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FEEDBACK CONTROL LAW USING THE EIGENFACTOR QUASI-COORDINATE VELOCITY VECTOR

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The use of the recently developed eigenfactor quasi-coordinate velocities (EQV) vectors in feedback control laws is examined. The equations of motion in these new coordinates do not require a mass matrix inverse to be taken and are ideally suited for massively parallel computation. It is shown that the Coriolis term of the EQV formulation does no mechanical work. This allows for simple velocity feedback laws which are globally asymptotically stable. The performance and convergence rate of the EQV feedback control law are compared to a traditional velocity feedback control law by using them to bring a three-link manipulator to rest. For a given maximum available control, the EQV feedback control law shows better performance than the traditional velocity feedback control law. The kinetic energy decays exponentially at an easily controllable rate. Further, numerical studies show that damping derived from an EQV feedback control law approximately decouples the nonlinear dynamics of a rigid multi-link system and brings each link to rest individually.

INTRODUCTION

Multi-body dynamical systems typically have configuration dependent mass matrices. Such dynamical systems include large nonlinear deformation models of arbitrary bodies, simple multi-body systems or multi-link robot arms. To obtain the instantaneous accelerations, these time varying mass matrices need to be inverted in some manner at each integration step. This process is computationally difficult, expensive and usually illsuited for use on massively parallel computer systems.

Mass matrix diagonalizing velocity coordinate transformations are introduced in References 1 and 2. Ref. 1 uses the innovations factorization approach to parameterize the mass matrix. One advantage of this method is that the mass matrix components can be found through a recursive algorithm. Ref. 2 parameterizes the mass matrix through a spectral decomposition. The mass matrix for a real mechanical system is guaranteed to be symmetric, positive definite. Therefore its eigenvector matrix is orthogonal and the eigenvalues are greater than zero. This fact is used to obtain simple first order differential equations whose solutions provide the eigenfactors. In essence, the problem of inverting a configuration variable dependent mass matrix is replaced with the problem of solving two auxiliary

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first order differential equations.[3]

Both methods in References 1 and 2 introduce a new quasi-coordinate velocity. The advantage of these new coordinates is a diagonalized dynamical system with no need of inverting any mass matrices. Further, describing the dynamics in terms of these new velocities provides for a natural and elegant splitting of momentum-level dynamics and kinematics. The classical Lagrange equations of motion are usually rather complicated second order differential equations. By introducing the new quasi-velocities the system is given by one first order differential equation describing the system momentum-level dynamics and by another first order differential equation describing the system kinematics. This structure is analogous to what is typically done in rigid body attitude dynamics. Let $\boldsymbol{\theta}$ be an attitude vector measured in some choice of Euler angles or similar coordinates. Then Euler's equations of motion are usually not written in terms of the $\dot{\boldsymbol{\theta}}$ and $\ddot{\boldsymbol{\theta}}$, but rather in terms of the body angular velocity and acceleration vectors $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$. This greatly simplifies the formulation and provides a convenient splitting of dynamics and kinematics.

Further, attitude control laws typically feed back the quasi-velocities $\boldsymbol{\omega}$ rather than $\dot{\boldsymbol{\theta}}$. This paper investigates the use of the analogous eigenfactor quasi-coordinate velocity (EQV) vectors presented in Ref. 2 as a feedback variable in multi-body control laws. In particular, the effect of the nonlinear generalized coriolis term is studied and a simple EQV feedback control law is presented.

PROBLEM FORMULATION

Let $\boldsymbol{x}$ be a configuration state vector. The classical Lagrange equations of motion are derived assuming expressions for the system kinetic energy $T(\boldsymbol{x}, \dot{\boldsymbol{x}}) = \frac{1}{2}\dot{\boldsymbol{x}}^T M(\boldsymbol{x})\dot{\boldsymbol{x}}$ and the potential energy $V(\boldsymbol{x})$ are available. Then the standard Lagrange equations of motion for an unconstrained natural system are

$$M(\boldsymbol{x})\ddot{\boldsymbol{x}} + \dot{M}(\boldsymbol{x}, \dot{\boldsymbol{x}})\dot{\boldsymbol{x}} - \frac{1}{2}\dot{\boldsymbol{x}}^T M_x(\boldsymbol{x})\dot{\boldsymbol{x}} + V_x = \boldsymbol{Q} \tag{1}$$

where the notation M_x and V_x symbolizes the partial derivative with respect to the configuration vector $\boldsymbol{x}$ and the expression $\dot{\boldsymbol{x}}^T M_{\boldsymbol{x}}(\boldsymbol{x}, t)\dot{\boldsymbol{x}}$ is the column vector $col\left(\dot{\boldsymbol{x}}^T \left[\frac{\partial M}{\partial x_i}\right] \dot{\boldsymbol{x}}\right)$. The vector $\boldsymbol{Q}$ is the generalized, nonconservative external force acting on the system. These second order differential equations are nontrivial to solve. In particular, the time and state dependence of the mass matrix poses a challenging difficulty, especially for systems of high dimensionality.

The system mass matrix $M(\boldsymbol{x})$ has the spectral decomposition

$$M = C^T DC \qquad C^T = [\boldsymbol{c}_1 \ldots \boldsymbol{c}_n] \qquad D = diag(\lambda_i) \tag{2}$$

where $\boldsymbol{c}_i$ is the i-th eigenvector. Since $M(\boldsymbol{x})$ is symmetric, positive definite for real mechanical systems, the eigenvector matrix C is orthogonal and all eigenvalues are positive. The diagonal matrix S is defined to be the positive square root of the eigenvalue matrix D.

$$D = S^T S \tag{3}$$

As proposed in Ref. 2, introducing the eigenfactor quasi-coordinate velocity vector $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = SC\dot{\boldsymbol{x}} \tag{4}$$

allows the kinetic energy expression to be simplified to

$$T = \frac{1}{2}\boldsymbol{\eta}^T\boldsymbol{\eta} \tag{5}$$

The equivalent mass matrix of the EQV formulation is the simple identity matrix. Analogous to the usual structure for the attitude dynamics kinematic differential equation $\dot{\boldsymbol{\theta}} = f(\boldsymbol{\theta})\boldsymbol{\omega}$, for the general dynamics problem, the corresponding kinematic equation becomes

$$\dot{\boldsymbol{x}} = C^T S^{-1}\boldsymbol{\eta} \tag{6}$$

Note that since the eigenvector matrix C is orthogonal and all eigenvalues are positive, the kinematic relationship between $\dot{\boldsymbol{x}}$ and $\boldsymbol{\eta}$ in Eq. (6) is defined singularity free. However, this formulation does require the instantaneous eigenfactors C and S of the mass matrix $M(\boldsymbol{x})$. As outlined in References 2 and 3, these are found by solving their corresponding first order differential equations. Since C is an orthogonal matrix, its differential equation is of the Poisson form[4–6]

$$\dot{C} = -[\Omega]C \tag{7}$$

where $[\Omega]$ is a skew-symmetric matrix. Each Ω_{ij} entry provides an eigenaxis angular velocity, analogous to the ω_i body angular velocity components in attitude dynamics.[3] Let the matrix μ be defined as

$$\mu = C\dot{M}C^T \tag{8}$$

The Ω_{ij} entries are then defined as[7,8]

$$[\Omega_{ij}] = \begin{cases} \frac{\mu_{ij}}{s_j^2 - s_i^2} & \text{for } |s_j^2 - s_i^2| \geq \epsilon \\ \Omega_{ij}(t_0) + \dot{\Omega}_{ij}(t_0)(t_1 - t_0) & \text{for } |s_j^2 - s_i^2| < \epsilon \end{cases} \tag{9}$$

If the eigenvalues λ_i and λ_j are within ϵ, then Ω_{ij} is approximated linearly. Reference 3 discusses this method for accommodating the rare case of crossing eigenvalues with minimal loss in accuracy. As it turns out, calculating these Ω_{ij} terms is the most costly operation performed in this formulation. However, the calculation of the $\mu_{ij} = \boldsymbol{c}_i^T\dot{M}\boldsymbol{c}_j$ term is ideally suited for massively parallel computer systems since each term can be calculated independent of other μ_{ij} terms. The time derivative of the eigenvalue square roots is given by[3,7–9]

$$\dot{S} = \frac{1}{2}\Gamma S^{-1} \tag{10}$$

where the matrix $\Gamma = diag(\mu_{ii})$. After solving the unconstrained Boltzmann-Hamel equations with the EQV vector $\boldsymbol{\eta}$ defined in Eq. (4), the system dynamics are given by[2,10,11]

$$\dot{\boldsymbol{\eta}} + S^{-1}\left([\Omega]S + \dot{S}\right)\boldsymbol{\eta} - S^{-1}C\left(\frac{1}{2}\dot{\boldsymbol{x}}^T\frac{\partial M}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}\right) = S^{-1}C(\boldsymbol{Q} - V_x) \tag{11}$$

At first glance Eq. (11) may appear more complicated than the classical Lagrange equations of motion. Note the most important feature that $\dot{\boldsymbol{\eta}}$ appears with an identity matrix coefficient. Also, note that S is a diagonal matrix with a trivial inverse which greatly simplifies the algebra. Let the vector $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{\eta})$ be the nonlinear generalized coriolis term of the new formulation defined as

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{\eta}) = S^{-1}\left([\Omega]S + \dot{S}\right)\boldsymbol{\eta} - S^{-1}C\left(\frac{1}{2}\dot{\boldsymbol{x}}^T\frac{\partial M}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}\right) \tag{12}$$

and let $\boldsymbol{\epsilon}$ be the new generalized force vector

$$\boldsymbol{\epsilon} = S^{-1}C(\boldsymbol{Q} - V_x) \tag{13}$$

then the dynamical system can be written in the compact form

$$\dot{\boldsymbol{\eta}} = -\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{\eta}) + \boldsymbol{\epsilon} \tag{14}$$

Thus the new formulation replaces the original Lagrange equations of motion, which is a second order differential equation, with two first order differential equations given in Eqs. (6) and (14).

EIGENFACTOR QUASI-VELOCITY COORDINATE FEEDBACK LAW

Because of the close relationship between this EQV formulation and the classical rigid body attitude dynamics formulation, the question arises if an EQV vector could have an important role in feedback laws as the $\boldsymbol{\omega}$ vector enjoys in attitude control. The Euler equation of motion of a rigid body is given as

$$I\dot{\boldsymbol{\omega}} = -[\tilde{\omega}]I\boldsymbol{\omega} + \boldsymbol{u} \tag{15}$$

where I is the inertia matrix, $\boldsymbol{u}$ is the external torque vector and the tilde matrix $[\tilde{\omega}]$ is defined as

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{16}$$

The kinetic energy of a rotating rigid body is given as

$$T = \frac{1}{2}\boldsymbol{\omega}^T I \boldsymbol{\omega} \tag{17}$$

It is a known that the term $[\tilde{\omega}]I\boldsymbol{\omega}$ is a non-working term and that the kinetic energy rate can be written as the power equation[4,12]

$$\dot{T} = \boldsymbol{\omega}^T \boldsymbol{u} \tag{18}$$

Using the kinetic energy as a Lyapunov function, the simple feedback control law

$$\boldsymbol{u} = -P^{\omega}\boldsymbol{\omega} \tag{19}$$

is found where P^{ω} is a positive definite angular velocity feedback gain matrix. This controller is globally asymptotically stable.

Analogous statements can be made for the EQV formulation. Similar as in Ref 1, the nonlinear coriolis term $\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{\eta})$ for the EQV formulation is also nonworking. To show this important truth, let us study the term $\boldsymbol{\eta}^T\boldsymbol{H}$. Using Eq. (4) it can be written as

$$\boldsymbol{\eta}^T\boldsymbol{H} = \dot{\boldsymbol{x}}^T\dot{C}^T S^2 C\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^T C^T \dot{S} S C \dot{\boldsymbol{x}} - \frac{1}{2}\dot{\boldsymbol{x}}^T\left(\dot{\boldsymbol{x}}^T M_{\boldsymbol{x}} \dot{\boldsymbol{x}}\right) \tag{20}$$

which is the simplified to

$$\boldsymbol{\eta}^T\boldsymbol{H} = \dot{\boldsymbol{x}}^T\left(\dot{C}^T S^2 C + C^T \dot{S} S C\right)\dot{\boldsymbol{x}} - \frac{1}{2}\left[\sum_i^n \dot{\boldsymbol{x}}_i \left(\dot{\boldsymbol{x}}^T M_{x_i} \dot{\boldsymbol{x}}\right)\right] \tag{21}$$

Note the following identity.

$$\sum_i^n \dot{\boldsymbol{x}}_i \left(\dot{\boldsymbol{x}}^T M_{x_i} \dot{\boldsymbol{x}} \right) = \dot{\boldsymbol{x}}^T \dot{M} \dot{\boldsymbol{x}} \tag{22}$$

After adding and subtracting some terms $\boldsymbol{\eta}^T \boldsymbol{H}$ becomes

$$\boldsymbol{\eta}^T \boldsymbol{H} = \dot{\boldsymbol{x}}^T \left(\dot{C}^T S^2 C + C^T S^2 \dot{C} + 2C^T \dot{S} S C - C^T S^2 \dot{C} - C^T \dot{S} S C \right) \dot{\boldsymbol{x}} - \frac{1}{2} \dot{\boldsymbol{x}}^T \dot{M} \dot{\boldsymbol{x}} \tag{23}$$

From Eq. (2) it is clear that

$$\dot{M} = \dot{C}^T S^2 C + C^T S^2 \dot{C} + 2C^T \dot{S} S C \tag{24}$$

This identity is used to reduce $\boldsymbol{\eta}^T \boldsymbol{H}$ to

$$\boldsymbol{\eta}^T \boldsymbol{H} = \dot{\boldsymbol{x}}^T \left(\dot{M} - (SC)^T (\dot{SC}) \right) \dot{\boldsymbol{x}} - \frac{1}{2} \dot{\boldsymbol{x}}^T \dot{M} \dot{\boldsymbol{x}} \tag{25}$$

which can be further manipulated to give the important result:

$$\boldsymbol{\eta}^T \boldsymbol{H} = \dot{\boldsymbol{x}}^T \left(\dot{M} - \frac{1}{2} \dot{M} \right) \dot{\boldsymbol{x}} - \frac{1}{2} \dot{\boldsymbol{x}}^T \dot{M} \dot{\boldsymbol{x}} = 0 \tag{26}$$

Since the $\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{\eta})$ term is nonworking, the kinetic energy rate can be expressed simply as

$$\dot{T} = \boldsymbol{\eta}^T \dot{\boldsymbol{\eta}} = \boldsymbol{\eta}^T (-\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{\eta}) + \boldsymbol{\epsilon}) = \boldsymbol{\eta}^T \boldsymbol{\epsilon} \tag{27}$$

Using the kinetic energy as a Lyapunov function, the following feedback control law can be shown to be globally asymptotically stabilizing.

$$\boldsymbol{\epsilon} = -P^{\eta} \boldsymbol{\eta} \tag{28}$$

The velocity feedback gain matrix P^{η} is positive definite. Note that Eq. (28) provides a linear velocity feedback control law in the $\boldsymbol{\eta}$ formulation. This renders $\dot{T}$ in Eq. (27) into the negative definite expression

$$\dot{T} = -\boldsymbol{\eta}^T P^{\eta} \boldsymbol{\eta} \tag{29}$$

Therefore $\boldsymbol{\epsilon} = -P^{\eta} \boldsymbol{\eta}$ is a globally asymptotically stabilizing feedback control law for Eq. (14). Eq. (13) is used to rewrite this control law in terms of the generalized external force vector $\boldsymbol{Q}$.

$$\boldsymbol{Q} = V_x(\boldsymbol{x}) - C^T S P^{\eta} \boldsymbol{\eta} \tag{30}$$

If the feedback gain P^{η} is assumed to be a scalar, then the control laws simplifies to

$$\boldsymbol{Q} = V_x(\boldsymbol{x}) - P^{\eta} M(\boldsymbol{x}) \dot{\boldsymbol{x}} \tag{31}$$

Note the physical significance of Eq. (31). Instead of just feeding back the $\dot{\boldsymbol{x}}$ vector, as would be done traditionally in velocity feedback (proportional damping), a momentum type quantity is fed back instead. Even though P^{η} is a constant scalar in this expression, the $M(\boldsymbol{x})$ term acts as a state dependent feedback gain matrix. As a comparison, the traditional method of constructing output velocity feedback laws would result in a control law of the type

$$\boldsymbol{Q} = V_x(x) - P^{\dot{x}} \dot{\boldsymbol{x}} \tag{32}$$

where $P^{\dot{x}}$ is the velocity feedback gain matrix.

To study stability, let us write the kinetic energy in the classical form and use it as a system Lyapunov function.

$$T = \frac{1}{2}\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} \tag{33}$$

The first time derivative of T is

$$\dot{T} = \dot{\boldsymbol{x}}^T M \ddot{x} + \frac{1}{2}\dot{\boldsymbol{x}}^T \dot{M} \dot{\boldsymbol{x}} \tag{34}$$

Using Eq. (1) this is reduced to

$$\dot{T} = \dot{\boldsymbol{x}}^T \left(\boldsymbol{Q} - V_x - \frac{1}{2}\dot{M}\dot{\boldsymbol{x}} + \frac{1}{2}\dot{\boldsymbol{x}}^T M_{\boldsymbol{x}} \dot{\boldsymbol{x}} \right) \tag{35}$$

which is then expanded to

$$\dot{T} = \dot{\boldsymbol{x}}^T (\boldsymbol{Q} - V_x) - \frac{1}{2}\dot{\boldsymbol{x}}^T \dot{M} \dot{\boldsymbol{x}} + \frac{1}{2}\sum_i \dot{\boldsymbol{x}}_i (\dot{\boldsymbol{x}}^T M_{x_i} \dot{\boldsymbol{x}}) \tag{36}$$

Using the identity in Eq. (22) this is simplified to the usual work/energy power expression

$$\dot{T} = \dot{\boldsymbol{x}}^T (\boldsymbol{Q} - V_x) \tag{37}$$

The traditional velocity feedback control law in Eq. (32) then yields the following negative definite expression

$$\dot{T} = -\dot{\boldsymbol{x}}^T P^{\dot{x}} \dot{\boldsymbol{x}} \tag{38}$$

where $P^{\dot{x}}$ is a positive definite matrix. This control law is also globally asymptotically stabilizing.

EXPONENTIAL CONVERGENCE

Both velocity feedback control laws in Eqs. (30) and (32) are shown to be globally asymptotically stabilizing. Here their convergence rates will be studied. First, assume that P^{η} is a scalar quantity. Then the time derivative of T can be written as

$$\dot{T} = -P^{\eta} \boldsymbol{\eta}^T \boldsymbol{\eta} \tag{39}$$

Using Eq. (5) this is rewritten as

$$\dot{T} = -2P^{\eta} T \tag{40}$$

This simple first order differential equation can be solved explicitly to yield

$$T(t) = e^{-2P^{\eta} t}\, T(0) \tag{41}$$

Therefore, for any choice of positive P^{η}, the total system kinetic energy will decay exponentially at a well defined rate.

To show exponential convergence for the case where P^{η} is a fully populated positive definite matrix, we make use of the Rayleigh-Ritz inequality[13,14]

$$\lambda_{min}^{P^{\eta}} \boldsymbol{\eta}^T \boldsymbol{\eta} \leq \boldsymbol{\eta}^T P^{\eta} \boldsymbol{\eta} \leq \lambda_{max}^{P^{\eta}} \boldsymbol{\eta}^T \boldsymbol{\eta} \tag{42}$$

This inequality allows $\dot{T}$ to be written as

$$\dot{T} = -\boldsymbol{\eta}^T P^\eta \boldsymbol{\eta} \leq -\lambda_{min}^{P^\eta} \boldsymbol{\eta}^T \boldsymbol{\eta} \tag{43}$$

Using Eq. (5) again, the following inequality for $\dot{T}$ is obtained.

$$\dot{T} \leq -2\lambda_{min}^{P^\eta} T \tag{44}$$

Note that in this more general case the kinetic energy is upwardly bounded by a exponentially decaying curve at a rate of $-2\lambda_{min}^{P^\eta}$.

$$T(t) \leq T(0) e^{-2\lambda_{min}^{P^\eta} t} \tag{45}$$

Here the kinetic energy decay rate cannot be easily determined for the entire maneuver. However, as time t grows sufficiently large it will approach the decay rate of the smallest eigenvalue $\lambda_{min}^{P^\eta}$.

Proving exponential stability for the feedback control law in Eq. (32) is more difficult. Let $\lambda_{min}^{P^{\dot{x}}}$ be the smallest eigenvalue of $P^{\dot{x}}$, then using Eq. (38) and (42) the following inequality must hold.

$$\dot{T} \leq -\lambda_{min}^{P^{\dot{x}}} \dot{\boldsymbol{x}}^T \dot{\boldsymbol{x}} \tag{46}$$

Let λ_{max}^M be the largest eigenvalue of M. If

$$\lambda_{min}^{P^{\dot{x}}} \geq \lambda_{max}^M \tag{47}$$

is true, then the above inequality can be expanded to

$$\dot{T} \leq -\lambda_{min}^{P^{\dot{x}}} \dot{\boldsymbol{x}}^T \dot{\boldsymbol{x}} \leq -\lambda_{max}^M \dot{\boldsymbol{x}}^T \dot{\boldsymbol{x}} \leq -\dot{\boldsymbol{x}}^T M \dot{\boldsymbol{x}} \tag{48}$$

Using Eq. (33), the kinetic energy derivative will be upwards limited by

$$\dot{T} \leq -2T \tag{49}$$

Note that with this control law, exponential stability is only shown for a sufficiently large set of $P^{\dot{x}}$ eigenvalues. Also, the convergence rate cannot be determined easily from the above analysis. Numerical analysis shows that the condition in Eq. (47) is very conservative. The velocity feedback control law in Eq. (32) is found to be exponentially stabilizing in the endgame even if the condition in Eq. (47) is violated.

The concept of feeding back a generalized momentum quantity instead of a configuration velocity coordinate to achieve a controlled exponential stability can easily be applied to the rigid body attitude control problem discussed earlier. Instead of using the traditional control torque $\boldsymbol{u}$ defined in Eq. (19), let the feedback gain P^ω be a scalar quantity and define the torque $\boldsymbol{u}$ instead as

$$\boldsymbol{u} = -P^\omega I \boldsymbol{\omega} \tag{50}$$

The kinetic energy derivative in Eq. (18) can now be written using Eq. (17) as

$$\dot{T} = -P^\omega \boldsymbol{\omega}^T I \boldsymbol{\omega} = -2P^\omega T \tag{51}$$

With this slight modification, a globally asymptotically stabilizing feedback control law is made exponentially stabilizing with an straight forward way to control the kinetic energy decay rate.

NUMERICAL RESULTS

To compare the performance of the feedback control laws in Eqs. (30) and (32), they are applied to a three-link manipulator system and a tumbling rigid body. In each simulation the goal of the control law is to bring the system to rest by dissipating all initial kinetic energy. For all cases, we consider only damping (velocity) feedback.

Three-Link Manipulator System

The rigid three-link manipulator system is shown in Figure 1. Each link has some initial rotational velocity and the goal of this example is to bring all the links to rest at an arbitrary orientation. The tip masses and rod lengths are set to 1. The numerical integration is performed with a fourth order Runge-Kutta method with an integration step size of 0.005 seconds and a simulation duration of 15 seconds.

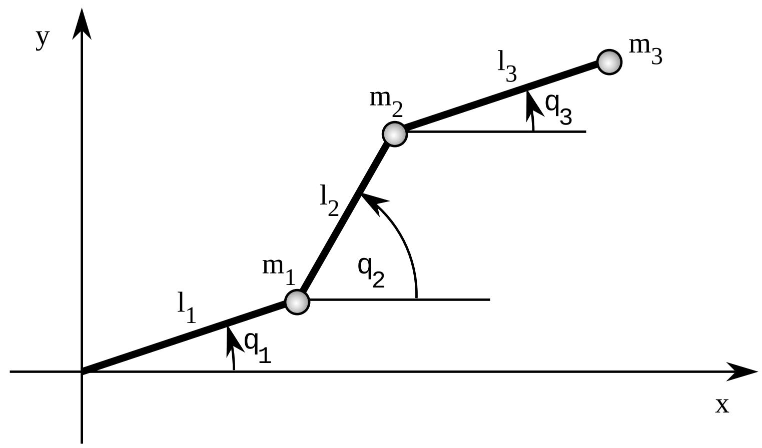

Figure 1 Three-Link Manipulator System Layout.

Choosing the inertial polar angles as generalized coordinates the state vector is $\boldsymbol{x} = (\theta_1, \theta_2, \theta_3)^T$, then the system mass matrix is

$$M(\boldsymbol{x}) = \begin{bmatrix} (m_1+m_2+m_3)l_1^2 & (m_2+m_3)l_1l_2\cos(\theta_2-\theta_1) & m_3l_1l_3\cos(\theta_3-\theta_1) \\ (m_2+m_3)l_1l_2\cos(\theta_2-\theta_1) & (m_2+m_3)l_2^2 & m_32_1l_3\cos(\theta_3-\theta_2) \\ m_3l_1l_3\cos(\theta_3-\theta_1) & m_3l_2l_3\cos(\theta_3-\theta_2) & m_3l_3^2 \end{bmatrix} \tag{52}$$

The feedback gains are held constant for each control law. To perform a reasonable comparison, the feedback gain magnitude for each control law is selected such that the maximum absolute control effort encountered is equal.

The first study is performed with a very large initial rotational motion of $\dot{\boldsymbol{x}}^T(0) = (93, -110, -73)$ degrees/second. The initial state vector is set the same for all simulation to be $x^T(0) = (-90, 30, 0)$ degrees. The gain P^{η} is set to 0.7 while $P^{\dot{x}}$ is set to 1. The magnitude of the control vector $\boldsymbol{Q}$ is shown in Figure 2. With this tuning of the gains, both control laws encounter a maximum control effort of about 3.5. The $\boldsymbol{\eta}$ control law magnitude starts to be reduced linearly on the base 10 logarithmic scale after only about 1 second of maneuver time. A linear logarithmic decay rate indicates an exponentially decaying quantity. The $\dot{\boldsymbol{x}}$ control law does not start to decay linearly on this scale until after 6 seconds into the maneuver. After an initial hump, the $\dot{\boldsymbol{x}}$ control effort drops off quickly at first and then decays relatively slowly. The kinetic energy T is decreased by about one order of magnitude every *five* seconds. The $\boldsymbol{\eta}$ control maintains a relatively large control effort for the initial portion of the maneuver where about 90% of the kinetic energy is being canceled. After this the $\boldsymbol{\eta}$ control effort drops off at a much faster than the $\dot{\boldsymbol{x}}$ control effort. The kinetic energy T is decreased by one order of magnitude in only *one* second. The only way

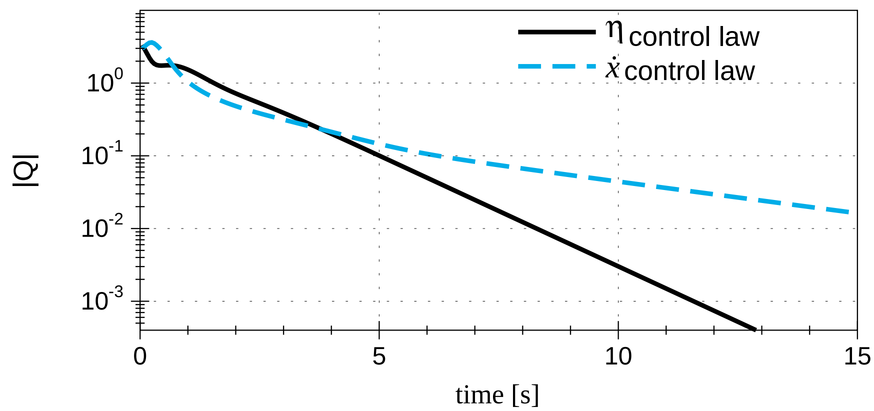

Figure 2 Control Vector Magnitude Time History.

for the $\dot{\boldsymbol{x}}$ control law to have a similar (five times as fast energy dissipation) performance as the $\boldsymbol{\eta}$ control law would be to make $P^{\dot{x}}$ time dependent, or introduce piecewise constant gain scheduling. The draw-back of feedback gain scheduling is that it makes the overall control law much more complicated. The $\boldsymbol{\eta}$ control law effectively performs some feedback gain scheduling implicitly as is seen in Eq. (31).

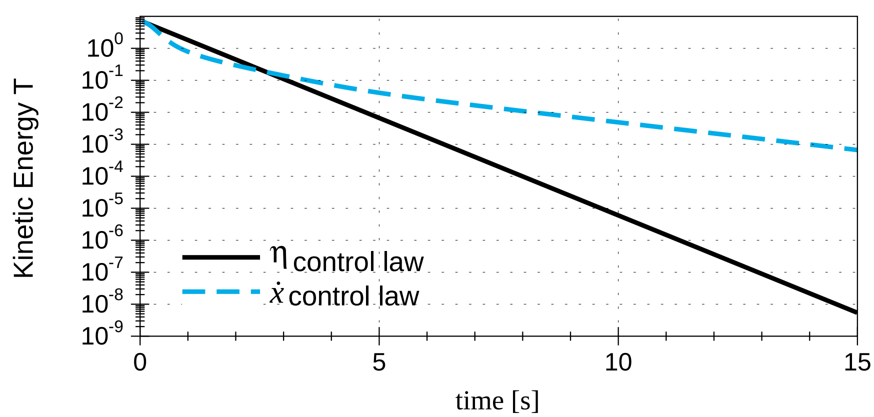

Figure 3 Kinetic Energy Time History.

The kinetic energy provides a scalar measure of the total system "error motion" and is plotted on a base 10 logarithmic scale in Figure 3. As predicted, the $\boldsymbol{\eta}$ control law has an exponentially decaying kinetic energy since the corresponding curve in Figure 3 is completely linear. The kinetic energy for the $\dot{\boldsymbol{x}}$ control law starts to decay linearly on the logarithmic scale after a few seconds, whereas the kinetic energy for the $\boldsymbol{\eta}$ control is linear from the outset. These results show very clearly that for a given maximum control effort, the $\boldsymbol{\eta}$ control law outperforms the constant gain $\dot{\boldsymbol{x}}$ control law by having a much larger final decay rate. While these results are for a particular example, the exponential convergence proof of the previous section is general. We therefore feel this pattern is representative.

The second simulation is performed with only the third link having an initial rotational motion of 10 degrees/second, the other two are at rest. The feedback gains are set to $P^{\eta} = 0.72$ and $P^{\dot{x}} = 1$. If left uncontrolled, then the coupled system dynamics would partially transfer the kinetic energy of the third link into the other two links and very quickly all three links would be rotating. The time history of the $\dot{\boldsymbol{x}}$ vector components for both control laws are shown in Figure 4.

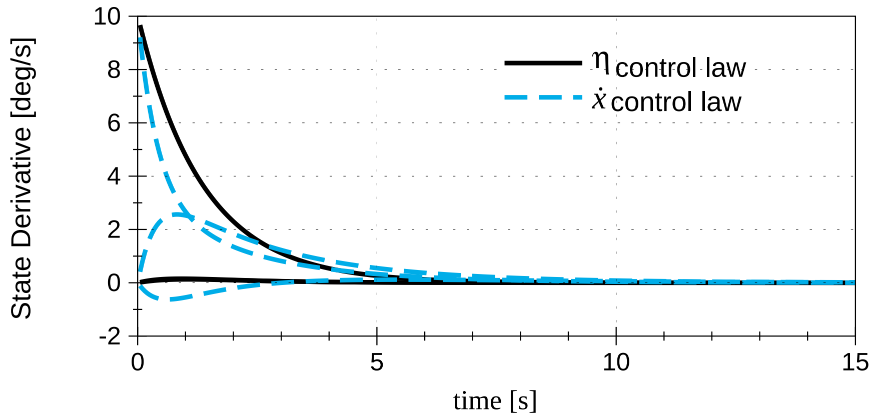

Figure 4 $\dot{\boldsymbol{x}}$ **Vector Component Time History.**

With the $\dot{\boldsymbol{x}}$ feedback control law, coupling causes all three links to start to rotate and then the control law brings them all to rest together. The $\boldsymbol{\eta}$ feedback control law performs quite differently. The later keeps the motion of the first two links very close two zero while exponentially reducing the initial 10 degrees/second motion of the third link. This pleasant surprise is essentially a nonlinear analog of "independent modal space control" popular for linear structural dynamical systems.[8,15] In effect, this feedback control law is able to decouple the rotation of each link and bring each *individually* to zero. Note that no explicit gain scheduling had to be performed with this control law to achieve this effect.

The magnitudes of the control efforts involved are shown in Figure 5. Both feedback control laws have the same maximum control effort. As is shown in the previous simulation, again the $\boldsymbol{\eta}$ control effort remains larger than the $\dot{\boldsymbol{x}}$ control effort until about 90% of the kinetic energy is canceled. After this the $\boldsymbol{\eta}$ control effort keeps on decreasing in an exponential manner while the $\dot{\boldsymbol{x}}$ control effort also decreases exponentially, but at a much slower rate.

The vector components of the control effort are shown in Figure 6. As expected, all three control components are active for the $\dot{\boldsymbol{x}}$ feedback law. The $\boldsymbol{\eta}$ control however keeps the first control component near zero while only using the second and third control component. They decay at the same exponential rate and differ in magnitude such that the motion of the third link is stopped while not starting any motion in the second link.

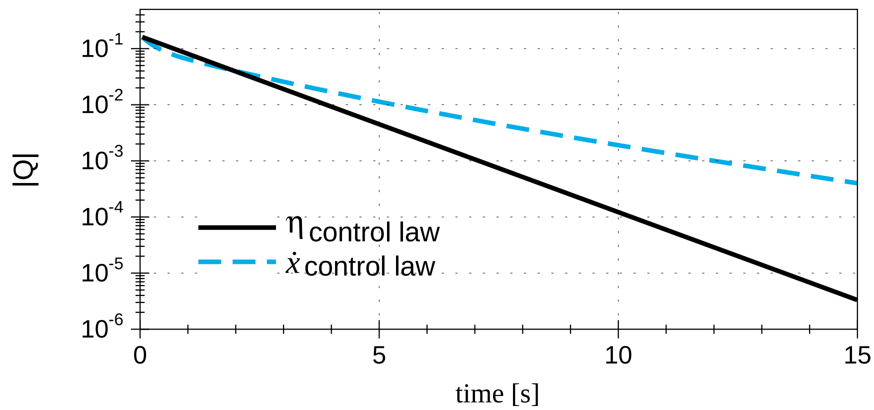

Figure 5 Control Effort Magnitude Time History.

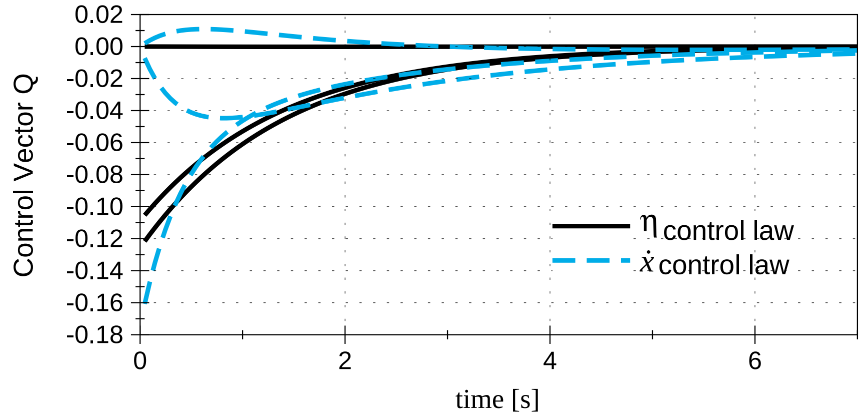

Figure 6 Vector Components of the Control Effort.

Tumbling Rigid Body

The second system studied is a tumbling rigid body. For this case, the body has a very large initial angular velocity and the feedback control law is designed to drive the kinetic energy to zero. The inertia matrix I is given by

$$I = \begin{bmatrix} 10 & 5 & 3 \\ 5 & 7 & 4 \\ 3 & 4 & 5 \end{bmatrix} \tag{53}$$

The Euler equations of motion for a rigid body are given in Eq. (15). A fourth order Runge-Kutta integration method is used to perform the numerical simulation with an integration step size of 0.1 seconds. Total maneuver time is 15 seconds. The initial body angular velocity vector is $\boldsymbol{\omega}(t_0) = (90, -70, 50)^T$ degrees/second.

The two control laws given in Eqs. (19) and (50) are compared here. The angular velocity feedback is chosen to be a scalar in both cases. The classical feedback law is

$$\boldsymbol{u} = -P_1 \dot{\boldsymbol{x}} = -P_1 \boldsymbol{\omega} \tag{54}$$

where the $\boldsymbol{\eta}$ feedback law reduces to

$$\boldsymbol{u} = -P_2 I \boldsymbol{\omega} \tag{55}$$

which is simply momentum feedback in this case since P_2 is a scalar. In an effort to make a fair comparison of the control laws, the feedback gains are chosen such that the maximum encountered control effort for both control laws is the same. Therefore P_1 is set to 2 and P_2=0.33. The magnitudes of each control law are shown in Figure 7. Both control laws have their gains tuned consistently so that they result in the same maximum control effort at the beginning of the simulation. As is observed in the three-link manipulator simulations, the $\boldsymbol{\eta}$ control law retains a larger control effort during the first segment of the maneuver and decays to a lower value than the $\dot{\boldsymbol{x}}$ control effort.

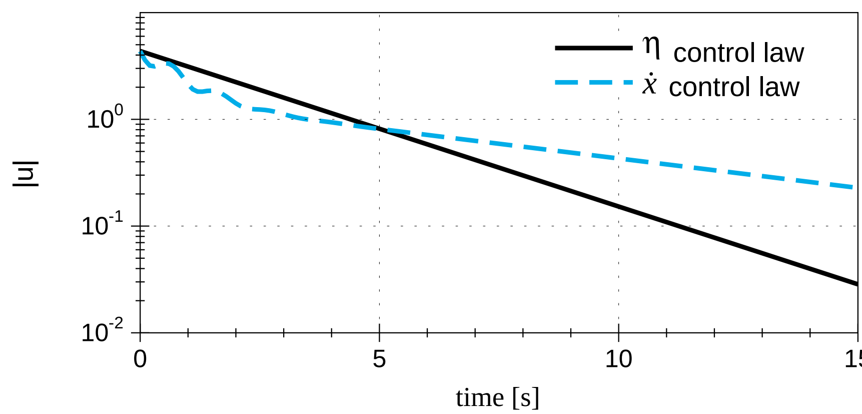

Figure 7 Control Torque Magnitudes.

The system kinetic energy is plotted in Figure 8. As anticipated, the $\boldsymbol{\eta}$ control law causes the system kinetic energy to decay exponentially at the prescribed rate. The $\dot{\boldsymbol{x}}$ control law appears to cause the kinetic energy to decay exponentially only after about 2 seconds into the maneuver at a slower rate. At the maneuver end the residual kinetic energy of the $\boldsymbol{\eta}$ control law simulations is over two orders of magnitude less then the $\dot{\boldsymbol{x}}$ control law simulation.

The corresponding body angular velocity vector magnitudes are plotted in Figure 9. Note that even though the kinetic energy of the $\boldsymbol{\eta}$ control law simulation is typically equal or lower than the $\dot{\boldsymbol{x}}$ control law kinetic energy, the angular velocity magnitude of the $\boldsymbol{\eta}$ control law is only smaller than the $\dot{\boldsymbol{x}}$ angular velocity magnitude after about six seconds into the simulation.

One reason why the $\boldsymbol{\eta}$ control law performs so much better in these simulations than the $\dot{\boldsymbol{x}}$ control law is that the inertia matrix is fully populated. If we repeat these simulations with the inertia matrix I near-diagonal, then there would be virtually no inertia matrix coupling to compensate for. In these cases the $\boldsymbol{\eta}$ and $\dot{\boldsymbol{x}}$ control laws perform almost identically.

CONCLUSION

The EQV Boltzmann-Hamel formulation provides an interesting new dynamical system formulation. The EQV feedback control law presented in this paper is shown to be globally

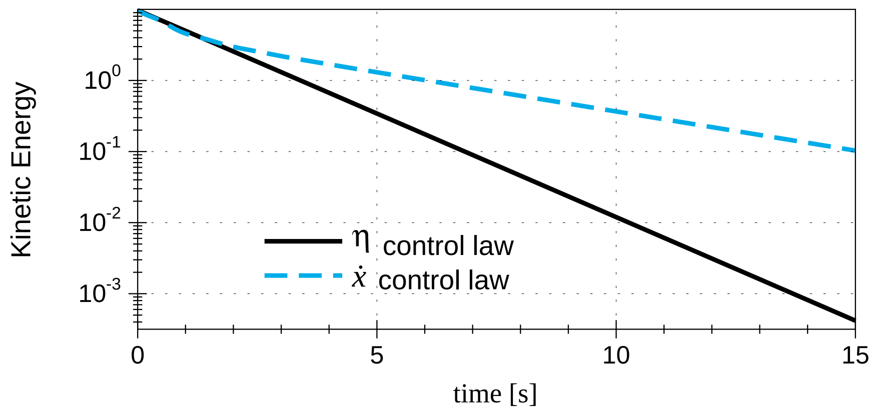

Figure 8 **Total Kinetic Energy Decay.**

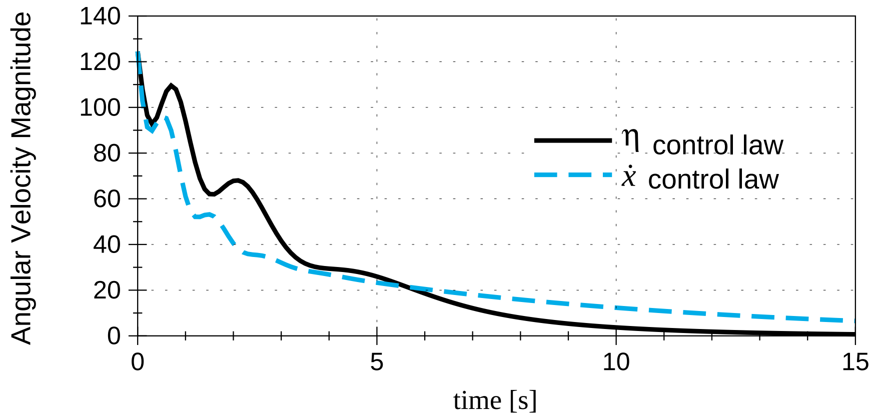

Figure 9 **Body Angular Velocity Magnitudes.**

exponentially stabilizing and have superior energy dissipation performance over traditional $\dot{\boldsymbol{x}}$ type feedback control laws. Instead of feeding back a configuration coordinate rate quantity, it feeds back a quantity proportional to a generalized momentum. For gains tuned to ensure the same maximum allowable control effort, the EQV feedback control law is shown to exhibit a faster final convergence rate than the traditional velocity feedback control laws. Numerical studies also show that this feedback control is able to decouple the motion of a nonlinear multi-link system and bring each link to rest individually.

REFERENCES

1. A. Jain and G. Rodriguez. "Diagonalized Lagrangian Robot Dynamics," *IEEE Trans. on Robotics and Automation*, Vol. 11, No. 4, Aug. 1995, pp. 571–584.

2. J. L. Junkins and H. Schaub. "An Instantaneous Eigenstructure Quasi-Coordinate Formulation For Nonlinear Multibody Dynamics," *7th AAS/AIAA Space Flight Meeting*, Huntsville, AL, Feb. 10–12 1997. paper No. AAS 97-119.

3. J. L. Junkins and H. Schaub. "Orthogonal Square Root Eigenfactor Parameterization of Mass Matrices," *35th Aerospace Sciences and Meeting*, Reno, NV, Jan. 1997. paper

No. 97-0248.

4. J. L. Junkins and J. D. Turner. *Optimal Spacecraft Rotational Maneuvers*. Elsevier Science Publishers, Netherlands, 1986.

5. H. Schaub, P. Tsiotras, and J. L. Junkins. "Principal Rotation Representations of Proper NxN Orthogonal Matrices," *International Journal of Engineering Science*, Vol. 33, No. 15, Elsevier Science Ltd., Great Britain, 1995, pp. 2277–2295.

6. I. Y. Bar-Itzhack and F. L. Markley. "Minimal Parameter Solution of the Orthogonal Matrix Differential Equation," *IEEE Transactions on Automatic Control*, Vol. 35, No. 3, March 1990, pp. 314–317.

7. Y. Oshman and I. Y. Bar-Itzhack. "Eigenfactor Solution of the Matrix Riccati Equation — A Continuous Square Root Algorithm," *IEEE Trans. on Automatic Control*, Vol. AC-30, No. 10, Oct. 1985, pp. 971–978.

8. J. L. Junkins and Y. Kim. *Introduction to Dynamics and Control of Flexible Structures*. AIAA Education Series, Washington D.C., 1993.

9. Y. Kim, S. Lee, and J. L. Junkins. "Eigenvector Derivatives for Mechanical Second - Order Systems," *Journal of Guidance, Control and Dynamics*, Vol. 18, No. 4, July–Aug. 1995, pp. 899–906.

10. M. H. Rheinfurth and H. B. Wilson. "Methods of Applied Dynamics," NASA Reference Publication 1262, NASA, May 1991.

11. J. G. Papastavridis. "On the Boltzmann-Hamel Equations of Motion: A Vectorial Treatment," *Journal of Applied Mechanics*, Vol. 61, June 1994, pp. 453–459.

12. H. Schaub, R. D. Robinett, and J. L. Junkins. "Globally Stable Feedback Laws for Near-Minimum-Fuel and Near-Minimum-Time Pointing Maneuvers for a Landmark-Tracking Spacecraft," *Journal of the Astronautical Sciences*, Vol. 44, No. 4, Oct.–Dec. 1996, pp. 443–466.

13. J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1991.

14. W. J. Rugh. *Linear System Theory*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1993.

15. H. Oz and L. Meirovitch. "Optimal Modal-Space Control of Flexible Gyroscopic Systems," *Journal of Guidance and Control*, Vol. 3, No. 3, 1980, pp. 218–226.