$$\frac{dR}{dt} = \sqrt{\frac{8\pi G\rho}{3R} + \frac{\Lambda}{3}R^2}$$

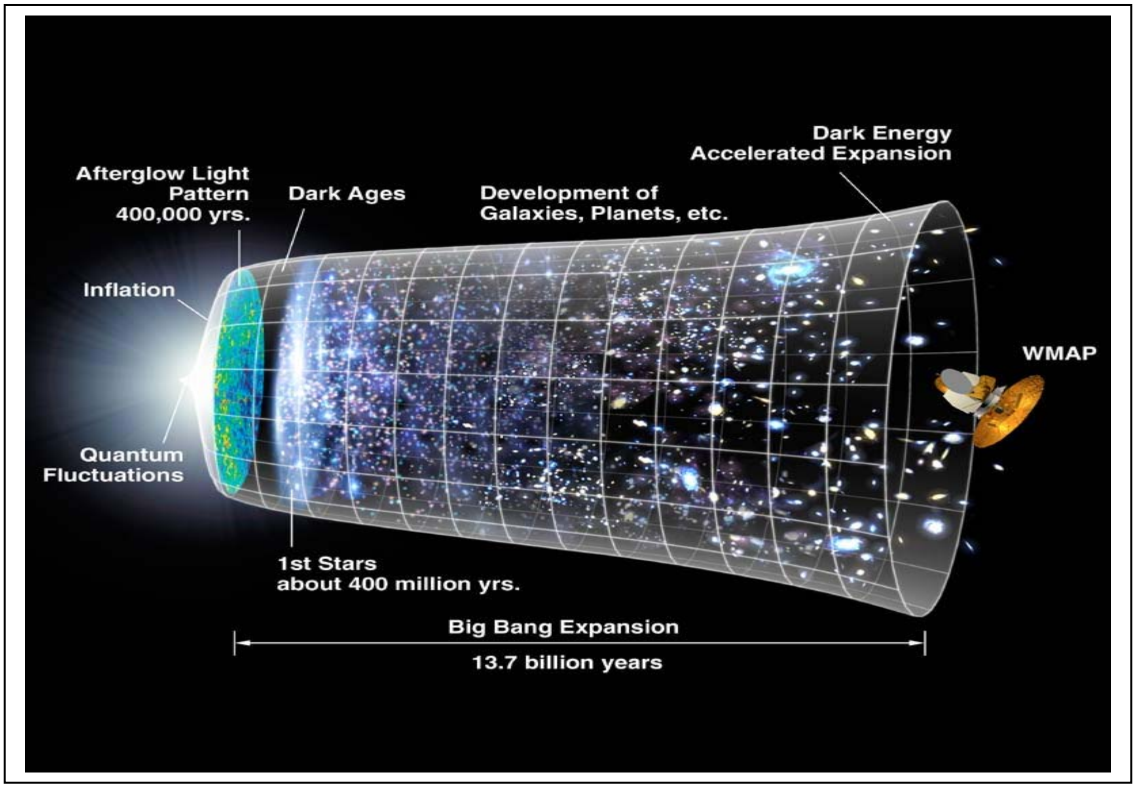
According to Big Bang theory, the scale of the universe increases with time at a rate that depends on the density of matter, ρ, and the size of the cosmological constant, Λ. This is defined by the fundamental equation to the left.

Problem 1 - Determine the general form of the integral that relates the time, t, to the value of the scale factor, R; Solve the integral for the time, t, but do not solve the integral for R.

Problem 2 - Transform the integral for R to a new variable, U, such that $U = (A/C)^{1/3} R$ where $A = \Lambda/3$ and $C = 8\pi G\rho/3$.

Problem 3 - Solve the integral for two special cases A) The Inflationary Universe case where U >> 1 and B) the matter-dominated universe case where U << 1.

Problem 4 - Hubbel's Constant is a measure of the rate of expansion of the universe. It is defined as H = 1/R (dR/dt). Find the formula for Hubbel's Constant for the two cosmological cases described in Problem 3.



Answer Key

Problem 1 - The integral equation is then

$$\int dt = \int \frac{dR}{\sqrt{\frac{8\pi G\rho}{3R} + \frac{\Lambda}{3}R^2}}$$

Problem 2 First clean up the rather cumbersome radical expression so that it only involves R to positive powers and the constants A and C, by factoring out $(1/R)^{1/2}$ to get $(1/R)^{1/2}(C + A R^3)^{1/2}$. Factor out the constant C from the square-root so that the denominator of the integrand becomes $(1/R)^{1/2} C^{1/2} (1 + A/C R^3)^{1/2}$ and replace with $U = (A/C)^{1/3} R$ to get

$$C^{1/2} (A/C)^{1/6} U^{-1/2} (1 + U^3)^{1/2}$$

Note that we have also transformed the $(1/R)^{1/2}$ factor by replacing it with $(A/C)^{1/6} U^{-1/2}$. Since $dU = (A/C)^{1/3} dR$, we can now re-write the complete integrand as

$$(1/C)^{1/2} (C/A)^{1/6} (C/A)^{1/3} U^{1/2} dU / (1 + U^3)^{1/2}$$

After combining the constants A and C and replacing them with their definitions the integrand simplifies to $(3/\Lambda)^{1/2} U^{1/2} dU / (1 + U^3)^{1/2}$ and the integral becomes

$$t = \sqrt{\frac{3}{\Lambda}} \int \frac{U^{1/2} dU}{\sqrt{U^3 + 1}}$$

Problem 3 A) If U > 1, then the term under the square-root is essentially U^3, so we get $U^{1/2} / U^{3/2} = 1/U$. This leads to an integrand of $(3/\Lambda)^{1/2}$ 1/U dU which is a fundamental integral whose solution is $t = (3/\Lambda)^{1/2} \ln U + C$. This can be re-written as $U(t) = e[(\Lambda/3)^{1/2} t]$. From the definition for U we get

$$R(t) = \left(\frac{8\pi G\rho}{\Lambda}\right)^{\frac{1}{3}} e^{\left(\frac{\Lambda}{3}\right)^{1/2} t}$$

This represents a universe that expands at an exponential rate because of the positive pressure provided by the cosmological constant - a property of the energy of empty space. This solution is thought to describe our universe during its' inflationary' era shortly after the Big Bang.

Problem 3 B) In this case, U << 1 so the term under the square-root is essentially 1, and the integrand becomes $(3/\Lambda)^{1/2} U^{1/2} dU$. This is easily integrated to get $t = (3/\Lambda)^{1/2} U^{3/2}$. After substituting for the definition of U we get $t = (3/\Lambda)^{1/2} (\Lambda/8\pi G\rho)^{1/2} R^{3/2}$ so that $t = (3/8\pi G\rho)^{1/2} R^{3/2}$. This can be easily inverted to get

$$\mathbf{R(t) = (8\pi G\tilde{\rho} / 3)^{1/3}\ t^{2/3}}$$

This solution is the 'matter-dominated' cosmology represented by Big Bang cosmology, and applies to the modern expansion of the universe.

Problem 4 A) $\mathbf{H = (\Lambda/3)^{1/2}}$ and B) **H = 2/3 (1/t)**. In the inflationary case, the rate of expansion is constant in time, but in the matter-dominated case, the expansion rate decreases in proportion to the age of the universe !