Market Efficiency and Inefficiency in Rational Expectations Equilibria: Dynamic Effects of Heterogeneous Information and Noise

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Abstract: The paper examines time series properties and efficiency of a securities market where disparately informed traders hold rational expectations and extract signals from the endogenous market price. Two equilibria are calculated, using a method of Sargent to handle the problem of infinite regress. When rational speculation is the sole source of potential trade, the market price reflects all private information, and zero trade occurs. When net supply is perturbed by unobserved noise, the market exhibits a broad range of characteristics cited in empirical literature, including excess volatility, mean reversion, dividend yield effects, trading volume and divergence of opinion.

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1.1. Introduction

Are securities prices too volatile? This question is central to evaluating how efficiently resources are allocated in a competitive market. A large body of literature suggests that securities markets will generate prices which reflect all available information, even if some traders are better informed than others. Recent empirical studies place this market efficiency result in dispute, and suggest that the volatility of securities prices is too great to be explained by volatility in expected future payouts. For example, the decline of over 20% in the Dow Jones Industrial Average on October 19, 1987 seems difficult to explain as a rational downward revision in expected future dividend streams. Also contrary to the implications of the efficient markets literature, investors commonly disagree about future returns, and this disagreement generates nonzero trading volume. This paper examines the efficiency of financial markets in the context of a dynamic model where traders have heterogeneous information, and extract additional information from the endogenous market price. Market efficiency and inefficiency emerge as special cases of this model.

The focus of the paper is two-fold. First, an approach of Sargent (1991) is modified to compute dynamic rational expectations equilibria in a securities market where individuals hold imperfect private information about "fundamentals" which are correlated over time. The approach is of significant interest in and of itself. Prior attempts to study such equilibria have encountered the problem of infinite regress, which emerges when traders must "forecast the forecasts of forecasts . . . " of others. By neatly circumventing the problem of infinite regress, rational expectations equilibria may be rapidly computed under a wide variety of informational assumptions and market structures.

Second, the resulting equilibria are examined in light of previous literature on market efficiency [including Grossman (1976), Radner (1979), Tirole (1982)], trading volume [Milgrom and Stokey (1982)], excess volatility [Shiller (1981)], mean reversion [DeBondt and Thaler (1985)], dividend yield effects [Fama and French (1987)] and noise trading [Grossman (1976), Campbell and Kyle (1988)]. In an equilibrium where fully rational traders are endowed with imperfect private information, the rational expectations equilibrium exhibits strong market efficiency, and zero trading volume. However, when the market is contaminated by a small amount of noise in the net supply of the security available to rational traders, market efficiency fails to hold. In these “noisy” rational expectations equilibria, the market exhibits a wide range of anomalies such as mean-reversion, “excess” volatility, trading volume, and significant correlation between dividend yields and subsequent returns.

With the notable exception of Singleton (1987), models of heterogeneous information in securities markets have generally been confined to a two-period market structure. When information is imperfect and true market fundamentals are correlated across time, the problem of infinite regress creates difficulties in obtaining rational expectations equilibria. Consider two classes of traders, a and b, each having access to private, but imperfect information. Each class has an incentive to infer the information of the other class by observing the market price. If the beliefs of class a are modelled as state variables which are unobserved by class b, then the beliefs of class b about the beliefs of class a are also unobserved state variables to class a, and so on. As Sargent (1991) notes, such an approach causes the dimension of the state vector describing the economy to become infinite.

Townsend (1983) and Singleton (1987) have approached this problem by limiting the information structure so that the true state variables in the economy are observed by all agents after a lag of j periods. Using a method of undetermined coefficients, Townsend analyzes a model motivated by Pigou (1929), where firms extract signals about the true state of the economy by observing endogenous output prices. Singleton extends Townsend’s approach to study a model where traders infer information from endogenous asset prices, and asset supplies are perturbed by unobserved and serially correlated shocks.

Sargent (1991) describes a method for computing equilibria for the case where $j = \infty$; that is, where the true state variables of the economy are never perfectly revealed to agents. The method is used to compute the equilibria presented in sections 1.4 and 1.5 of this paper. Instead of modelling the beliefs of each class of agents as unobserved state variables, agents are modelled as forecasting by fitting vector $ARMA$ models for all information available to them, including endogenous variables such as prices. By imposing market clearing, these forecasts generate the actual laws of motion for the endogenous variables in the economy. Equilibrium is then defined as the fixed point of a finite dimensional operator that maps perceived laws of motion to actual laws of motion. The method draws on Muth’s (1960) observation that the space spanned by a pure infinite order autoregression may, in some cases, be replicated by a finite order $ARMA$ process.

The paper is structured as follows. Section 1.2 describes a simple stock market in which dividends are the sum of persistent and purely temporary components. An analytical solution for the stock price is derived, yielding a "full information" equlibrium. It will be useful to compare this solution with subsequent equilibria calculated under imperfect information. Section 1.3 introduces this imperfect information in the form of noise-ridden signals received by traders about various components of the dividend process. A rational expectations equilibrium is defined, and the method for computing this equibrium is described.

Section 1.4 analyzes the rational expectations equilibrium obtained in a market which is free of "noise" in the net supply of the security. This equilibrium is found to be consistent with a large volume of literature regarding market efficiency. Section 1.5 analyzes a "noisy" rational expectations equilibrium. This equilibrium is found to exhibit a number of inefficiencies and anomalies which have been cited in empirical literature on the U.S. stock market. Section 1.6 concludes the paper, and suggests extensions for further research.

1.2. Market equilibrium with complete information

Consider a stock market in which the process for dividends is driven by persistent and purely temporary components. The persistent components follow first order autoregressive processes. The most persistent component is denoted θ_{1t}. A second, less persistent component is denoted θ_{2t}, and a third, purely transitory component ϵ_t also perturbs the dividend process. The dividend D_t is the sum of these three components.

$$D_t = \theta_{1t} + \theta_{2t} + \epsilon_t \tag{1}$$

The shocks to each component, ν_{1t}, ν_{2t} and ϵ_t are mutually orthogonal white noise innovations with mean zero and variances $\sigma^2_{\nu_1}$, $\sigma^2_{\nu_2}$, and σ^2_{ϵ}. The persistence of shocks to θ_{1t} and θ_{2t} is governed by the parameters ρ_1 and ρ_2.

$$\begin{aligned} \theta_{1t} &= \rho_1 \theta_{1t-1} + \nu_{1t} \\ \theta_{2t} &= \rho_2 \theta_{2t-1} + \nu_{2t} \end{aligned} \tag{2}$$

In an economy with complete information, each trader observes each of these components directly, so that no trader faces a problem of signal extraction. There are N traders, indexed by j. Following Grossman (1976), traders are assumed to have constant absolute risk aversion (exponential utility). This assumption results in security demand functions which are conveniently linear in price, with the drawback that they are independent of wealth. Each trader chooses a quantity Q_t^j of shareholdings to maximize the expected utility of next-period wealth W_{t+1}^j. A share of stock is a claim on the infinite stream of future dividends. The share price is denoted P_t, and the constant gross interest rate is denoted R. Each trader then faces

$$\max_{Q_t^j} E_t^j \left[-\exp\left(\frac{-W_{t+1}^j}{\phi} \right) \right] \tag{3}$$

subject to

$$W_{t+1}^j = RW_t^j + Q_t^j \left(P_{t+1} + D_{t+1} - RP_t \right) \tag{4}$$

where ϕ is a constant coefficient of risk tolerance (the inverse of risk aversion), and E_t^j denotes the expectation of trader j conditional on the set of information Ω_t^j available to j at time t.

The solution of this problem has a simple return/risk interpretation. Traders hold a position that is linear in the difference between the expected return on stocks and the rate of interest. Risk aversion and the existence of forecast errors prevent traders from taking an infinite position based on expected return differentials. The demand of each trader is

$$Q_t^j = \frac{\phi \, E_t^j \left[P_{t+1} + D_{t+1} - RP_t \right]}{\sigma^2_{P_{t+1}+D_{t+1}} \,|\, \Omega_t^j} \tag{5}$$

where the denominator is simply the variance of forecast errors for trader j, conditional on information available at time t. Since all traders share the same information set, and therefore hold identical expectations, the denominator may be abbreviated to σ^2, and the index j may be omitted from the expectations operator.

The total amount of stock is denoted $\bar{Q}$. Market clearing then implies

$$P_t = R^{-1}\left(E_t[P_{t+1} + D_{t+1}] - \frac{\bar{Q}\sigma^2}{N\phi}\right) \tag{6}$$

which may be solved forward to yield

$$P_t = E_t\left[\sum_{i=1}^{\infty} R^{-i} D_{t+i}\right] - \Psi \tag{7}$$

where

$$\Psi = \frac{\bar{Q}\sigma^2}{N\phi(R-1)}. \tag{8}$$

Note that the equilibrium price is simply the present discounted value of expected future dividends, less a constant risk factor Ψ which is linear in the total supply of the security. Using the lag operator L as outlined in Appendix 1A at the end of this chapter, the stock price may then be expressed as a function of current and past shocks to the dividend process:

$$P_t = \frac{\rho_1}{(R-\rho_1)(1-\rho_1 L)}\nu_{1t} + \frac{\rho_2}{(R-\rho_2)(1-\rho_2 L)}\nu_{2t} - \Psi \tag{9}$$

Finally, as a benchmark for evaluating the equilibria calculated in subsequent sections, it will be useful to derive analytical expressions for price and dividend variances under full information. Using (1), (2) and (9), the expected return on the stock in equilibrium is found to be

$$E_t\left[P_{t+1} + D_{t+1}\right] = RP_t + \frac{\bar{Q}\sigma^2}{N\phi}, \tag{10}$$

the variance of forecast errors σ^2 is

$$\sigma^2 = \left(\frac{R}{R-\rho_1}\right)^2 \sigma_{\nu_1}^2 + \left(\frac{R}{R-\rho_2}\right)^2 \sigma_{\nu_2}^2 + \sigma_\epsilon^2, \tag{11}$$

the covariance of the stock price and dividend is

$$Cov(P_t, D_t) = \left(\frac{\rho_1}{R-\rho_1}\right)\left(\frac{1}{1-\rho_1^2}\right)\sigma_{\nu_1}^2 + \left(\frac{\rho_2}{R-\rho_2}\right)\left(\frac{1}{1-\rho_2^2}\right)\sigma_{\nu_2}^2, \tag{12}$$

the variance of the stock price is

$$Var(P_t) = \left(\frac{\rho_1}{R-\rho_1}\right)^2 \left(\frac{1}{1-\rho_1^2}\right)\sigma_{\nu_1}^2 + \left(\frac{\rho_2}{R-\rho_2}\right)^2 \left(\frac{1}{1-\rho_2^2}\right)\sigma_{\nu_2}^2, \qquad (13)$$

and the variance of the dividend is

$$Var(D_t) = \left(\frac{1}{1-\rho_1^2}\right)\sigma_{\nu_1}^2 + \left(\frac{1}{1-\rho_2^2}\right)\sigma_{\nu_2}^2 + \sigma_\epsilon^2. \qquad (14)$$

1.3. Computing market equilibria with imperfect information and signal extraction

In the foregoing section, it was possible to derive an analytical solution for price because all variables were directly observable and traders shared the same information set. When the underlying components of the dividend process θ_{1t}, θ_{2t} and and ϵ_t are not directly observable, it is still possible to derive an analytical solution for price by treating these components as unobserved state variables and using equations associated with the Kalman filter. Unfortunately, this technique breaks down when information is both imperfect and heterogeneous. In this case, traders have an incentive to extract information from the endogenous price. If the beliefs of one class of traders are modelled as hidden state variables to each other class of traders, the problem of infinite regress causes the state vector associated with the Kalman filtering problem to become unbounded in dimension.

Sargent (1991) provides an alternative approach to calculating rational expectations equilibria when imperfectly informed agents extract signals from endogenous variables. The idea is to model agents as forecasting using vector $ARMA$ models for all variables which are observable. Equilibrium is calculated as the fixed point in the mapping from these perceived $ARMA$ processes to the actual $ARMA$ process induced by these perceptions. Sargent also provides a method for checking whether the $ARMA$ models which agents fit are "full order"; that is, whether agents would have an incentive to fit higher order models in equilibrium. When a full order equilibrium is achieved, the forecasts using the finite dimensional $ARMA$ models are identical to the forecasts which would be generated if agents were to condition on the infinite history of their observables. This section describes how to map a simple stock market into the objects described by Sargent.

Further details are presented in Appendix 1B at the end of this chapter. The notation and organization is largely consistent with the original paper to facilitate comparison.

There are two classes of rational traders, indexed by $j = a, b$. The total number of rational traders is N, with a proportion α_a in class a, and α_b in class b. The risk tolerance of each class, respectively, is denoted ϕ_a, and ϕ_b. In some versions of the model analyzed here, there is also a class of noise traders who demand a random quantity Q_t^n of the security. Following Grossman (1976) and others, this random element may alternatively be interpreted as unobserved variability in the aggregate supply of the security.

Information in this stock market is imperfect and heterogeneous. Traders in class a receive a signal S_t^a on the most persistent component of dividends θ_{1t}, contaminated by a white noise error term μ_t^a. Traders in class b receive a signal S_t^b on the less persistent component θ_{2t}, contaminated by a white noise error term μ_t^b.

$$
\begin{aligned}
S_t^a &= \theta_{1t} + \mu_t^a \\
S_t^b &= \theta_{2t} + \mu_t^b
\end{aligned}
\tag{15}
$$

Each trader j observes the stock price P_t, the private signal S_t^j, and the dividend D_t. The vector of these observables is denoted z_{jt}. So a trader in class a observes a record of current and past

$$
z_{at} = \begin{bmatrix} P_t \\ S_t^a \\ D_t \end{bmatrix}.
$$

Trader a has a perceived law of motion for z_{at} which is a first order vector $ARMA$ of the form

$$
z_{at} = A_a z_{at-1} + \zeta_{at} + C_a \zeta_{at-1} \tag{16}
$$

where ζ_{at} is the innovation in z_{at} conditional on the history ${z_a}^{t-1}$. This perceived law of motion may be written as

$$
\begin{bmatrix} z_{at} \\ \zeta_{at} \end{bmatrix} = \begin{bmatrix} A_a & C_a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_{at-1} \\ \zeta_{at-1} \end{bmatrix} + \begin{bmatrix} I \\ I \end{bmatrix} \zeta_{at} \tag{17}
$$

or

$$x_{at} = \beta_a x_{at-1} + \begin{bmatrix} I \\ I \end{bmatrix} \zeta_{at} \tag{18}$$

where $x_{at} = \begin{bmatrix} z_{at} \\ \zeta_{at} \end{bmatrix}$ and $\beta_a = \begin{bmatrix} A_a & C_a \\ 0 & 0 \end{bmatrix}$. The perceived laws for traders of class b are symmetric. Given these perceptions, traders form period-ahead forecasts according to

$$\begin{aligned} Ex_{at}|x_{at-1} &= \beta_a x_{at-1} \\ Ex_{bt}|x_{bt-1} &= \beta_b x_{bt-1} \end{aligned} . \tag{19}$$

The actual law of motion for price results from the asset demands arising from these expectations. In a rational expectations equilibrium, trader's perceived laws of motion must give rise to an actual law of motion consistent with those perceptions.

Sargent describes the approach to calculating a rational expectations equilibrium as follows. The state vector of the economy is denoted z_t. The state evolves according to

$$z_t = T(\beta) z_{t-1} + V(\beta)\varepsilon_t \tag{20}$$

The observables x_{at}, x_{bt} are subvectors of z_t. For a given set of perceptions $\beta = (\beta_a, \beta_b)$, the actual law of motion (20) may be used to obtain the projections of x_{jt} on x_{jt-1} for $j = a, b$. These projections are denoted

$$Ex_{jt}|x_{jt-1} = S_j(\beta) x_{jt-1} \tag{21}$$

where $S_j(\beta)$ is obtained using the linear least squares projection formula. Marcet and Sargent (1989a) define a limited information rational expectations equilibrium to be a fixed point where $\beta_a = S_a(\beta_a, \beta_b)$, $\beta_b = S_b(\beta_a, \beta_b)$.

The stock market model considered here fits conveniently into the fixed point mapping just described. We define state vector of the entire system as z_t, where

$$z_t = \begin{bmatrix} P_t \\ S_t^a \\ S_t^b \\ D_t \\ \theta_{1t} \\ \theta_{2t} \\ Q_t^n \\ \zeta_{at} \\ \zeta_{bt} \end{bmatrix} \quad (22).$$

The vector of shocks ε_t is defined as

$$\varepsilon_t = \begin{bmatrix} \mu_t^a \\ \mu_t^b \\ \nu_{1t} \\ \nu_{2t} \\ \epsilon_t \\ \omega_t \end{bmatrix} \quad (23)$$

where the ω_t is a serially uncorrelated shock to Q_t^n. The inclusion of Q_t^n in the state vector allows us to examine equilibria with serially correlated noise. When noise is serially uncorrelated, it may be excluded from the state. When there is no noise, ω_t may be omitted from the ε_t vector. It is convenient to consider price in terms of deviation from the mean, which corresponds to setting the supply constant $\bar{Q} = 0$.

Appendix 1B details the mapping from the perceptions of traders (β_a, β_b) to the actual law of motion for the state vector. The appendix includes equations for the covariance matrix M_z of the state vector, and the optimal projection laws $S(\beta) = (S_a(\beta), S_b(\beta))$. Equilibrium in this market is defined as follows:

Definition ▷ A rational expectations equilibrium with imperfect, heterogeneous information is the fixed point $\beta = S(\beta)$. This equilibrium satisfies the following conditions:

i) the equilibrium stockholdings Q_t^a, Q_t^b solve (3), (4) for traders a, b;

ii) the market for the security clears; $\bar{Q} = \alpha_a N Q_t^a + \alpha_b N Q_t^b + Q_t^n$.

The following sections present a number of equilibria computed using this method.

1.4. Market efficiency in a "pure" rational expectations equilibrium

The first equilibrium under consideration is a market unperturbed by noise in the supply of stock available to rational traders. Traders differ only in the signal which they receive about the dividend process. The issue of market efficiency is central to our investigation of this equilibrium. In the context of a speculative market where there is no insurance motive for trade, Tirole (1982) proves that the market price will fully communicate private information. The current price P_t must be equal to $R^{-1}E\left[P_{t+1}+D_{t+1}\right]$. This equality holds for every trader, so that no trader expects an excess return conditional on observing both the private signal and the market price. As Milgrom and Stokey (1982) have shown, the volume of trade in the security must be identically zero.

While Tirole's result characterizes the properties of a rational expectations equilibrium in the absence of supply noise, it does not show how to compute such an equilibrium. The fixed point approach outlined in section 1.3 allows us to address this question directly, by giving a convenient regression interpretation to the inference problems faced by traders.

The following parameter values are used to calculate a rational expectations equilibrium in the absence of supply noise.

Risk tolerances:	$\phi_a = 1$	$\phi_b = 1$
Trader proportions:	$\alpha_a = .5$	$\alpha_b = .5$
θ persistences:	$\rho_1 = .8$	$\rho_2 = .4$
Constants:	$N = 1$	$\bar{Q} = 0$
Gross interest rate	$R = 1.1000$	
Innovation covariances:		

$$E\begin{bmatrix}\mu_t^a\\ \mu_t^b\\ \nu_{1t}\\ \nu_{2t}\\ \epsilon_t\end{bmatrix}\begin{bmatrix}\mu_t^a\\ \mu_t^b\\ \nu_{1t}\\ \nu_{2t}\\ \epsilon_t\end{bmatrix}' = \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

Recall that traders fit first order $ARMA$ models for their observable variables. These models may be written

$$\begin{bmatrix}z_t^j\\ \zeta_t^j\end{bmatrix} = \beta_j\begin{bmatrix}z_{t-1}^j\\ \zeta_{t-1}^j\end{bmatrix} + \begin{bmatrix}I\\ I\end{bmatrix}\zeta_t^j$$

where

$$z_t^j = \begin{bmatrix} P_t \\ S_t^j \\ D_t \end{bmatrix} \qquad \beta_j = \begin{bmatrix} A_j & C_j \\ 0 & 0 \end{bmatrix}.$$

Using the parameter values given above, the equilibrium values are calculated to be

$$\beta_a = S_a = \begin{bmatrix} 1.3172 & -0.8551 & -0.5241 & 0.0317 & 0.3310 & 0.0000 \\ 0.6879 & -0.6413 & -0.3931 & 0.0238 & 0.2482 & 0.0000 \\ -0.2172 & 0.8551 & 0.5241 & -0.0317 & -0.3310 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$\beta_b = S_b = \begin{bmatrix} 0.8315 & -0.1626 & -0.0840 & 0.0011 & 0.0786 & 0.0000 \\ -0.0551 & 0.2845 & 0.1470 & -0.0019 & -0.1375 & 0.0000 \\ 0.2685 & 0.1626 & 0.0840 & -0.0011 & -0.0786 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

These equilibrium values reveal a striking fact. Even though traders have different information, and use different *formulas* to compute projections for their observables, both traders can be shown to hold identical *expectations* about future returns in equilibrium. The equilibruim exhibits full communication. To see this, note that for each trader j, the first row of β_j is the vector of coefficients used to compute the expectation of P_{t+1}. The third row of coefficients is used to compute the projection of D_{t+1}. Summing the first and third rows to compute $E[P_{t+1} + D_{t+1}]$ for each trader j, we notice that all of the coefficients exactly offset eachother, except the coefficient on the first element of $x_t^j : P_t$. The sum of these initial coefficients, not coincidentally, is 1.1000. This is the parameter value which was chosen for the gross interest rate R. Thus Tirole's result holds exactly. For both traders, $E[P_{t+1} + D_{t+1}] = RP_t$ so that no trader expects an excess return from trading on his information.

The moment matrix of the state vector M_z may be used to investigate the properties of this equilibrium in greater detail. For each trader, the covariance matrix of forecast errors ζ_t^j (corresponding to P_t, S_t^j and D_t) is a submatrix of M_z. The computed values are

$$E\zeta_t^a\zeta_t^{a\prime} = \begin{bmatrix} 6.5765 & 3.3428 & 3.8265 \\ 3.3428 & 2.2635 & 1.2172 \\ 3.8265 & 1.2172 & 3.2334 \end{bmatrix}$$

$$E\zeta_t^b\zeta_t^{b\prime} = \begin{bmatrix} 6.5765 & 0.4837 & 3.8265 \\ 0.4837 & 2.0625 & 1.0162 \\ 3.8265 & 1.0162 & 3.2334 \end{bmatrix}.$$

It is straightforward to show that

$$\sigma_a^2 = \sigma_b^2 = 17.4628$$

which is equal to the actual variance of $P_{t+1} + D_{t+1} - RP_t$. This confirms that excess returns (measured as the difference between actual returns and the rate of interest) are unforecastable in this market. When the asset supply $\bar{Q}$ is strictly positive, the return to holding the stock is augmented by a constant risk premium.

An attractive feature of this equlibrium is that it is "full order", meaning that traders would have no incentive to fit higher order $ARMA$ models to improve their forecasts. The Kalman filter provides a simple way to confirm this. For each trader j, the law of motion for the state vector z_t and the selector matrix u_j (defined in appendix 1B) can be used to compute the Kalman gain and an associated covariance matrix of forecast errors. This matrix includes the error covariances on P_t, S_t^j, and D_t which would emerge if traders were allowed to condition on the entire history of their observables. When the equilibrium is full order, the error covariances given by the Kalman filter are identical to $E\zeta_t^j\zeta_t^{j\prime}$ for $j = a, b$. This condition is satisfied in the equilibrium calculated above.

The moment matrix M_z also provides the following covariance matrix for the variables $[P_t \, S_t^a \, S_t^b \, D_t \, \theta_{1t} \, \theta_{2t}]$.

$$\begin{bmatrix} 17.5277 & 7.4074 & 0.6803 & 8.0877 & 6.3923 & 0.8430 \\ 7.4074 & 3.7778 & 0.0000 & 2.7778 & 2.7778 & 0.0000 \\ 0.6803 & 0.0000 & 2.1905 & 1.1905 & 0.0000 & 1.1905 \\ 8.0877 & 2.7778 & 1.1905 & 4.9683 & 2.7778 & 1.1905 \\ 6.3923 & 2.7778 & 0.0000 & 2.7778 & 2.7778 & 0.0000 \\ 0.8430 & 0.0000 & 1.1905 & 1.1905 & 0.0000 & 1.1905 \end{bmatrix}$$

Note that $Cov(P_t, D_t) = 8.0877$ and $Var(D_t) = 4.9683$. These are equivalent to the values obtained by substituting the parameter values for this market into equations (12) and (14), derived under the assumption of full observability. Since the information errors μ_t^a and μ_t^b are uncorrelated with D_t, any effect of these errors on P_t will not be picked up by $Cov(P_t, D_t)$. Given the current dividend D_t, the conditional expectation of P_t is

$$E\left[P_t \mid D_t\right] = \left[\frac{Cov(P_t, D_t)}{Var(D_t)}\right] D_t = 1.6279\, D_t.$$

Any deviation of the actual price from this conditional expectation is attributable to information and errors carried in private signals. Since excess returns are unforecastable, such deviations do not present opportunities for profit. We will return to this issue in section 1.5.

The information errors μ_t^a and μ_t^b do cause a number of differences between the equilibrium calculated here and the equilibrium derived analytically (under full information) in section 1.2. In the limited information equilibrium, the variance of excess returns is $\sigma^2 = 17.4628$. This compares to $\sigma^2 = 16.9138$ calculated using (11) under full information. At the same time, $Var(P_t) = 17.5277$ with limited information, but using (13), we find that $Var(P_t) = 20.1418$ under full information.

The lower variance of P_t under limited information is consistent with market efficiency, and underlies the excess volatility tests of Shiller (1981), West (1988), and Durlauf and Hall (1989). In an efficient market, P_t is equal to the present value of rationally expected dividends. If agents had perfect foresight, the variance of P_t would be identical to the variance of the discounted stream of future dividends. If information is imperfect and agents forecast future dividend streams rationally, their forecast errors must be orthogonal to their fitted values. It then follows that the variance of the fitted values — $Var(P_t)$ — must be less than the variance of the ex-post streams actually realized. In an efficient market, greater imperfection in information drives the variance of P_t downwards, and raises the variance of forecast errors.

Table 1.1

Impulse response of P_t with zero noise trade					
t	μ^a	μ^b	ν_1	ν_2	ϵ
1	1.0151	-0.1627	1.8676	0.6897	0.8524
2	0.2700	-0.1110	1.9231	0.3239	0.1590
3	0.0748	-0.0392	1.6488	0.1258	0.0355
4	0.0211	-0.0121	1.3491	0.0472	0.0090
5	0.0060	-0.0036	1.0876	0.0178	0.0024
6	0.0017	-0.0010	0.8725	0.0068	0.0007
7	0.0005	-0.0003	0.6987	0.0026	0.0002
8	0.0001	-0.0001	0.5591	0.0010	0.0001
9	0.0000	-0.0000	0.4474	0.0004	0.0000
10	0.0000	-0.0000	0.3579	0.0002	0.0000

In a market with full information, P_t is affected only by "fundamental" shocks ν_{1t}, ν_{2t} which are relevant to the future stream of dividends. Evidently, in a market with imperfect information, errors in information μ_t^a, μ_t^b and purely transitory dividend shocks ϵ_t also have an effect on equilibrium prices. This is because these shocks are partially interpreted as fundamental information. The effect on price may also persist for several periods, as new information slowly helps agents to reinterpret past shocks. The impulse responses of P_t to unit shocks in μ_t^a, μ_t^b, ν_{1t}, ν_{2t} and ϵ_t are presented in Table 1.1.

Finally, it is possible to address the issues of "learning" and "implementation" in the context of this rational expectations equilibrium. Bray and Kreps (1981) have posed the question "Could agents in the model who initially don't know how to form rational expectations learn how to do so by using standard statistical techniques on the data generated by the model?" Bray (1982) investigates this question in the context of the Grossman-Stiglitz (1980) model with informed and uninformed traders. Although traders are estimating models which are misspecified while they learn, Bray shows that an ordinary least squares learning procedure will converge almost surely to a rational expectations equilibrium. Marcet and Sargent (1989b) demonstrate this convergence result in a model where all agents are imperfectly informed. This convergence property of OLS learning is the basis for the fixed point approach presented in section 1.3.

The question of "implementation" is related. If traders expect zero excess returns from trading on their information, how does information become impounded in price? A useful insight is suggested by the recursive procedure used here to compute equilibria. For any market which is some small distance δ from the efficient rational expectations equilibrium, there are subtle profits available to traders who act on their information. The subtle profit opportunities cause slight alterations in the projection laws β_a, β_b and the resulting asset demands of traders. These alterations, in turn, force the market to converge towards an efficient equilibrium. The efficiency of the market in a rational expectations equilibrium is therefore enforced by the profit opportunities which exist in a stable neighborhood of that equilibrium.

1.5. Market inefficiency in a "noisy" rational expectations equilibrium

A growing volume of literature suggests that while market efficency theorems are robust with respect to the quality and allocation of private information, the existence of a seemingly trivial amount of "noise" can substantially undermine market efficiency. Grossman (1976), Hellwig (1980), Verrecchia (1982), Admati (1985), and Wang (1990) model this noise as variability in the unobserved supply of the security. Noise may also be modelled as unobserved variability in demand by "noise traders" as in Campbell and Kyle (1988). Noise traders are defined by Shleifer and Summers (1990) as individuals who "are not fully rational" in that "their demand for risky assets is affected by beliefs or sentiment that are not fully justified by fundamental news".

Regardless of whether noise is considered a supply or a demand phenomenon, the key characteristic is that the net supply available to fully rational traders is a random variable. When noise is unobservable, rational traders are faced with the signal extraction problem of discerning whether a change in price is due to other rational traders acting on private information about fundamentals, or whether it is due to fluctuations in unobserved supply. The breakdown of market efficiency in the presence of noise is analogous to the monetary non-neutrality result in Lucas (1972). In the Lucas model, changes in the unobserved supply of money in the economy generate real economic effects by causing changes in output prices which are partly interpreted as "fundamental" information about the terms of trade.

The parameters from section 1.4 are used to compute the market equilibrium in the presence of noise. In addition, there is a serially uncorrelated noise disturbance ω_t with variance $\sigma_\omega^2 = .01$. The equilibrium projection laws are calculated to be

$$\beta_a = S_a = \begin{bmatrix} 0.4081 & 1.0497 & -0.0046 & -0.3406 & -0.3748 & 0.5565 \\ 0.0000 & 0.8000 & -0.0000 & 0.0091 & -0.4404 & 0.1537 \\ -0.0000 & 0.4000 & 0.4000 & -0.0013 & -0.1354 & -0.0713 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$\beta_b = S_b = \begin{bmatrix} 0.4436 & -0.9742 & 0.9504 & -0.2845 & 0.6364 & -0.1078 \\ -0.0000 & 0.4000 & 0.0000 & -0.0104 & -0.2281 & 0.0799 \\ -0.0000 & -0.4000 & 0.8000 & 0.0462 & 0.3717 & -0.4045 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

Note that unlike the equilibrium presented in section 1.4, the elements in rows one and three do not offset eachother, which implies that in this noisy equilibrium, traders a and b have divergent expectations regarding $[P_{t+1} + D_{t+1}]$. By the fact that traders a and b are *rational*, we then conclude that excess market returns, measured as $[P_{t+1} + D_{t+1} - RP_t]$, must be partially correlated with information available to traders. Further calculations using the moment matrix M_z verify this conclusion. The variance of excess market returns is 28.7029, but the variances of forecast errors for traders a and b are

$$\sigma_a^2 = 22.2667 \qquad \sigma_b^2 = 23.7299$$

In terms of the coefficient of determination, it is easily verified that trader a has an $R^2 = .2242$, and trader b has an $R^2 = .1733$ in forecasting excess returns. This predictability is not solely due to the posession of private information. Restricting the information set to P_t and D_t alone yields an $R^2 = .1290$ in forecasting excess returns.

The existence of predictable excess returns in an *equilibrium* with rational traders seems a bit odd. There are two factors which drive inefficiency in the market considered here. The first, clearly, is the existence of noise, which forces rational traders to hold a randomly varying supply of the security. This unobservable variation confounds the inference problems faced by traders, and causes them to rely heavily on their own information signals. When the equilibrium is calculated with a lower noise variance $\sigma_\omega^2 = .005$, the variance of excess market returns drops to 22.4252, and the R^2 drops to .1358 for the excess return forecasts of trader a, .0761 for the forecasts of trader b, and .0455 for forecasts based on fully public information P_t and D_t.

Campbell and Kyle (1988) analyze a model in which noise traders are able to have an effect on price because rational traders are risk averse. This risk aversion effect also appears in the model presented here. The demand function (5) for each trader implies that, in the presence of forecast errors, no trader chooses to hold an infinite position based on an expected excess return. Indeed, when the equilibrium is calculated with higher risk tolerance $\phi_a = \phi_b = 2$, the variance of excess market returns drops to 20.0446, and the R^2 drops to .0872 for trader a, .0313 for trader b, and .0154 for forecasts based on public information.

In a noisy rational expectations equilibrium, Tirole's (1982) efficiency result fails to hold. Rational traders expect a gain from trading on their information. Excess returns are available even on the basis of public information, because risk averse traders must receive compensation for the disutility of providing liquidity to other market participants. Unobservable noise also creates an externality in that rational traders are able to extract rents from costless private information.

A further interaction between noise and private information is that greater imperfection in private information need not drive down the variance of price monotonically, as is the case in a market free of noise. Wang (1990) notes in a model of informed versus uninformed traders that in the extreme case where all rational traders are uninformed, "the fact that nobody knows anything enables everybody to know something". Specifically, the absence of heterogeneity in the information signals of rational traders allows them to perfectly infer the amount of noise in the market. The introduction of private information enables the market price to reflect the dividend process more accurately, but destroys the revelation of supply shocks. The effect of increased information on the variance of price is then ambiguous.

The covariance matrices of forecast errors ζ_t^j (corresponding to P_t, S_t^j and D_t) for $j = a, b$ are

$$E\zeta_t^a\zeta_t^{a\prime} = \begin{bmatrix} 11.1479 & 2.5175 & 3.9412 \\ 2.5175 & 2.2913 & 1.2111 \\ 3.9412 & 1.2111 & 3.2365 \end{bmatrix}$$

$$E\zeta_t^b\zeta_t^{b\prime} = \begin{bmatrix} 11.8165 & 0.6341 & 4.2710 \\ 0.6341 & 2.0695 & 0.9853 \\ 4.2710 & 0.9853 & 3.3715 \end{bmatrix}.$$

These matrices match the covariance matrices which would emerge if traders were allowed to condition on the infinite history of their observables. This equilibrium is therefore full order, and traders have no incentive to fit higher order models to improve their forecasts.

The moment matrix M_z also provides the following covariance matrix for the variables $[P_t\, S_t^a\, S_t^b\, D_t\, \theta_{1t}\, \theta_{2t}]$.

$$\begin{bmatrix} 21.2247 & 6.3530 & 0.9629 & 8.0881 & 5.8515 & 1.1463 \\ 6.3530 & 3.7778 & 0.0000 & 2.7778 & 2.7778 & 0.0000 \\ 0.9629 & 0.0000 & 2.1905 & 1.1905 & 0.0000 & 1.1905 \\ 8.0881 & 2.7778 & 1.1905 & 4.9683 & 2.7778 & 1.1905 \\ 5.8515 & 2.7778 & 0.0000 & 2.7778 & 2.7778 & 0.0000 \\ 1.1463 & 0.0000 & 1.1905 & 1.1905 & 0.0000 & 1.1905 \end{bmatrix}$$

Note that the variance of P_t is 21.2247, which is higher than the value of 20.1418 which would emerge with zero noise and full information. In this model, given our parameter values, the existence of noise raises the variance of P_t. This is not a necessary result. When noise is modelled with serial correlation, some parameter values generate a variance for P_t which is below the full information value. Since excess returns are predictable in both cases, movements in price will not be attributable solely to surprises in rationally expected dividends.

Shiller (1981) used a test based on the variances of prices and ex-post dividend streams to demonstrate that U.S. stock prices display "excess volatility". As Durlauf and Hall (1988) point out, the excess volatility test of Shiller will detect noise which significantly raises the variance of P_t, but will not be able to detect variance-lowering noise. The Durlauf-Hall test is more powerful, and exploits an orthogonality restriction implied by the dividend discount model: the *covariance* between price and the discounted value of ex-post dividends should be equal to the variance of price itself. Using this test, Durlauf and Hall (1989) find that the variance of U.S. stock prices is dominated almost entirely by a noise component unrelated to the variance of expected future dividends. These findings of "excess volatility" in the U.S. stock market are fully consistent with the model at hand.

The ability of dividend yields (or dividend price ratios) to predict subsequent stock returns has been noted in empirical studies including Shiller (1986) and Flood, Hodrick and Kaplan (1986). Fama and French (1987) find that dividend yields can explain more than 25% of the variance of 2 to 4 year returns. The noisy rational expectations equilibrium presented here offers insights into these findings.

The interpretation given by Fama and French is that dividend yields "track" time varying risk premia. That is, an increase in the dividend yield is a signal that the required return of investors has increased. An increase in required return drives the current price down, and simultaneously raises the expected future return. The argument is analogous to the exchange rate overshooting model of Dornbusch (1976). In the Dornbusch model, a monetary shock which depresses the domestic interest rate relative to the foreign rate is associated with an immediate "overshooting" depreciation of the domestic currency, followed by an expected subsequent appreciation.

While the excess returns available to rational traders are a result of risk aversion in this model, it is misleading to interpret these excess returns as time-varying risk premia. The notion of time-varying risk premia implies that discount factors or the second moments of the data are in question, and that the required return of *all* traders fluctuates over time. In the model presented here, all variables are covariance stationary, and the discount factor R is a constant. It is more appropriate to interpret excess returns as compensation demanded by rational traders for the *service* of providing liquidity and information to other market participants in the presence of uncertainty. The size of this compensation is a function of risk aversion and the level of noise.

The explanation of dividend yield effects offered here is therefore quite simple. The dividend yield is a useful predictor of excess returns because noise generates significant mean-reverting swings from fundamental value. A proxy for this fundamental value is $E[P_t \mid D_t]$; the expectation of price conditional on the current dividend. When the stock price is temporarily depressed relative to this measure, the dividend yield is relatively high, and positive excess returns are observed as price subsequently advances towards fundamental value. When the stock price is temporarily elevated relative to this measure, the dividend yield is relatively low, and negative excess market returns are observed as price subsequently falls towards fundamental value.

The matrices M_z and $T(\beta)$ provide an elegant method of supporting this assertion. The expected excess return, conditional on P_t, D_t is calculated to be

$$E\left[P_{t+1} + D_{t+1} - RP_t \,|\, P_t,\, D_t\right] = \begin{bmatrix}-0.6778 & 1.1034\end{bmatrix} \begin{bmatrix} P_t \\ D_t \end{bmatrix}. \tag{24}$$

In the efficient market considered in section 1.4, these coefficients are identically zero. The expectation of P_t conditional on D_t is

$$E\left[P_t \,|D_t\right] = \left[\frac{Cov\left(P_t, D_t\right)}{Var\left(D_t\right)}\right] D_t = 1.6279\, D_t$$

If P_t is subject to excessive, but mean reverting swings relative to its conditional expectation $E[P_t \,|\, D_t]$, then we would expect the magnitude of expected excess returns to be proportional to the deviation of price from this value. That is, we would expect to find some adjustment coefficient γ such that

$$E\left[P_{t+1} + D_{t+1} - RP_t \,|\, P_t,\, D_t\right] = \gamma \begin{bmatrix}-1 & 1.6279\end{bmatrix} \begin{bmatrix} P_t \\ D_t \end{bmatrix}. \tag{25}$$

It is easily seen that (25) satisfies (24) with $\gamma = .6778$. Recent evidence of mean reversion in U.S. financial markets [DeBondt and Thaler (1985), Lo and MacKinlay (1988), Poterba and Summers (1988)] therefore appears consistent with a noisy rational expectations equilibrium.

The impulse responses of P_t to innovations in μ_t^a, μ_t^b, ν_t^a, ν_t^b, ϵ_t and ω_t are presented in Table 1.2. The shock to ω_t is .1 unit (representing 1 standard deviation). The remaining innovations are unit shocks.

Table 1.2

Impulse response of P_t in a noisy market						
t	μ^a	μ^b	ν_1	ν_2	ϵ	ω
1	0.5015	-0.1834	1.5917	0.9068	1.0903	2.3772
2	0.1783	-0.1438	1.7268	0.4940	0.2751	0.1218
3	0.0708	-0.0701	1.5430	0.2183	0.0908	0.0503
4	0.0291	-0.0312	1.2983	0.0908	0.0348	0.0205
5	0.0121	-0.0135	1.0648	0.0369	0.0141	0.0082
6	0.0050	-0.0058	0.8626	0.0148	0.0058	0.0033
7	0.0020	-0.0024	0.6945	0.0059	0.0024	0.0013
8	0.0008	-0.0010	0.5574	0.0023	0.0010	0.0005
9	0.0003	-0.0004	0.4467	0.0009	0.0004	0.0002
10	0.0001	-0.0002	0.3576	0.0004	0.0002	0.0001

Trading volume is generated in this noisy rational expectations equilibrium. Even rational traders will transact with eachother. The no-trade theorem of Milgrom and Stokey does not hold here because not all asset demands are rationally conditioned on payoff relevant information, so traders cannot perfectly infer this information from prices. Still, a high proportion of total trade is due to rational traders transacting with noise traders, and only a small amount of trade is generated by rational traders transacting among themselves. Trading volume is calculated as half the sum of absolute changes in the positions of individual traders (including noise traders):

$$Volume = .5\left(|\Delta\alpha_a N Q_t^a| + |\Delta\alpha_b N Q_t^b| + |\Delta Q_t^n|\right).$$

If noise is serially uncorrelated, $Q_t^n = \omega_t$. Table 1.3 presents the impulse responses of trading volume and the total shareholdings of a and b ($\alpha_a N Q_t^a$, $\alpha_b N Q_t^b$) induced by a unit shock to noise ω_t.

Table 1.3

Impulse response of trading volume and holdings to noise			
t	Volume	$\alpha_a N Q_t^a$	$\alpha_b N Q_t^b$
1	1.0000	-0.5519	-0.4481
2	1.0000	-0.0131	0.0131
3	0.0079	-0.0053	0.0053
4	0.0031	-0.0022	0.0022
5	0.0013	-0.0009	0.0009
6	0.0006	-0.0004	0.0004
7	0.0002	-0.0001	0.0001
8	0.0001	-0.0001	0.0001
9	0.0000	-0.0000	0.0000
10	0.0000	-0.0000	0.0000

The equilibrium of this model was also computed in the presence of a noise disturbance Q_t^n which is correlated over time. Using an $ARMA(1,1)$ model for trader's expectations, the equilibrium was found to be of reduced order, meaning that traders would have an incentive to fit higher order models. The innovation covariance matrix was identical to the matrix given by the Kalman filter, except for a very slight difference in the error variance of P_t. The behavior of the market under correlated noise is qualitatively similar to the behavior reviewed above, except that the impulse response functions exhibit somewhat smoother geometric decay.

The fact that excess returns have a predictable component does not imply that rational traders earn consistent profits each period. Rather, the returns of rational traders contain a predictable component which is disturbed by a substantial amount of unpredictable variation. Over short horizons, it may be difficult to make an observational distinction between rational traders and noise traders. Over longer horizons, noise traders must experience returns which are below the rate of interest. If $\bar{Q} > 0$ and the average position of noise traders is strictly positive, these returns will be augmented by a risk premium.

Shleifer and Summers (1990) argue that noise traders may not be driven from the market asymptotically. The arguments are slightly outside the context of the current model, but are useful if one wishes to extend the results of this section to actual financial markets. One argument is that when noise traders earn a high return in some period, other traders may imitiate them. That is, traders may not adhere to a pure strategy, and may pursue strategies which have recently exhibited positive returns. Another argument is that new noise traders may constantly be entering the market, and old traders may return after replenishing their capital.

1.6. Concluding remarks

In his General Theory, Keynes (1936) suggested that financial markets are driven by traders devoted to "anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees". This paper extends Sargent's (1991) method of signal extraction from endogenous variables to calculate equilibria in markets where this degree of inference is unbounded. By transforming the infinite dimensional inference problem of traders into a finite dimensional problem which spans the same information set, it has been possible to calculate rational expectations equilibria under a variety of assumptions regarding market structure. The matrices which result from these computations have a natural regression interpretation. This allows the systematic investigation into the properties of various market equilibria.

In a market where all traders are rational, no trader is able to profit from private information because price conveys a signal to other market participants. This result is extremely robust with respect to the quality and allocation of private information, but quite fragile in the presence of unobserved variability of asset supply. This "noise" complicates the inference problems of traders, resulting in a number of features consistent with empirical evidence on actual markets. These features include divergence of opinion, trading volume, mean reversion, dividend yield effects, and "excess" volatility. Clearly, a similarity between model implications and empirical data does not necessarily imply that the data were generated by the model. Still, the concept of noisy rational expectations is interesting because these empirically consistent implications are broadly incompatible with alternative models which suggest strong efficiency and zero trade.

The modelling approach used here is also a natural framework in which to analyze contagion in markets. King and Wadhwani (1990) model the international transmission of price volatility as a result of rational traders extracting information from price changes in other markets. Pozdena (1991) analyzes banking panics as a result of depositors attempting to infer information about one bank from the behavior of other banks that share imperfectly overlapping characteristics. Sargent (1991) applies the signal extraction framework to a macroeconomic model of output fluctuations across industries. Firms in this model make inferences about common economic factors by observing endogenous output prices of other firms. It is not difficult to think of other important settings in which economic variables are not only generated by expectations, but simultaneously provide information which condition those expectations. These settings appear to be promising areas for future research.

Appendix 1A

This appendix derives the expression for P_t shown in equation (9) in the text. Since $E_t[\theta_{1t+i}] = \rho_1^i \theta_{1t}$ and $E_t[\theta_{2t+i}] = \rho_2^i \theta_{2t} \, \forall i > 0$, equation (9) may be obtained by treating the bracketed expectation in equation (7) as the sum of two infinite geometric series. The following alternative is useful for a wide range of problems involving prediction or signal extraction, in which the annihilation operator $[\quad]_+$ must be evaluated.

Using equation (7) and substituting (1) and (2) for dividends, the equilibrium full information stock price may be written as

$$P_t = \left[\frac{R^{-1}L^{-1}}{(1 - R^{-1}L^{-1})(1 - \rho_1 L)} \right]_+ \nu_{1t} + \left[\frac{R^{-1}L^{-1}}{(1 - R^{-1}L^{-1})(1 - \rho_2 L)} \right]_+ \nu_{2t} - \Psi$$

where L denotes the lag operator and $[\quad]_+$ is the annihilation operator which ignores negative powers of L. Each bracketed expression contains a pole inside the unit circle at R^{-1}, which prevents invertibility to an expansion in strictly positive powers of L. Ignoring negative powers of L amounts to expanding each expression by partial fractions, and subtracting off the principal part of the Laurent expansion corresponding to the "offensive" pole. For the first bracketed expression, the offending term is

$$\frac{-R}{(R - \rho_1)(1 - RL)},$$

and for the second, the term is

$$\frac{-R}{(R - \rho_2)(1 - RL)}.$$

The stock price may then be expressed as a function of current and past shocks to the dividend process.

$$P_t = \frac{\rho_1}{(R - \rho_1)(1 - \rho_1 L)} \nu_{1t} + \frac{\rho_2}{(R - \rho_2)(1 - \rho_2 L)} \nu_{2t} - \Psi$$

Appendix 1B

This appendix describes the procedure used to calculate the equilibria presented in the text. We wish to derive a mapping from trader's perceptions

$$x_{at} = \beta_a x_{at-1} + \begin{bmatrix} I \\ I \end{bmatrix} \zeta_{at} \qquad\qquad x_{bt} = \beta_b x_{bt-1} + \begin{bmatrix} I \\ I \end{bmatrix} \zeta_{bt} \tag{1b}$$

to the actual law of motion for the state vector

$$z_t = T(\beta) z_{t-1} + V(\beta)\varepsilon_t. \tag{2b}$$

Traders use the forecasting rules (β_a, β_b) in conjunction with equation (5) in the text to determine their desired stock holdings Q_t^a and Q_t^b. In the presence of a potentially nonzero noise disturbance Q_t^n, market clearing implies

$$P_t = \Lambda^{-1}\left(\sigma_b^2 \alpha_a N \phi_a E_a[P_{t+1} + D_{t+1}] + \sigma_a^2 \alpha_b N \phi_b E_b[P_{t+1} + D_{t+1}] + Q_t^n \sigma_a^2 \sigma_b^2 - \bar{Q} \sigma_a^2 \sigma_b^2\right) \tag{3b}$$

where

$$\Lambda = RN(\sigma_b^2 \alpha_a \phi_a + \sigma_a^2 \alpha_b \phi_b),$$

and σ_a^2 and σ_b^2 are the error variances of a and b in forecasting $[P_{t+1} + D_{t+1}]$. To avoid inducing a unit eigenvalue in the $T(\beta)$ matrix, the supply constant $\bar{Q}$ is not carried as part of the state vector. Price is then measured in terms of deviation from the mean.

Define the selector matrices e_a, e_b, u_a, u_b so that

$$\begin{aligned} z_{at} &= e_a z_t \quad x_{at} = u_a z_t \\ z_{bt} &= e_b z_t \quad x_{bt} = u_b z_t \end{aligned}. \tag{4b}$$

Using (3b), we can also define weighted selector matrices c_a, c_b, u_n so that price may be defined in terms of its deviation from mean

$$P_t = [c_a \beta_a u_a + c_b \beta_b u_b + u_n] z_t \tag{5b}$$

where c_a, c_b select and weight the forecasts of a and b of $[P_{t+1} + D_{t+1}]$, and u_n selects and weights the noise disturbance Q_t^n.

There are two problems here which did not emerge in the model investigated by Sargent. Since the first element of z_t is P_t, it is clear from (5b) that the equilibrium price depends upon itself. Moreover, the forecast variances σ_a^2 and σ_b^2 are unknown. The required modification to Sargent's original approach is to compute a fixed point not only in β, but also in the top row of $T(\beta)$ and $V(\beta)$, as well as in σ_a^2 and σ_b^2.

The matrices $T(\beta)$ and $V(\beta)$ are constructed as follows. The first rows of T and V, and the variances σ_a^2 and σ_b^2 are initially chosen arbitrarily (each successive iteration refines these choices). The second and third rows of T and V, corresponding to S_t^a and S_t^b are implied by (15) in the text. Rows four through six, corresponding to $D_t, \theta_{1t}, \theta_{2t}$ are implied by (1) and (2). Row seven of V, corresponding to the noise term Q_t^n is

$$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$$

while row seven in T is a zero row when noise is serially uncorrelated, and has an autoregressive parameter ρ_n in column seven if noise is modelled with serial correlation.

Rewriting (16), substituting from (2b) and (5b), and defining a selector matrix e_{za} such that $\zeta_{at} = e_{za} z_t$ gives

$$\zeta_{at} = [e_a T(\beta) - A_a e_a - C_a e_{za}]\, z_{t-1} + e_a V(\beta)\varepsilon_t \tag{6b}$$

The equation for ζ_{bt} is symmetric. Since the first rows of T and V are already fixed, and e_a does not select from the last six rows of T and V, it is unneccessary to compute a separate fixed point in the rows of T and V corresponding to ζ_{at} and ζ_{bt}.

With these T and V matrices in hand, (2b) and (5b) may be combined to yield

$$P_t = [c_a\beta_a u_a + c_b\beta_b u_b + u_n]T(\beta)z_{t-1} + [c_a\beta_a u_a + c_b\beta_b u_b + u_n]V(\beta)\varepsilon_t \tag{7b}$$

This equation is then used to compute new first rows in T and V. This completes the construction of T and V.

It is assumed that the eigenvalues of T are all inside the unit circle. This allows the computation of a stationary covariance matrix for z_t, where the matrix $M_z(\beta) = E z_t z_t'$ satisfies

$$M_z(\beta) = T(\beta) M_z(\beta) T(\beta)' + V(\beta) \Omega V(\beta)' \tag{8b}$$

where $\Omega = E\varepsilon_t \varepsilon_t'$. This is a "discrete Lyapunov equation" which may be solved using a variety of algorithms.

Note that the moment matrix of ζ_{jt} gives the forecast error variances and covariances needed to calculate σ_a^2 and σ_b^2. Given the moment matrix M_z, the selector vectors u_{sj} may be defined so that

$$\sigma_j^2 = u_{sj} M_z u_{sj}' \qquad j = a, b. \tag{9b}$$

The optimal projection laws $S_j(\beta)$ for $j = a, b$ are given by

$$E x_{jt} | x_{jt-1} = S_j(\beta) x_{jt-1} \tag{10b}$$

where

$$S_j(\beta) = u_j T(\beta) M_z u_j' \left[u_j M_z u_j' \right]^{-1} \tag{11b}$$

In some cases, the matrix $\left[u_j M_z u_j' \right]$ will become singular, due to linear dependence in the observables of a or b. This problem may be circumvented following Sargent (1991) by choosing matrices u_{jj} which restrict the set of regressors used to compute $S_j(\beta)$. The columns in A_j, C_j corresponding to the excluded regressors are assigned zero values. Using the resulting equilibrium, it is straightforward to compute the coefficient of determination in the regression of the excluded regressors onto included regressors. If this coefficient is unity, the exclusion does not restrict the information set of a and b in equilibrium.

A rational expectations equilibrium with imperfect, heterogeneous information is the fixed point $\beta = S(\beta)$. Calculation of a fixed point equilibrium proceeds by repeated iterations on this "S mapping". Initial arbitrary values for $A_a, C_a, A_b, C_b, \sigma_a^2, \sigma_b^2$ and the first rows of $T(\beta)$ and $V(\beta)$ are updated at each iteration by a convex combination of the previous value and the newly computed value.

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