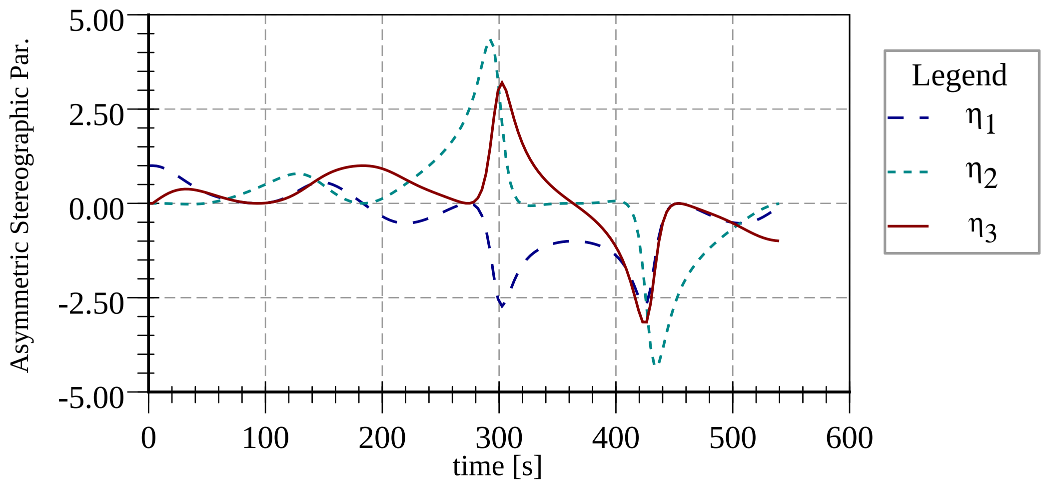

Figure 6: Spinning Body Description with Asymmetric Stereographic Parameters.

As Figure 6 shows, the asymmetric stereographic parameters $\underline{\eta}$ are smooth and continuous at all time. The sample motion performs 1.5 revolutions without encountering any singularity.

To compare the asymmetric with the symmetric stereographic parameter description for this spinning body, the polar plot in Figure 7 was generated. The magnitude of each parameter vector is plotted versus the principal rotation angle ϕ. As expected, the symmetric stereographic parameters go singular at certain ϕ, namely $\pm 180°$ for the classical Rodrigues parameters and $\pm 360°$ for the modified Rodrigues parameters. On the other hand, the asymmetric parameter vector $\underline{\eta}$ remains bounded at all times.

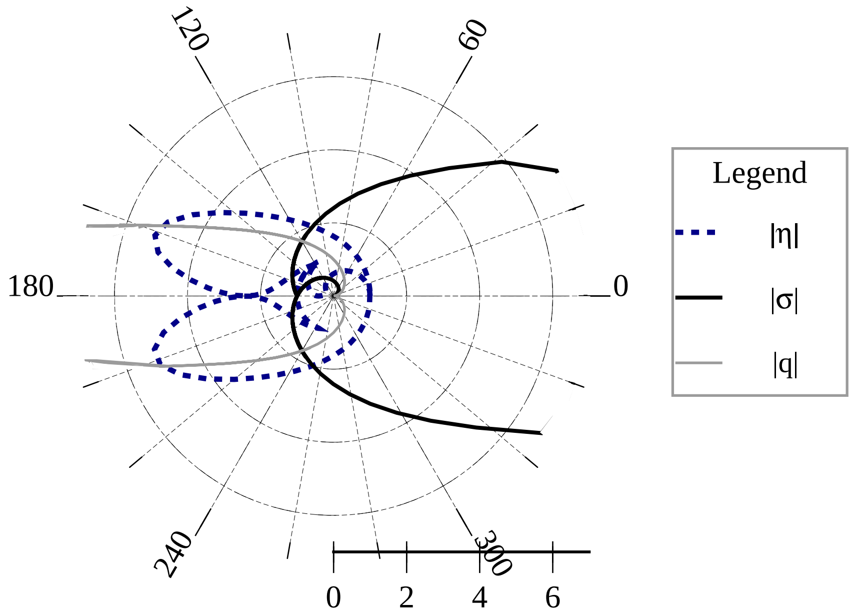

Figure 7: Comparison of Symmetric and Asymmetric Stereographic Parameters.

Figure 8 shows the time history of the principal rotation angle ϕ for this spinning body maneuver. Because of the oscillations about the first and second body axis, ϕ gets reduced during some portions of the maneuver. Because the magnitude of the symmetric stereographic parameters depends only on the principal rotation angle, these "backing up" phases are not visible on the polar plot in Figure 7. However, the magnitude of the asymmetric stereographic parameters depends on both the principal rotation angle and the direction of the principal rotation axis. This explains the more irregular features of the $|\underline{\eta}|$ plot in Figure 7.

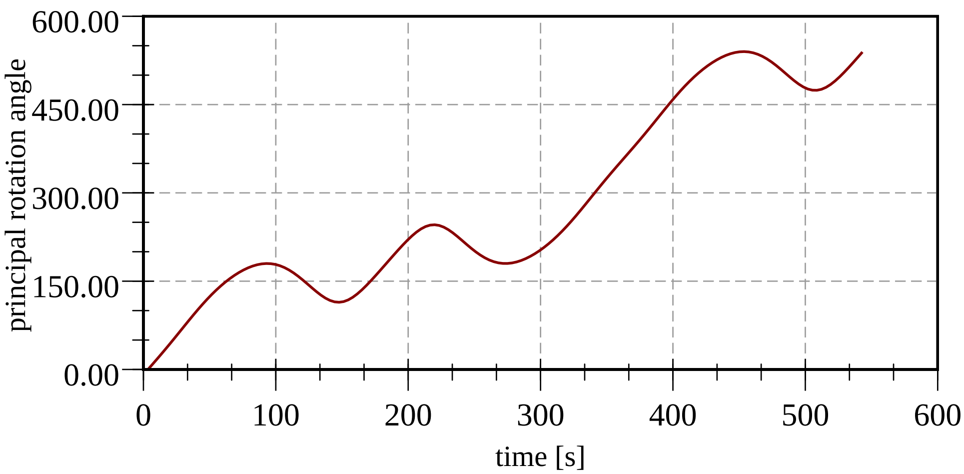

Figure 8: Principal Rotation Angle Time History of Spinning Body Maneuver.

While some loss in symmetry and elegance of the equations results, asymmetric sets of stereographic parameters are able to represent the motion of a spinning body without switching between the shadow and the original parameters, like the modified Rodrigues parameters would require.

GLOBALLY STABLE CONTROL USING MODIFIED RODRIGUES PARAMETERS

The combined set of modified Rodrigues parameters and their shadow counterparts lend themselves very well for regulator type control design. Adopting the switching surface $\underline{\sigma}^T\underline{\sigma} = 1$ has a surprising benefit in designing control laws. Consider the dynamics of a generally tumbling rigid body. The Lyapunov function[2]

$$V(\underline{\omega}, \underline{\sigma}) = \frac{1}{2}\underline{\omega}^T J\underline{\omega} + 2K\log(1 + \underline{\sigma}^T\underline{\sigma}) \tag{41}$$

will not have any discontinuities at the switching surface, since both the original $\underline{\sigma}$ and its shadow $\underline{\sigma}^S$ point have unit magnitude there! $V(\underline{\omega}, \underline{\sigma})$ is by inspection only zero if both $\underline{\omega}$ and $\underline{\sigma}$ are zero. As a consequence, it is easy to establish a globally stable Lyapunov controller with a three rotation parameter set which never encounters a singularity! J in Eq. (41) denotes the 3x3 inertia matrix in body axis. The scalar K is a positive feedback gain. For this nonlinear regulator type problem, the external control torque $\underline{u}$ is found by setting the time derivative of Eq. (41) equal to

$$\dot{V} = -\underline{\omega}^T P\underline{\omega} \tag{42}$$

with P being a positive definite matrix, and using Eq. (28) and Euler's equation of motion:

$$J\dot{\underline{\omega}} = -[\tilde{\underline{\omega}}]J\underline{\omega} + \underline{u} \tag{43}$$

to solve for the torque $\underline{u}$. Using the logarithm of $\underline{\sigma}^T\underline{\sigma}$ in Eq. (41) results in a globally stabilizing feedback control law for the torque $\underline{u}$ which is *linear* in both $\underline{\omega}$ and $\underline{\sigma}$ (Ref. 2,5).

$$\underline{u} = -P\underline{\omega} - K\underline{\sigma} \tag{44}$$

The control law in Eq. (44) is valid for any arbitrary departure motion $\underline{\sigma}$. Conventional sets of three parameters would encounter singular orientations. Another problem with conventional parameter sets is that they have no inherent mechanism to accommodate tumbling situations when the object has performed a principal rotation beyond ±180° away from the desired state. When this happens, it would probably be desirable to "help" the object complete the revolution, rather than to attempt to force it back the way it came. The only set of parameters that can "almost" handle this scenario is the classical set of Rodrigues parameters. They fail because they go singular near the "up-side-down" orientation at $\Phi=\pm180°$. The combined set of $\underline{\sigma}$ and $\underline{\sigma}^S$, however, are well behaved up to and well beyond $\Phi = \pm180°$. Since $\underline{\sigma}(t)$ and $\underline{\sigma}^S(t)$ satisfy exactly the same differential equation Eq. (29), it is obvious that switching to the incoming "shadow trajectory" using the transformation of Eq. (34) [i.e., upon encountering $\underline{\sigma}^T\underline{\sigma} > 1$] can be accomplished easily with little programming or computational cost. Switching at $\underline{\sigma}^T\underline{\sigma} = 1$ makes it possible for the control law to let the object go past the "up-side-down" orientation and then let it rotate back to the origin the short way, as we illustrate in an example below.

The angular velocity $\underline{\omega}$ feedback is required for global stability, and the P matrix should be chosen to achieve satisfactory damping of the nonlinear oscillations.

The results of a single-axis spin maneuver using the control law in Eq. (44) are presented. The inertia J used was 12000 kgm^2; the feedback gains were chosen as K=300 and P=1800. Initial angular velocity was +60°/s. Figure 9 below shows the time history of the principal angle of rotation. The object clearly spins beyond the "up-side-down" point of Φ=+180° and then returns back to the origin by continuing the motion and completing the revolution. The $\underline{\omega}$ feedback sufficiently dampens the system to prevent excessive oscillations about the origin.

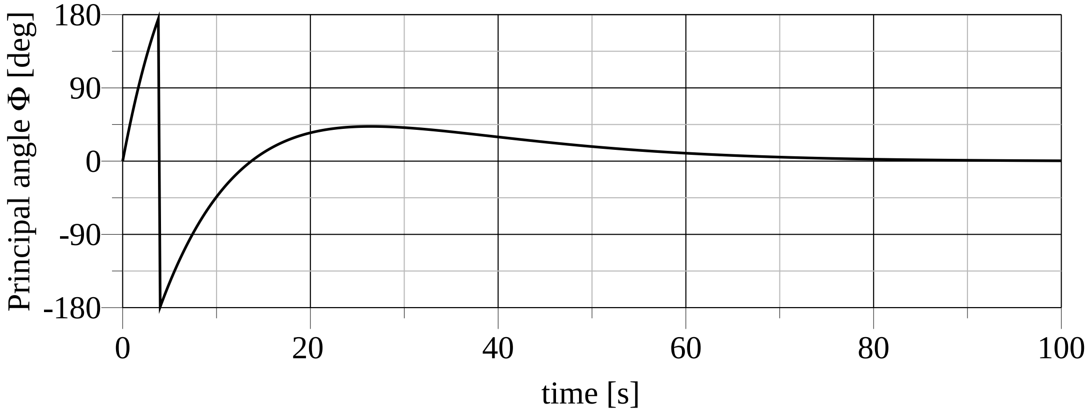

Figure 9: Principal Angle of Rotation of Spin Maneuver.

The angular velocity, shown in Figure 10, decreases steadily from +60°/s and converges to zero. Where the Φ goes beyond 180° there is a discontinuity in the slope of $\underline{\omega}$.

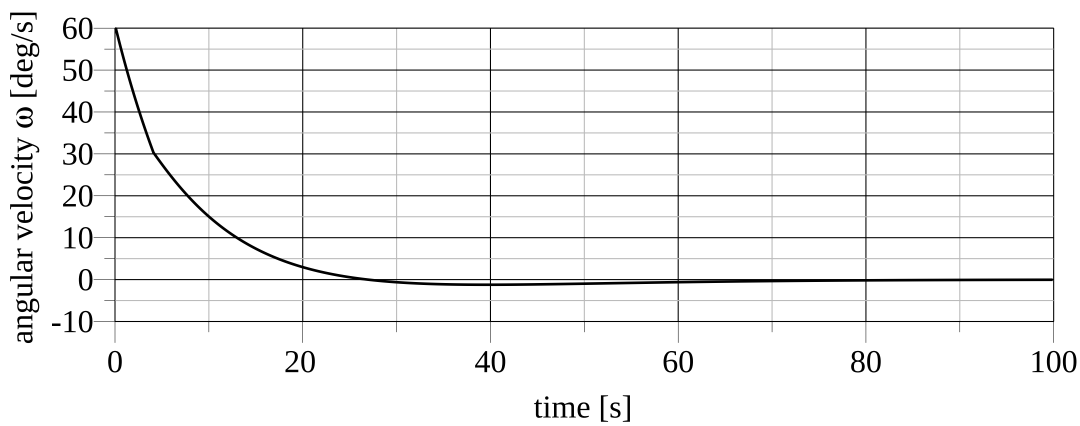

Figure 10: Angular Velocity of Spin Maneuver.

The corresponding external control torque is presented in Figure 11. A large torque is demanded initially because of the large initial angular velocity $\underline{\omega}$. As $\underline{\omega}$ decreases, so does the torque. There is a discontinuity where the modified Rodrigues parameter switch from the original to the shadow point trajectory. This is because the position error $\underline{\sigma}$ reversed its sign, driving the object towards the origin about the other way. However, the control torque does not jump to a negative value because of the $\underline{\omega}$ feedback. It keeps the torque positive; i.e. the controller is still slowing down the spin, even during the switching.

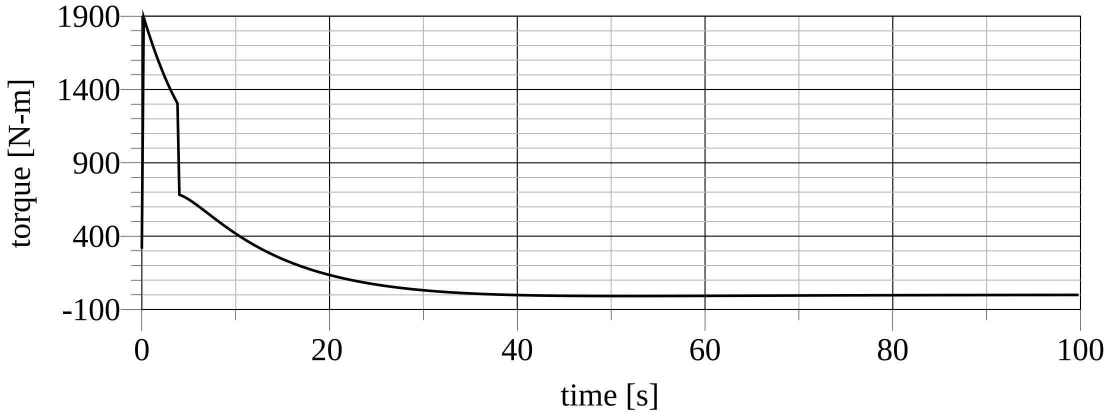

Figure 11: External Control Torque of Spin Maneuver.

The position error and the associated torque discontinuity due to switching to the shadow trajectory may be troublesome for highly flexible bodies. However, this is easily addressed in practice by replacing the instantaneous switch by a smooth one. Also, introducing a simple digital filter will effectively smooth out such jump discontinuities.

It is conceptually easy to introduce a reference trajectory and design analogous tracking-type feedback control with, using the methods of reference 5, global stability guaranteed. This is useful in achieving global control shaping, and also to permit selection of feedback gains sufficiently large to reject disturbances.

CONCLUSION

A new family of stereographic parameters has been presented including the general transformation from and to the Euler parameters. The general stereographic parameters are not unique and have a corresponding set of shadow point parameters whose singular behavior is different from the original parameters.

The classical Rodrigues parameters are a special set of the symmetric stereographic parameters where the original parameters and their shadow points coincide. The modified Rodrigues parameters are also a special case of the symmetric stereographic parameters. They have the largest non-singular principal angle range of ±360°. Their associated shadow points are singular at the zero rotation and zero and $\Phi = \pm 360°$. This combined set of stereographic parameters and their shadow point parameters are able to describe any rotation without encountering a singularity, but with one discontinuity.

The asymmetric stereographic parameters have their singular orientations defined both by an axis and a principal rotation angle. The two singular angles do not have equal magnitude as with the symmetric stereographic parameter. Asymmetric parameters do allow rotations beyond ±360° and are therefore attractive to spinning body type problems.

The globally stable control law presented implicitly "knows" when an object has rotated beyond ±180° from the target state, and to let it complete the revolution back to the desired state. This control implicitly seeks out the smallest principal rotation angle to the target state. This control law was developed by making use of the modified Rodrigues parameter and their shadow points.

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