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NEW ATTITUDE PENALTY FUNCTIONS FOR SPACECRAFT OPTIMAL CONTROL PROBLEMS*

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A universal attitude penalty function *g()* is presented which renders spacecraft optimal control problem solutions independent of attitude coordinate choices. This function will return the same scalar penalty for a given attitude regardless of the choice of attitude coordinates used to describe this attitude. The only singularities the *g()* function might encounter are solely due to the choice of attitude coordinates. A second attitude penalty function *G()* is considered which depends specifically on the modified Rodrigues parameter (MRP) vector $\vec{\sigma}$. The function *G()* is also globally nonsingular and of simpler form than *g()*. A theorem is presented which allows MRPs, along with a switching condition to their "image or shadow" trajectory, to be used in optimal control problems.

I. Introduction

SOLUTIONS of spacecraft optimal control problems, whose cost function rely on an attitude description, usually depend on the choice of attitude coordinates used. Coordinate choices are often considered a matter of taste, but the question of coordinate "optimality" arises. For example, a problem could be solved using 3-2-1 Euler angles or using classical Rodrigues parameters and yield two different "optimal" solutions, unless the performance index is invariant with respect to the attitude coordinate choice. Another problem arising with many attitude coordinates is that the resulting control formulation has no intrinsic sense of when a body has tumbled beyond ±180° from the reference attitude. In many such cases it would be simpler and cheaper to let the body complete the revolution rather than force it to reverse the rotation and return to the desired attitude.

This paper develops a universal attitude penalty function *g()* whose value is independent of the attitude coordinates chosen to represent it. Furthermore, this function achieves its maximum value for any principal rotation of ±180° from the target state. This will implicitly permit the *g()* function to sense the shortest rotational distance back to the reference state.

An attitude penalty function *G()* which depends on the Modified Rodrigues Parameters (MRP) will also be presented. These recently discovered MRPs[1–6] are a non-singular three-parameter set which can describe any three dimensional attitude. This MRP penalty function is simpler than the attitude coordinate independent *g()* function, but retains the useful property of avoiding lengthy principal rotations of more than ±180° and being nonsingular. A theorem is presented which allows discontinuous MRPs to be used in optimal control problems.

II. Problem Statement

A. Optimal Control Problem

Most spacecraft optimal control problems have a cost function J which depends on the control effort, the body angular velocity and the attitude. Let $\vec{u}$ be the control torque vector, $\vec{\omega}$ be the body angular velocity vector and $\vec{\eta}$ be a generic attitude coordinate vector in the following general optimal control formulation with fixed maneuver time t_f

$$\begin{aligned} &\min J = h\left(t_f\right) + \int_0^{t_f} p(\vec{\eta},\vec{\omega},\vec{u},t)dt \\ &subject\ to \\ &(\dot{\vec{\eta}},\dot{\vec{\omega}})^T = F(\vec{\eta},\vec{\omega},\vec{u},t) \end{aligned} \quad (1)$$

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$$h\left(t_f\right) = \frac{1}{2}K_1 g\left(\vec{\eta}_{t_f}\right) + \frac{1}{2}\vec{\omega}_{t_f}^T K_2 \vec{\omega}_{t_f} \tag{2}$$

and

$$p(\vec{\eta},\vec{\omega},\vec{u},t) = \frac{1}{2}\left(K_3 g(\vec{\eta}) + \vec{\omega}^T K_4 \vec{\omega} + \vec{u}^T R \vec{u}\right) \tag{3}$$

The weights K_1 and K_3 are scalars, the weights K_2, K_4 and R are 3x3 matrices. The function $g(\vec{\eta})$ is a general, non-negative attitude penalty function. For spacecraft optimal control problems, the equations of motion are usually imposed as an equality constraint. They are given in Eq. (4) and (5) below, where the function $f(\vec{\eta})$, obtained from kinematic analysis, returns a matrix dependent on the choice of attitude coordinates. The equations of motion are

$$\dot{\vec{\eta}} = f(\vec{\eta})\vec{\omega} \tag{4}$$

$$\Im\dot{\vec{\omega}} = -[\tilde{\omega}]\Im\vec{\omega} + \vec{u} \tag{5}$$

The matrix $\Im$ is the spacecraft inertia matrix. The tilde matrix is the cross-product operator

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{6}$$

The Hamiltonian H for this system is

$$\begin{aligned} H = &\frac{1}{2}K_3 g(\vec{\eta}) + \frac{1}{2}\vec{\omega}^T K_4 \vec{\omega} + \frac{1}{2}\vec{u}^T R \vec{u} \\ &+ \vec{\Lambda}_1^T f(\vec{\eta})\vec{\omega} + \vec{\Lambda}_2^T \Im^{-1}(-[\tilde{\omega}]\Im\vec{\omega} + \vec{u}) \end{aligned} \tag{7}$$

The costate equations are given by[7,8]

$$\dot{\vec{\Lambda}}_1 = -\frac{\partial H}{\partial \vec{\eta}} = -\frac{1}{2}K_3\frac{\partial g}{\partial \vec{\eta}} - \frac{\partial}{\partial \vec{\eta}}(f(\vec{\eta})\vec{\omega})^T\vec{\Lambda}_1 \tag{8}$$

$$\begin{aligned} \dot{\vec{\Lambda}}_2 &= -\frac{\partial H}{\partial \vec{\omega}} \\ &= -K_4\vec{\omega} - f(\vec{\eta})^T\vec{\Lambda}_1 - \left(\Im[\tilde{\omega}] - \left[\widetilde{\Im\omega}\right]\right)\Im^{-1}\vec{\Lambda}_2 \end{aligned} \tag{9}$$

For unbounded control torque $\vec{u}$, the optimality condition $\partial H/\partial\vec{u} = 0$ leads to the following optimal control torque[7,8]

$$\vec{u} = -R^{-1}\Im^{-1}\vec{\Lambda}_2 \tag{10}$$

The transversality conditions for a free final state are[7,8]

$$\vec{\Lambda}_1\left(t_f\right) = \frac{\partial h}{\partial \vec{\eta}}\left(t_f\right) = \frac{1}{2}K_1\frac{\partial g}{\partial \vec{\eta}}\left(t_f\right) \tag{11}$$

$$\vec{\Lambda}_2\left(t_f\right) = \frac{\partial h}{\partial \vec{\omega}}\left(t_f\right) = K_2\vec{\omega}\left(t_f\right) \tag{12}$$

Given good estimates of initial conditions, this nonlinear optimal control problem can be solved using various standard techniques.

B. Attitude Coordinates

Attitude coordinates define the rotational orientation of a rigid body relative to some reference frame. Just as there are a number of different coordinates which describe translation (cartesian, cylindrical, spherical), there are an infinite number of different ways to describe an attitude. Common examples are the Euler angles, the classical Rodrigues parameters or the Euler parameters (quaternions). Describing a rotation differs though from describing a translation in a fundamental way. The largest difference between two orientations corresponds to a principal rotation of $\pm 180°$, a finite value. Whereas the difference in two positions can can grow to infinity.

Minimal three-coordinate attitude representations usually contain singularities. These are specific attitudes at which the coordinates are not defined. This singularity can be avoided by using the Euler parameters at the cost of adding another coordinate. This redundant set has an equality constraint which restrains the attitude vector to be of unit magnitude. Therefore, if Euler parameters are used in a simple optimal control problem without any constraints, an equality constraint is automatically added.

Among others, this paper will use the very elegant set of recently developed Modified Rodrigues Parameters (MRP) with their "shadow"[1,3–5] counterpart. They are a non-singular, minimal attitude coordinate representation of rigid body attitudes with several useful attributes. They can be defined through a transformation from the Euler parameters as

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1,2,3 \tag{13}$$

or in terms of the principal rotation axis $\hat{e}$ and the principal rotating angle ϕ as

$$\vec{\sigma} = \hat{e}\cdot\tan(\phi/4) \tag{14}$$

Like the Euler parameters, the modified Rodrigues parameters are not unique. A second set of modified Rodrigues parameters, called the "shadow" set, can be used to avoid the singularity at $\phi = \pm 360°$ at the cost of a discontinuity at a switching point. The "shadow" set is found by reversing the sign of the β_i's in Eq. (13). The transformation between the "original" and "shadow" sets of MRPs for any arbitrary switching surface $\vec{\sigma}^T\vec{\sigma}$ is[1,3,4]

$$\sigma_i^S = -\sigma_i/\vec{\sigma}^T\vec{\sigma} \qquad i = 1,2,3 \tag{15}$$

Keep in mind that distinguishing between "original" and "shadow" set is purely arbitrary. Both sets describe the same physical orientation. If the switching condition is set to $\vec{\sigma}^T\vec{\sigma} = 1$, the magnitude of the MRP orientation vector is bounded between $0 \leq \vec{\sigma} \leq 1$ and the principal rotation angle is restricted between $-180° \leq \phi \leq +180°$. Note that this combined set of "original" and "shadow" parameters implicitly "knows" the shortest rotation back to the origin.[1] Principal rotations of more than 180° are typically avoided.

The differential kinematic equation for the modified Rodrigues parameters is given below.[1,2,6] The equation only contains second order polynomial nonlinearities in $\vec{\sigma}$.

$$\frac{d\vec{\sigma}}{dt} = \frac{1}{2}\left[I\left(\frac{1-\vec{\sigma}^T\vec{\sigma}}{2}\right) + [\tilde{\sigma}] + \vec{\sigma}\vec{\sigma}^T\right]\vec{\omega} \qquad (16)$$

Eq. (16) holds for both the "original" and the "shadow" set. This means that the derivative is well defined even at the switching point. Let us introduce the notation $\sigma^{2n} = (\vec{\sigma}^T\vec{\sigma})^n$. Then the general relationship between $d\vec{\sigma}/dt$ and $d\vec{\sigma}^S/dt$ for an arbitrary switching condition is

$$\frac{d\vec{\sigma}^S}{dt} = \frac{1}{\sigma^4}\left(\frac{d\vec{\sigma}}{dt} - \frac{1}{2}\left(1+\sigma^2\right)[\tilde{\sigma}]\vec{\omega} - \frac{(1-\sigma^4)}{4}\vec{\omega}\right) \qquad (17)$$

The partial derivative of Eq. (16) with respect to $\vec{\sigma}$ is

$$\frac{\partial}{\partial\vec{\sigma}}(f(\vec{\sigma})\vec{\omega}) = \frac{1}{2}\left(\vec{\sigma}\vec{\omega}^T - [\tilde{\omega}] - \vec{\omega}\vec{\sigma}^T + \vec{\sigma}^T\vec{\omega}I\right) \qquad (18)$$

The direction cosine matrix in terms of the modified Rodrigues parameters is[1,2,6]

$$C(\vec{\sigma}) = \frac{1}{(1+\sigma^2)^2}\begin{bmatrix} 4\left(2\sigma_1^2-\sigma^2\right)+\Sigma^2 & 8\sigma_1\sigma_2+4\sigma_3\Sigma & 8\sigma_1\sigma_3-4\sigma_2\Sigma \\ 8\sigma_1\sigma_2-4\sigma_3\Sigma & 4\left(2\sigma_2^2-\sigma^2\right)+\Sigma^2 & 8\sigma_2\sigma_3+4\sigma_1\Sigma \\ 8\sigma_1\sigma_3+4\sigma_2\Sigma & 8\sigma_2\sigma_3+4\sigma_1\Sigma & 4\left(2\sigma_3^2-\sigma^2\right)+\Sigma^2 \end{bmatrix}$$

$$\sigma^2 = \vec{\sigma}^T\vec{\sigma} \qquad \Sigma = 1-\sigma^2 \qquad (19)$$

III. Universal Attitude Penalty Function

A scalar attitude penalty function is sought which is independent of the choice of attitude coordinates. This allows for an universal solution to many spacecraft optimal control problems. We introduce the following non-negative measure of attitude displacement from a reference orientation.

$$g([C]) = \frac{1}{4}(3 - trace([C])) \quad \in \Re^+ \qquad (20)$$

This penalty function is given in terms of a proper orthogonal direction cosine matrix $[C]$. This rotation matrix is the most fundamental way to describe a rotation; unfortunately, also the most redundant. If there is no rotational displacement, the $[C]$ matrix is the identity matrix and $g([C]) = 0$.

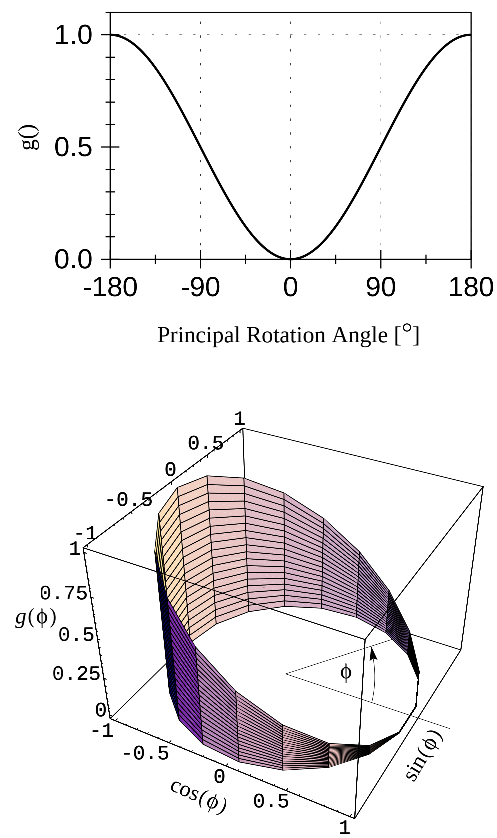

Fig. 1 Universal Attitude Penalty Function $g()$

The largest difference between two attitudes is a principal rotation of ±180°. Here the $[C]$ matrix is a diagonal matrix with two entries being -1 and one being +1. In this case $g([C]) = 1$. Therefore the $g()$ function is bounded for all possible motion between

$$0 \leq g() \leq 1 \qquad (21)$$

This penalty function can be written explicitly in terms of the principal rotation angle ϕ as

$$g(\phi) = \sin^2(\phi/2) \qquad (22)$$

This description makes the bound in Eq. (21) very easy to understand. The attitude cost is the highest only if the body is turned ±180° from the reference state. The $g()$ function is plotted relative to the principal rotation angle ϕ in Fig. 1. Using this type of attitude penalty function will typically avoid lengthy rotations. It intrinsically lowers the cost once the attitude has moved beyond ±180°. The advantage of defining the $g()$ function initially in terms of the $[C]$ matrix is that this rotation matrix can be parameterized by any attitude coordinates and thus making it universally valid for any choice of attitude coordinates. The following are a few sample parameterizations. Let $\vec{\beta}$ be an Euler parameter vector, then

$$g(\vec{\beta}) = \beta_1^2 + \beta_2^2 + \beta_3^2 \tag{23}$$

and

$$\frac{\partial g}{\partial \vec{\beta}} = [0 \quad 2\beta_1 \quad 2\beta_2 \quad 2\beta_3]^T \tag{24}$$

The Euler parameters are a non-singular once-redundant set of attitude coordinates. Their draw-back for optimization problems is that the redundancy introduces an additional equality constraint on the attitude vector. This redundancy also requires care to avoid other numerical problems when inverting some sets of equations.

Another popular attitude coordinate set is the classical Rodrigues parameter vector $\vec{q}$. It parameterizes the $g()$ function as

$$g(\vec{q}) = \frac{1}{4} \cdot \frac{\vec{q}^T\vec{q}}{1+\vec{q}^T\vec{q}} \tag{25}$$

and

$$\frac{\partial g}{\partial \vec{q}} = \frac{\vec{q}}{2}\frac{1}{\left(1+\vec{q}^T\vec{q}\right)^2} \tag{26}$$

These coordinates are a minimal three coordinate set and don't have any problems with redundancies. However, like most three-parameters sets, they contain a singular orientation. The classical Rodrigues parameters go singular for any principal rotation of $\pm 180°$. If it is a priori known that no such rotations will be encountered, this choice for attitude coordinates does allow for a large range of possible rotations.

Probably the most popular choice of attitude coordinates are any one of the 12 sets of Euler angles. Let $(\theta_1, \theta_2, \theta_3)$ be the set of 3-1-3 Euler angles. They would parameterize the $g()$ function as

$$g(\vec{\theta}) = \frac{1}{4}(3 - (1+\cos\theta_2)\cos(\theta_1+\theta_3) - \cos\theta_2) \tag{27}$$

and

$$\frac{\partial g}{\partial \vec{\theta}} = \frac{1}{4}\begin{bmatrix} (1+\cos\theta_2)\sin(\theta_1+\theta_3) \\ \sin\theta_2\cos(\theta_1+\theta_3)+\sin\theta_2 \\ (1+\cos\theta_2)\sin(\theta_1+\theta_3) \end{bmatrix} \tag{28}$$

The advantage of the Euler angles is that they are easy to visualize, especially for small angles. However, any attitude description with Euler angles is never more than 90° away from a singularity. This makes these coordinates difficult to use for large arbitrary rotations. Further, the kinematic equations for the Euler angles are in terms of trigonometric functions, making them more computationally intensive than having only polynomial equations.

A very attractive attitude description are the modified Rodrigues parameters. They parameterize the $g()$ function as

$$g(\vec{\sigma}) = 4\frac{\vec{\sigma}^T\vec{\sigma}}{\left(1+\vec{\sigma}^T\vec{\sigma}\right)^2} \tag{29}$$

and

$$\frac{\partial g}{\partial \vec{\sigma}} = 8\vec{\sigma}\left(\frac{1-\vec{\sigma}^T\vec{\sigma}}{\left(1+\vec{\sigma}^T\vec{\sigma}\right)^3}\right) \tag{30}$$

The MRPs are a minimal attitude description which are also non-singular when combined with their corresponding "shadow set." They are well suited to describe any large arbitrary rotation while their equations retain a simple polynomial form.

The $g()$ attitude penalty function could be parameterized by any other attitude coordinate description. All equations for $g()$ shown return the same penalty for a given reference orientation. This effectively removes the dependency of the optimal control solution on the choice of attitude coordinates. However, the optimal costate vector $\vec{\Lambda}_1$ will depend on the attitude coordinates used since Eqs. (8), (11) depend on the partial derivative of $g()$ with respect to the particular attitude coordinates.

IV. MRP Attitude Penalty Function

While the universal attitude penalty function $g()$ has some very appealing properties, it is usually more complicated than just using the standard sum squared of the attitude coordinates typically seen as an optimal control performance measure. For example, using the simpler attitude penalty function

$$G(\vec{\sigma}) = \vec{\sigma}^T\vec{\sigma} \tag{31}$$

where $\vec{\sigma}$ is a MRP vector, retains all properties of $g()$ defined in Eq. (20), except being universal with respect to attitude coordinate choice. By switching between "original" and "shadow" MRP trajectories on the $\vec{\sigma}^T\vec{\sigma} = 1$ surface, the attitude penalties in Eq. (31) are bounded within [0,1]. Using Eq. (14) the penalty function can be written in terms of the principal rotation angle ϕ as

$$G(\phi) = \tan^2(\phi/4) \tag{32}$$

The $G()$ function is plotted relative to the principal rotation angle ϕ in Fig. 2. Note that like the universal attitude penalty function $g()$, the maximum

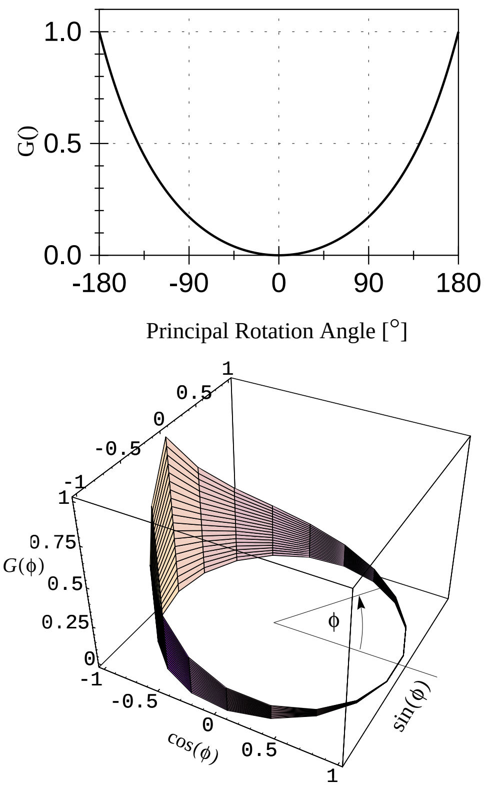

Fig. 2 MRP Attitude Penalty Function $G()$

attitude penalty is also attained at a principal rotation of $\pm 180°$.

By using the MRPs, this penalty function is globally non-singular using a minimal attitude coordinate description. The non-singularity comes at the price of having the switching condition defined in Eq. (15). However, both $\vec{\sigma}$ and $\vec{\sigma}^S$ are well defined for both the original and the "shadow" parameters. By choosing the switching surface $\vec{\sigma}^T\vec{\sigma} = 1$ Eqs. (15), (17) are simplified to

$$\vec{\sigma}^S = -\vec{\sigma} \tag{33}$$

$$\dot{\vec{\sigma}}^S = \dot{\vec{\sigma}} - [\tilde{\sigma}]\vec{\omega} \tag{34}$$

Observe from Eq. (34) that for pure single axis rotations $\dot{\vec{\sigma}}$ simply equals $\dot{\vec{\sigma}}^S$ on the $\vec{\sigma}^T\vec{\sigma} = 1$ switching surface. Since the derivatives of the costates depend on the attitude coordinates, they will also have a discontinuity as the attitude vector is switched.

V. MRP Costate Switching Condition

The MRP have many useful attributes. However, to avoid a singularity, this minimal attitude coordinate description needs to switch between the original and the "shadow" MRP set.[1,4] This switching can occur on any surface $\vec{\sigma}^T\vec{\sigma} = c^2$, where $c \geq 0$. The most attractive switching surface is $\vec{\sigma}^T\vec{\sigma} = 1$. During this switching the MRP attitude vector and its derivative are generally not continuous as shown earlier. The optimality conditions for the optimal control problem used in this paper were derived assuming that all states were smooth and continuous. This is no longer guaranteed with the MRP.

The Weierstrass-Erdmann corner conditions were developed for the case where the state derivative is discontinuous.[7,8] The same initial assumptions used in deriving the Weierstrass-Erdmann corner conditions also hold if the state, not the derivative of the state, is discontinuous. Without loss of generality, let us assume that $\vec{\sigma}$ is only discontinuous at t_1, where $0 < t_1 < t_f$. The cost function J, in terms of the system Hamiltonian H and the costates Λ, can now be written as

$$\begin{aligned} J &= h(t_f) + \int_0^{t_1^-} \left(H - \vec{\Lambda}^T\dot{\vec{x}}\right) dt \\ &+ \int_{t_1^+}^{t_f} \left(H - \vec{\Lambda}^T\dot{\vec{x}}\right) dt \\ &= J_1 + J_2 + J_3 \end{aligned} \tag{35}$$

where $\vec{x} = (\vec{\sigma}, \vec{\omega})^T$ and $\vec{\Lambda} = (\vec{\Lambda}_1, \vec{\Lambda}_2)^T$. For notational compactness, lets define $\vec{\sigma}_- = \vec{\sigma}(t_1^-)$, $\vec{\sigma}_+ = \vec{\sigma}(t_1^+)$, $\vec{\Lambda}_{i-} = \vec{\Lambda}_i(t_1^-)$ and $\vec{\Lambda}_{i+} = \vec{\Lambda}_i(t_1^+)$. Each integral can now be evaluated without state discontinuity problems. The first variation of J must satisfy[7]

$$\partial J = 0 = \delta J_1 + \delta J_2 + \delta J_3 \tag{36}$$

The first variation of J_1 is

$$\delta J_1 = \frac{\partial h}{\partial \vec{x}}(\vec{x}(t_f))^T \delta\vec{x}(t_f) \tag{37}$$

Taking the first variation of J_2 it must be taken into account that t_1 is a free final time of the integral.

$$\begin{aligned} \delta J_2 &= \left(H(t_1^-) - \vec{\Lambda}(t_1^-)^T \dot{\vec{x}}(t_1^-)\right)\delta t_1 \\ &+ \int_0^{t_1^-} \left(\frac{\partial H}{\partial \vec{x}}^T \delta\vec{x} - \vec{\Lambda}^T\delta\dot{\vec{x}} + \frac{\partial H}{\partial \vec{u}}^T \delta\vec{u}\right) dt \end{aligned} \tag{38}$$

Since the states are smooth and continuous within the integral $\delta\dot{\vec{x}}$ can be written as $\frac{d}{dt}(\delta\vec{x})$. This permits δJ_2 to be integrated by parts.

$$\begin{aligned} \delta J_2 &= \left(H(t_1^-) - \vec{\Lambda}(t_1^-)^T \dot{\vec{x}}(t_1^-)\right)\delta t_1 \\ &- \vec{\Lambda}(t_1^-)^T \delta\vec{x}(t_1^-) \\ &+ \int_0^{t_1^-} \left(\left(\frac{\partial H}{\partial \vec{x}} + \dot{\vec{\Lambda}}\right)^T \delta\vec{x} + \frac{\partial H}{\partial \vec{u}}^T \delta\vec{u}\right) dt \end{aligned} \tag{39}$$

Let $\delta\vec{x}(t_1^-) = (\delta\vec{\sigma}(t_1^-), \delta\vec{\omega}(t_1^-))^T$. Then the state variations at $t_1^- + \delta t_1$ are defined as[7,8]

$$\delta\vec{\sigma}_- = \delta\vec{\sigma}(t_1^- + \delta t_1) = \delta\vec{\sigma}(t_1^-) + \dot{\vec{\sigma}}(t_1^-)\delta t_1 \quad (40)$$

$$\delta\vec{\omega}_- = \delta\vec{\omega}(t_1^- + \delta t_1) = \delta\vec{\omega}(t_1^-) + \dot{\vec{\omega}}(t_1^-)\delta t_1 \quad (41)$$

which reduces δJ_2 to the simple form of

$$\begin{aligned}\delta J_2 &= H\left(t_1^-\right)\delta t_1 \\ &- \vec{\Lambda}_{1-}^T\delta\vec{\sigma}_- - \vec{\Lambda}_{2-}^T\delta\vec{\omega}_- \\ &+ \int_0^{t_1^-}\left(\left(\frac{\partial H}{\partial \vec{x}} + \dot{\vec{\Lambda}}\right)^T\delta\vec{x} + \frac{\partial H}{\partial \vec{u}}^T\delta\vec{u}\right)dt\end{aligned} \quad (42)$$

Similarly δJ_3 can be found assuming that the initial and final states and the time t_1 are free.

$$\begin{aligned}\delta J_3 &= \vec{\Lambda}_{1+}^T\delta\vec{\sigma}_+ + \vec{\Lambda}_{2+}^T\delta\vec{\omega}_+ \\ &- H\left(t_1^+\right)\delta t_1 - \vec{\Lambda}(t_f)^T\delta\vec{x}_f \\ &+ \int_{t_1^+}^{t_f}\left(\left(\frac{\partial H}{\partial \vec{x}} + \dot{\vec{\Lambda}}\right)^T\delta\vec{x} + \frac{\partial H}{\partial \vec{u}}^T\delta\vec{u}\right)dt\end{aligned} \quad (43)$$

Since the body angular velocity is continuous $\delta\vec{\omega}_- = \delta\vec{\omega}_+ = \delta\vec{\omega}$. After enforcing the optimality and transversality conditions, the total variation δJ becomes

$$\begin{aligned}\delta J &= \vec{\Lambda}_{1+}^T\delta\vec{\sigma}_+ - \vec{\Lambda}_{1-}^T\delta\vec{\sigma}_- \\ &+ \left(\vec{\Lambda}_{2+} - \vec{\Lambda}_{2-}\right)^T\delta\vec{\omega} \\ &- \left(H\left(t_1^+\right) - H\left(t_1^-\right)\right)\delta t_1 = 0\end{aligned} \quad (44)$$

Since the variations $\delta\vec{\omega}_1$ and δt_1 in Eq. (44) are independent from other variations, the following conclusions can be made.

$$\vec{\Lambda}_{2+} = \vec{\Lambda}_{2-} \quad (45)$$

$$H(t_1^+) = H(t_1^-) \quad (46)$$

Before any conclusions can be made about $\vec{\Lambda}_{1-}$ and $\vec{\Lambda}_{1+}$, further development is needed to establish what constitutes an admissible variation $\delta\vec{\sigma}_-$ and $\delta\vec{\sigma}_+$, what is their relationship and if they are non-zero.

It is assumed that at time t_1^- the optimal attitude $\vec{\sigma}_-^*$ is on the constraint surface $\vec{\sigma}^T\vec{\sigma} = c^2$. Let $\vec{\sigma}(t_1^- + \delta t_1)$ be a variation of $\vec{\sigma}_-^*$. Since the variation of $\vec{\sigma}_-^*$ must also be on the unit sphere surface, the following condition must hold

$$|\vec{\sigma}_-^*|^2 = |\vec{\sigma}(t_1^- + \delta t_1)|^2 = c^2 \quad (47)$$

Let $\vec{\sigma}(t_1 + \delta t_1)$ be related to the optimal $\vec{\sigma}_-^*$ through

$$\vec{\sigma}(t_1 + \delta t_1) = \vec{\sigma}^*(t_1) + \delta\vec{\sigma}_1 \quad (48)$$

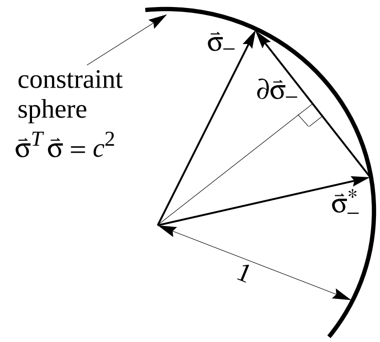

Fig. 3 Constraint Illustration of $\delta\vec{\sigma}_1$

as illustrated in Fig. 3. The variation $\delta\vec{\sigma}_-$ must be such that $\vec{\sigma}_-$ still lies on the unit sphere. Using Eqs. (47) and (48) it can be shown that

$$\left(\vec{\sigma}_-^* + \frac{1}{2}\delta\vec{\sigma}_-\right)^T\delta\vec{\sigma}_- = 0 \quad (49)$$

The condition in Eq. (49) is satisfied if the orthogonality is satisfied or if $\delta\vec{\sigma}_-$ is zero. The variation $\delta\vec{\sigma}_+$ must satisfy the same type of condition. For general rotations Eq. (49) is satisfied through the orthogonality condition. However, if a single-axis rotations is being done, then Eq. (49) is satisfied by forcing $\delta\vec{\sigma}_-$ and $\delta\vec{\sigma}_+$ to be zero.

Lemma: The variations $\delta\vec{\sigma}_-$ and $\delta\vec{\sigma}_+$ must be zero if a single-axis rotation is being performed.

Proof: From the definition of the MRPs in Eq. (14) it is clear that for a single-axis rotation all MRP vectors will lie along the constant principal rotation vector. This straight line will touch the unit sphere constraint surface only at two points as illustrated in Figure 4. It is impossible to be at such a surface point and have a small variation while remaining on the constraint surface. ∎

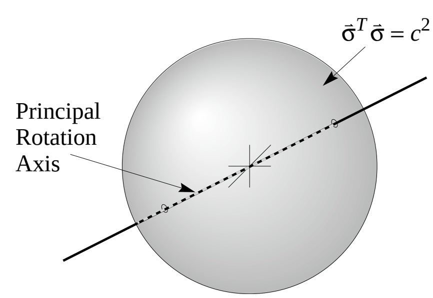

Fig. 4 Lemma Illustration

Since a switch occurs at t_1, $\vec{\sigma}^+$ would be the "shadow" set of $\vec{\sigma}^-$. This mapping from $\vec{\sigma}^-$ to $\vec{\sigma}^+$ is well defined in Eq. (15), therefore the variations $\delta\vec{\sigma}_-$

and $\delta\vec{\sigma}_+$ must be related. Their relative mapping is found by taking the first variation of Eq. (15).

$$\delta\vec{\sigma}_- = \left[2\vec{\sigma}_-\vec{\sigma}_-^T - \vec{\sigma}_-^T\vec{\sigma}_- I\right]\delta\vec{\sigma}_+ \tag{50}$$

Let us first examine the case where $\delta\vec{\sigma}_-$ and $\delta\vec{\sigma}_+$ are not zero. In this case Eq. (44) shows that

$$\vec{\Lambda}_{1+}^T\delta\vec{\sigma}_+ = \vec{\Lambda}_{1-}^T\delta\vec{\sigma}_- \tag{51}$$

which can be expanded using Eq. (50) to

$$\left(\vec{\Lambda}_{1+} - \left[2\vec{\sigma}_-\vec{\sigma}_-^T - \vec{\sigma}_-^T\vec{\sigma}_- I\right]\vec{\Lambda}_{1-}\right)^T\delta\vec{\sigma}_+ = 0 \tag{52}$$

Since Eq. (52) must hold for any admissible variations $\delta\vec{\sigma}_+$ the following costate switching condition is found.

$$\vec{\Lambda}_{1+} = \left[2\vec{\sigma}_-\vec{\sigma}_-^T - \left(\vec{\sigma}_-^T\vec{\sigma}_-\right) I\right]\vec{\Lambda}_{1-} \tag{53}$$

Note that Eq. (53) yields a general mapping for the switching of a costate $\vec{\Lambda}_{1-}$ to its "shadow" costate $\vec{\Lambda}_{1+}$. This mapping is valid for any switching condition $\vec{\sigma}^T\vec{\sigma} = c^2$, but has its simplest form if the switching surface $\vec{\sigma}^T\vec{\sigma} = 1$ is chosen.

Eq. (46) provides another condition that must be satisfied for optimality. Since the attitude penalty functions $g(\vec{\sigma})$ and $G(\vec{\sigma})$ are such that $p(t_1^-) = p(t_1^+)$, Eq. (46) can be further reduced using Eqs. (45) to

$$\vec{\Lambda}_{1-}^T\dot{\vec{\sigma}}(t_1^-) = \vec{\Lambda}_{1+}^T\dot{\vec{\sigma}}\left(t_1^+\right) \tag{54}$$

By making use of the costate switching condition in Eq. (53) and of Eq. (16), the above condition can be shown to be always true. Therefore Eq. (46) provides no further information for the case where $\delta\vec{\sigma}_-$ and $\delta\vec{\sigma}_+$ are non-zero.

By our Lemma, if the optimal rotation is a single-axis rotation then $\delta\vec{\sigma}_- \equiv 0$ and $\delta\vec{\sigma}_+ \equiv 0$. Because of this, Eq. (44) does not reveal any information about the costate $\vec{\Lambda}_1$ at t_1. To derive the necessary costate switching condition for $\vec{\Lambda}_1$ Eq. (46), which was reduced to Eq. (54), will be used.

Let $\hat{e}$ be the constant axis of rotation. Using Eq. (15) the attitude vector can be written as $\vec{\sigma} = \hat{e}|\vec{\sigma}| = \hat{e}\sigma$. The body angular velocity vector is given by $\vec{\omega} = \hat{e}|\vec{\omega}| = \hat{e}\omega$. Using Eqs. (16) and (17), the following costate switching condition can be found for the single-axis rotation case.

$$\vec{\Lambda}_{1+} = \left(\vec{\sigma}_-^T\vec{\sigma}_-\right)\vec{\Lambda}_{1-} \tag{55}$$

The above condition shows that the only instance for which $\vec{\Lambda}_1$ does not have a discontinuity during the switching is the case of a single-axis rotation with the switching surface $\vec{\sigma}^T\vec{\sigma} = 1$. Note that even though the costate switching condition in Eq. (53) was not derived for the case of a pure single-axis rotation, it does simplify to Eq. (55) when a single-axis rotation is imposed. This allows the costate switching condition for both cases to be unified into one costate switching condition.

Theorem (MRP Costate Switching Condition) Let the MRP switching surface be $\vec{\sigma}^T\vec{\sigma} = c^2$ and let the attitude penalty function be continuous with respect to $\vec{\sigma}$, then the costate $\vec{\Lambda}_2$ will remain continuous during the switching of the MRPs to their "shadow" set. The costate $\vec{\Lambda}_1$ however will have a discontinuity defined by Eq. (53).

This theorem leads directly to the following corollary regarding the costate magnitude $|\vec{\Lambda}_1|$ during the MRP switching.

Corollary: The costate magnitude $|\vec{\Lambda}_1|$ will remain continuous during the MRP switching if the $\vec{\sigma}^T\vec{\sigma} = 1$ switching surface is used.

Proof: This corollary is verified by using the theorem to find $\vec{\Lambda}_1^T\vec{\Lambda}_1$ before and after the MRP switching and using the fact that during the switching $\vec{\sigma}^T\vec{\sigma}$ is equal to 1. ■

This corollary shows that the MRP costates $\vec{\Lambda}_1$ behave very similarly to the MRP during the switching. Both switch states on the surface of a sphere. The difference is that the MRP switch on a unit sphere, where the MRP costates $\vec{\Lambda}_1$ switch on a sphere of arbitrary radius.

VI. Single-Axis Analytical Result

To verify the MRP costate switching conditions, a simple single-axis optimal control problem is solved analytically using the MRPs as an attitude parameters. For generality, the switching surface is set to $\sigma^2 = c^2$. Lets minimize the cost function J which depends solely on the control u

$$J = \int_0^1 u^2 dt \tag{56}$$

subject to the simple one-dimensional equations of motion for a body with unit inertia

$$\dot{\sigma} = \frac{1}{4}\left(1 + \sigma^2\right)\omega \tag{57}$$

$$\dot{\omega} = u \tag{58}$$

and subject to the state constraints

$$\begin{aligned}\sigma(t_0 = 0) = \sigma_0 \qquad \sigma(t_f = 1) = \sigma_f \\ \omega(t_0) = \omega(t_f) = 0\end{aligned} \tag{59}$$

The optimal control torque u^* for this cost function J is known to be of the form

$$u^*(t) = k(1-2t) \tag{60}$$

where k is simply a scaling factor that guarantees that the body is at σ_f at t_f. Note that the optimal trajectory is independent of the choice of attitude coordinates. This allows the optimal control problem to be solved using either the "original" MRP set or their "shadow" set. By comparing the resulting costate Λ_1 history for the "original" and "shadow" costates, we will be able to verify the costate switching condition for single-axis rotations in Eq. (55). The optimality condition in Eq. (10) states that

$$u^*(t) = -\Lambda_2(t) \tag{61}$$

Since u^* is continuous, so is Λ_2 as predicted in Eq. (45). If at some point in time the MRP are switched to their "shadow" set it obviously has no effect on the continuity of Λ_2.

To find a time history of the costate Λ_1, Eq. (9) is used.

$$\dot{\Lambda}_2 = -\frac{1}{4}\left(1+\sigma^2\right)\Lambda_1 \tag{62}$$

Since $\dot{\Lambda}_2 = 2k$ this can be solved for Λ_1.

$$\Lambda_1 = -\frac{8k}{1+\sigma^2} \tag{63}$$

Let σ^S and Λ_1^S be the "shadow" attitude and costate. Analogously to above, the solution for Λ_1^S would be

$$\Lambda_1^S = -\frac{8k}{1+\left(\sigma^S\right)^2} \tag{64}$$

Eq. (64) can be written in terms of σ by using Eq. (15).

$$\Lambda_1^S = -\frac{8k}{1+\frac{1}{\sigma^2}} = -\frac{8k}{1+\sigma^2}\sigma^2 \tag{65}$$

Substituting Eq. (63) into Eq. (65) a direct relationship between Λ_1^S and Λ_1 is obtained.

$$\Lambda_1^S = \sigma^2\Lambda_1 \tag{66}$$

By switching between the two possible MRP attitude description the costate Λ_1 would have to be switched as well according to Eq. (66). This result verifies the single-axis rotation MRP costate switching condition found in Eq. (55).

VII. 3-D Numerical Result

To verify the general transformation given in the MRP costate switching condition theorem, a three-dimensional optimal control problem was solved as outlined in the problem statement. The attitude penalty function was chosen to be the *g()* given in Eq. (20). With this penalty function the answer did not depend on the attitude coordinate choice. Therefore the optimal solution using the combined set of $\vec{\sigma}$ and $\vec{\sigma}^S$ should be the same as the optimal solution obtained by using only $\vec{\sigma}$ or $\vec{\sigma}^S$.

The optimization problem was solved numerically by a steepest descent gradient method. The only modification needed to use the combined set of original and shadow MRP vectors was to check whether $\vec{\sigma}^T\vec{\sigma}$ had grown larger than one. If yes, then the attitude vector was switched to its shadow counter part. At the same time the corresponding attitude costate vector was also switched to its shadow counter part using the MRP costate switching condition.

The three-dimensional optimal control problem had a fixed maneuver time of $t_f = 10$ seconds. The body inertia matrix was $\Im = diag(0.5, 1.0, 0.7)$ kgm^2. The cost function weights were $K_1 = 2$, $K_2 = 10$, $K_3 = 1$, $K_4 = 5$ and $R = 20$. The initial states were $\vec{\sigma}(0) = (0.87, 0, 0)$ and $\vec{\omega}(0) = (80.21, 51.57, 45.84)°/\text{s}$. Note that the initial orientation has the body almost turned up-side-down with a large initial angular velocity driving it to the up-side-down orientation. This optimal control problem will penalize any non-zero state and torque during the maneuver and any non-zero final state. Note that the final state is left free though. Trying to minimize torque for this maneuver, it would be intuitively reasonable to let the body rotate trough the up-side-down orientation and then reduce the states instead of forcefully reversing the existing motion. We show key results in Figures 5–8.

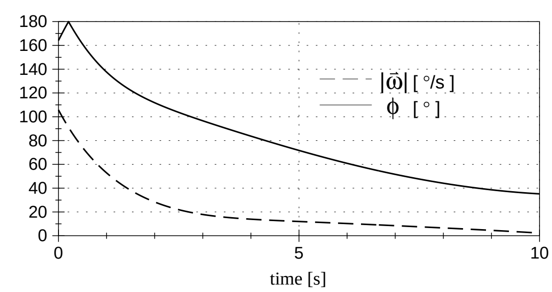

Fig. 5 Optimal States for all Three Cases

Three separate optimal control problems were solved using either $\vec{\sigma}/\vec{\sigma}^S$, $\vec{\sigma}$ or $\vec{\sigma}^S$ as the attitude coordinates. As expected, all three optimizations converged to the same solution. The principal rotation angle ϕ and the magnitude of the angular velocity are shown in Figure 5. The optimal solution indeed let the body rotate through the $\phi = 180°$ point and diminished the angular velocity and attitude at the final maneuver time.

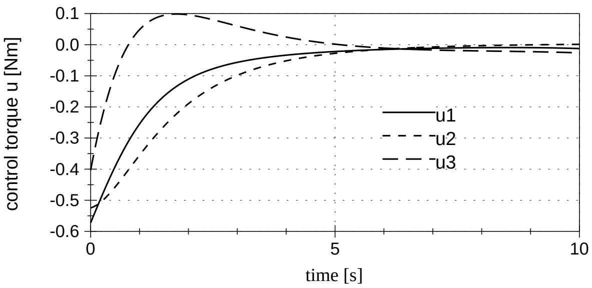

Fig. 6 Optimal Control Torque $\vec{u}$

The optimal control torque for the maneuver is shown in Figure 6. The attitude coordinate vector time histories were different for each problem, since different attitude coordinates were used. The combined set $\vec{\sigma}/\vec{\sigma}^S$ started out identical to $\vec{\sigma}$, since the initial attitude vector had less than unit magnitude. As $|\vec{\sigma}|$ grew larger than one, the combined set $\vec{\sigma}/\vec{\sigma}^S$ trajectory is switched to the shadow set $\vec{\sigma}^S$ trajectory. This is illustrated in Figure 7. The black

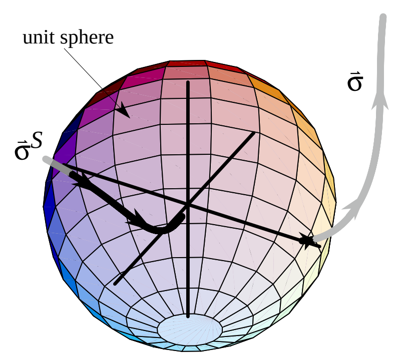

Fig. 7 3-D Illustration of Attitude Vectors

line denotes the trajectory of the combined $\vec{\sigma}/\vec{\sigma}^S$ set which remains within the unit sphere. Note that this trajectory converged exactly with the $\vec{\sigma}$ and $\vec{\sigma}^S$ trajectories whenever they too were within the unit sphere.

The ultimate test of the MRP costate switching condition theorem is to see if the costate $\vec{\lambda}_1$ exhibits the same behavior. Its trajectories are shown in Figure 8. Again the black line is the solution obtained using the combined $\vec{\sigma}/\vec{\sigma}^S$ set and using the MRP costate switching condition. Indeed the costate $\vec{\Lambda}_1$ switches exactly from the costate trajectory of the pure $\vec{\sigma}$ solution to the costate trajectory of the pure $\vec{\sigma}^S$ solution, thus verifying the theorem presented in this paper.

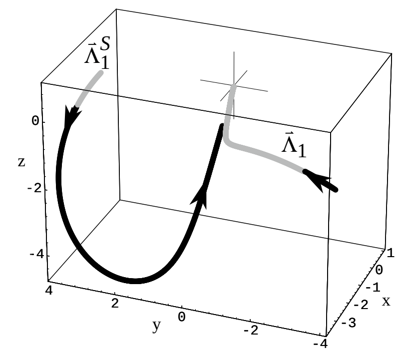

Fig. 8 3-D Illustration of Costate $\vec{\Lambda}_1$

VIII. Conclusion

An universal attitude penalty function *g()* for optimal control problems is presented which makes the optimization independent of the choice on attitude coordinates. This function also has other beneficial properties such as being bounded between 0 and 1 and being non-singular. Another attitude penalty function *G()* was presented which made use of many good properties of the MRP. The MRP costate switching condition introduced in this paper makes the use of MRP combined with their "shadow" set possible for general motion optimal control problems.

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