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Outline of a Generalized Theory of Relativity and of a Theory of Gravitation

I. Physical Part

by Albert Einstein

II. Mathematical Part

by Marcel Grossmann

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I
Physical Part

The theory expounded in what follows derives from the conviction that the proportionality between the inertial and the gravitational mass of bodies is an exactly valid law of nature that must already find expression in the very foundation of theoretical physics. I already sought to give expression to this conviction in several earlier papers by seeking to reduce the *gravitational* mass to the *inertial* mass;[1] this endeavor led me to the hypothesis that, from a physical point of view, an (infinitesimally extended, homogeneous) gravitational field can be completely replaced by a state of acceleration of the reference system. This hypothesis can be expressed pictorially in the following way: An observer enclosed in a box can in no way decide whether the box is at rest in a static gravitational field, or whether it is in accelerated motion, maintained by forces acting on the box, in a space that is free of gravitational fields (equivalence hypothesis). [2]

We know the fact that the law of proportionality of inertial and gravitational mass is satisfied to an extraordinary degree of accuracy from the fundamentally important investigation by Eötvös,[2] which is based on the following argument. A body at rest on the surface of the Earth is acted upon by gravity as well as by the centrifugal force resulting from Earth's rotation. The first of these forces is

[1]A. Einstein, *Ann. d. Phys.* 35 (1911): 898; 38 (1912):355; 38 (1912): 443. [1]

[2]B. Eötvös, *Mathematische und naturwissenschaftliche Berichte aus Ungarn* 8 (1890); Wiedemann's *Beiblätter* 15 (1891): 688. [3]

proportional to the gravitational mass, and the second to the inertial mass. Thus, the direction of the resultant of these two forces , i.e., the direction of the apparent gravitational force (direction of the plumb) would have to depend on the physical nature of the body under consideration if the proportionality of the inertial and gravitational mass were not satisfied. In that case the apparent gravitational forces acting on parts of a heterogeneous rigid system would, in general, not merge into a resultant; instead, in general, there would still be a torque associated with the apparent gravitational forces that would have to make itself noticeable if the system were suspended from a torsion-free thread. By having established the absence of such torques with great care, Eötvös proved that, for the bodies that he investigated, the relationship of the two masses was independent of the nature of the body to such a degree of exactness that the relative difference in this relationship that might still exist from one substance to another must be smaller than one twenty-millionth.

The decomposition of radioactive substances occurs with a release of such significant quantities of energy that the change in the inertial mass of the system that corresponds to that energy decrease according to the theory of relativity is not very small relative to the total mass.[3] In the case of the decay of radium, for example, this decrease amounts to one ten-thousandth of the total mass. If these changes of the inertial mass did not correspond to changes in the *gravitational* mass, then there would have to be deviations of the inertial mass from the gravitational mass much
[5] greater than those allowed by Eötvös's experiments. Hence it must be considered very probable that the identity of the inertial and gravitational mass is exactly satisfied. For these reasons it seems to me that the equivalence hypothesis, which asserts the essential physical identity of the gravitational with the inertial mass, possesses a high degree of probability.[4]

§1. Equations of Motion of the Material Point in the Static Gravitational Field

According to the customary theory of relativity,[5] in the absence of forces a point moves according to the equation

$$\delta\left\{\int ds\right\} = \delta\left\{\int\sqrt{-dx^2 - dy^2 - dz^2 + c^2dt^2}\right\} = 0. \tag{1}$$

For this equation states that the material point moves rectilinearly and uniformly. This is the equation of motion in the form of Hamilton's principle; for we can also

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[3]The decrease of the inertial mass corresponding to the released energy E is, as we
[4] know, E/c^2, if c denotes the velocity of light.

[4]Cf. also §7 of this paper.

[6] [5]Cf. M. Planck, *Verh. d. deutsch. phys. Ges.* (1906): 136.

set

(1a)
$$\delta\{\int H dt\} = 0,$$

where

$$H = -\frac{ds}{dt}m$$

is posited, if m designates the rest mass of the material point. From this we obtain, in the familiar way, the momentum J_x, J_y, J_z, and the energy E of the moving point:

(2)
$$\begin{cases} J_x = m\frac{\partial H}{\partial \dot{x}} = m\frac{\dot{x}}{\sqrt{c^2 - q^2}}; \text{ etc.} \\ E = \frac{\partial H}{\partial \dot{x}}\dot{x} + \frac{\partial H}{\partial \dot{y}}\dot{y} + \frac{\partial H}{\partial \dot{z}}\dot{z} - H = m\frac{c^2}{\sqrt{c^2 - q^2}}. \end{cases}$$ [7]

This mode of representation differs from the customary one only by the fact that in the latter J_x, J_y, J_z, and E contain also a factor c. But since c is constant in the customary theory of relativity, the system given here is equivalent to the ordinary one. The only difference is that J and E possess dimensions other than those in the customary mode of representation.

I have shown in previous papers that the equivalence hypothesis leads to the consequence that in a static gravitational field the velocity of light c depends on the gravitational potential. This led me to the view that the customary theory of relativity [8] provides only an approximation to reality; it should apply only in the limit case where differences in the gravitational potential in the space-time region under consideration are not too great. In addition, I found again equations (1) or (1a) as the [9] equations of motion of a mass point in a static gravitational field; however, c is not to be conceived of here as a constant but rather as a function of the spatial coordinates that represents a measure for the gravitational potential. From (1a) we obtain in the familiar fashion the equations of motion

(3)
$$\frac{d}{dt}\left\{\frac{m\dot{x}}{\sqrt{c^2 - q^2}}\right\} = -\frac{mc\frac{\partial c}{\partial x}}{\sqrt{c^2 - q^2}}.$$

It is easy to see that the momentum is represented by the same expression as above. In general, equations (2) hold for the material point moving in the static gravitational field. The right-hand side of (3) represents the force $\mathfrak{K}_x$ exerted on the mass point by the gravitational field. For the special case of rest ($q = 0$) we have

$$\mathfrak{K}_x = -m\frac{\partial c}{\partial x}.$$

From this one sees that c plays the role of the gravitational potential.

From (2) it follows that for a slowly moving point

(4)
$$J_x = \frac{m\dot{x}}{c},$$
$$E - mc = \frac{\frac{1}{2}mq^2}{c}.$$

At a given velocity, the momentum and the kinetic energy are thus inversely proportional to the quantity c; in other words: the inertial mass, as it enters into the momentum and energy, is $\frac{m}{c}$, where m denotes a constant that is characteristic of the mass point and independent of the gravitational potential. This is consonant with Mach's daring idea that inertia has its origin in an interaction between the mass point under consideration and all of the other mass points; for if we accumulate masses in the vicinity of the mass point under consideration, we thereby decrease the gravitational potential c, thus increasing the quantity $\frac{m}{c}$ that is determinative of
[10] inertia.

§2. Equations of Motion of the Material Point in an Arbitrary Gravitational Field. Characterization of the Latter

By introducing a spatial variability of the quantity c, we have breached the frame of the theory presently designated as the "relativity theory"; for now the expression designated by ds no longer behaves as an invariant with respect to orthogonal linear transformations of the coordinates. Thus, if the relativity principle is to be maintained—which is not to be doubted—then we must generalize the relativity theory in such a way that the theory of the static gravitational field whose elements have been indicated above will be contained in it as a special case.

If we introduce a new space-time system $K'(x', y', z', t')$ by means of an arbitrary substitution

$$x' = x'(x, y, z, t)$$
$$y' = y'(x, y, z, t)$$
$$z' = z'(x, y, z, t)$$
$$t' = t'(x, y, z, t),$$

and if the gravitational field in the original system K was static, then, upon this substitution, equation (1) goes over into an equation of the form

$$\delta\left\{\int ds'\right\} = 0,$$

where

$$ds'^2 = g_{11}dx'^2 + g_{22}dy'^2 + \ldots + 2g_{12}dx'dy' + \ldots,$$

and where the quantities $g_{\mu\nu}$ are functions of x', y', z', t'. If we put x_1, x_2, x_3, x_4 in place of x', y', z', t', and write again ds instead of ds', then the equations of motion [11]
of the material point with respect to K' take the form

(1″)
$$\begin{cases} \delta\left\{\int ds\right\} = 0, \quad \text{where} \\ ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu . \end{cases}$$

We thus arrive at the view *that in the general case the gravitational field is characterized by ten space-time functions*

$$\begin{matrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{matrix} \qquad (g_{\mu\nu} = g_{\nu\mu})$$

which in the case of the customary theory of relativity reduce to [12]

$$\begin{matrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +c^2, \end{matrix}$$

where c denotes a constant. The same kind of degeneration occurs in the static gravitational field of the kind considered above, except that in the latter case $g_{44} = c^2$
is a function of x_1, x_2, x_3. [13]

The Hamiltonian function H thus has the following value in the general case:

(5)
$$H = -m\frac{ds}{dt} = -m\sqrt{g_{11}\dot{x}_1^2 + \cdot + \cdot + 2g_{12}\dot{x}_1\dot{x}_2 + \cdot + \cdot + 2g_{14}\dot{x}_1 + \cdot + \cdot + g_{44}}.$$

The corresponding Lagrangian equations

(6)
$$\frac{d}{dt}\left(\frac{\partial H}{\partial \dot{x}}\right) - \frac{\partial H}{\partial x} = 0$$

yield directly the expression for the momentum J of the point and for the force $\mathfrak{K}$

exerted on it:

[14] (7)
$$J_x = -m\frac{g_{11}\dot{x}_{11} + g_{12}\dot{x}_2 + g_{13}\dot{x}_3 + g_{14}}{\frac{ds}{dt}} = -m\frac{g_{11}d_1x + g_{12}dx_2 + g_{13}dx_3 + g_{14}dx_4}{ds},$$

(8)
$$\mathfrak{K}_x = -\frac{1}{2}m\frac{\sum_{\mu\nu}\frac{\partial g_{\nu\mu}}{\partial x_1}dx_\mu dx_\nu}{ds\cdot dt} = -\frac{1}{2}m\cdot\sum_{\mu\nu}\frac{\partial g_{\nu\mu}}{\partial x_1}\cdot\frac{dx_\mu}{ds}\cdot\frac{dx_\nu}{dt}. *$$

Further, for the energy E of the point, one obtains

(9)
$$-E = -\left(\dot{x}\frac{\partial H}{\partial \dot{x}} + \cdot + \cdot\right) + H = -m\left(g_{41}\frac{dx_1}{ds} + g_{42}\frac{dx_2}{ds} + g_{43}\frac{dx_3}{ds} + g_{44}\frac{dx_4}{ds}\right).$$

In the case of the customary theory of relativity only linear orthogonal substitutions are permissible. It will turn out that we are able to set up equations for the influence of the gravitational field on the material processes that are covariant with respect to arbitrary substitutions.

First, from the role that ds plays in the law of motion of the material point, we can draw the conclusion that ds must be an absolute invariant (scalar); from this it
[15] follows that the quantities $g_{\mu\nu}$ form a covariant tensor of the second rank,[6] which we call the covariant fundamental tensor. This tensor determines the gravitational field. Further, it follows from (7) and (9) that the momentum and the energy of the material point form together a covariant tensor of the first rank, i.e., a covariant vector.[7]

§3. The Significance of the Fundamental Tensor of the $g_{\mu\nu}$ for the Measurment of Space and Time

From the foregoing, one can already infer that there cannot exist relationships between the space-time coordinates x_1, x_2, x_3, x_4 and the results of measurements obtainable by means of measuring rods and clocks that would be as simple as those in the old relativity theory. With regard to time, this has already found to be true in the case of the static gravitational field.[8] The question therefore arises, what is the

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[6]Cf. Part II, §1.
[7]Cf. Part II, §1.
[16] [8]Cf., e.g., A. Einstein, *Ann. d. Phys.* 35 (1911): 903 ff.

physical meaning (measurability in principle) of the coordinates x_1, x_2, x_3, x_4.

We note in this connection that ds is to be conceived as the invariant measure of the distance between two infinitely close space-time points. For that reason, ds must also possess a physical meaning that is independent of the chosen reference system. We will assume that ds is the "naturally measured" distance between the two space-time points, and by this we will understand the following. [17]

The immediate vicinity of the point (x_1, x_2, x_3, x_4) with respect to the coordinate system is determined by the infinitesimal variables dx_1, dx_2, dx_3, dx_4. We assume that, in their place, new variables $d\xi_1, d\xi_2, d\xi_3, d\xi_4$ are introduced by means of a linear transformation in such a way that

$$ds^2 = d\xi_1^2 + s\xi_2^2 + d\xi_3^2 - d\xi_4^2.$$

In this transformation the $g_{\mu\nu}$ are to be viewed as constants; the real cone $ds^2 = 0$ appears referred to its principal axes. Then the ordinary theory of relativity holds in this elementary $d\xi$ system, and the physical meaning of lengths and times shall be the same in this sytem as in the ordinary theory of relativity, i.e., ds is the square of the four-dimensional distance between two infinitely close space-time points, measured by means of a rigid body that is not accelerated in the $d\xi$-system, and by means of unit measuring rods and clocks at rest relative to it. [18]

From this one sees that, for given dx_1, dx_2, dx_3, dx_4, the natural distance that corresponds to these differentials can be determined only if one knows the quantities $g_{\mu\nu}$ that determine the gravitational field. This can also be expressed in the following way: the gravitational field influences the measuring bodies and clocks in a determinate manner. [19]

From the fundamental equation

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu$$

one sees that, in order to fix the physical dimensions of the quantities $g_{\mu\nu}$ and x_ν, yet another stipulation is required. The quantity ds has the dimension of a length. Likewise, we wish to view the x_ν (x_4 too) as lengths, and thus we do not ascribe any physical dimension to the quantities $g_{\mu\nu}$.

§4. The Motion of Continuously Distributed Incoherent Masses in an Arbitrary Gravitational Field

In order to derive the law of motion of continuously distributed incoherent masses, we calculate the momentum and the ponderomotive force per unit volume and apply the law of the conservation of momentum. [20]

To this end, we must first calculate the three-dimensional volume V of our mass point. We consider an infinitely small (four-dimensional) piece of the space-time thread of our material point. Its volume is

$$\iiiint dx_1 dx_2 dx_3 dx_4 = Vdt.$$

If we introduce the natural differentials $d\xi$ in place of the dx, assuming that the measuring body is at rest with respect to the material point, we have to set

$$\iiint d\xi_1 d\xi_2 d\xi_3 = V_0,$$

i.e., equal to the "rest volume" of the material point. Further, we have

$$\int d\xi_4 = ds,$$

where ds has the same meaning as above.

If the dx are related to the $d\xi$ by the substitution

$$dx_\mu = \sum_\sigma \alpha_{\mu\sigma} d\xi_\sigma,$$

then we have

$$\iiiint dx_1 dx_2 dx_3 dx_4 = \iiiint \frac{\partial(dx_1, dx_2, dx_3, dx_4)}{\partial(d\xi_1, d\xi_2, d\xi_3, d\xi_4)} \cdot d\xi_1 d\xi_2 d\xi_3 d\xi_4$$

or

$$Vdt = V_0 ds \cdot |\alpha_{\rho\sigma}|.$$

But since

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu = \sum_{\mu\nu\rho\sigma} g_{\mu\nu} \alpha_{\mu\rho} \alpha_{\nu\sigma} d\xi_\rho d\xi_\sigma = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 - d\xi_4^2,$$

there obtains the following relation between the determinant

$$g = |g_{\mu\nu}|,$$

i.e., the discriminant of the quadratic differential form ds^2, and the substitution determinant $|\alpha_{\rho\sigma}|$:

$$g \cdot (|\alpha_{\rho\sigma}|)^2 = -1,$$

$$|\alpha_{\rho\sigma}| = \frac{1}{\sqrt{-g}}.$$

Thus, one obtains the following relation for V:

$$Vdt = V_0 ds \cdot \frac{1}{\sqrt{-g}}.$$

From this one obtains with the help of (7), (8), and (9), if one substitutes ρ_0 for $\frac{m}{V_0}$,

$$\frac{J_x}{V} = -\rho_0\sqrt{-g}\cdot\sum_\nu g_{1\nu}\frac{dx_\nu}{ds}\cdot\frac{dx_4}{ds},$$

$$-\frac{E}{V} = -\rho_0\sqrt{-g}\cdot\sum_\nu g_{4\nu}\frac{dx_\nu}{ds}\cdot\frac{dx_4}{ds},$$

$$\frac{\mathfrak{K}_x}{V} = -\frac{1}{2}\rho_0\sqrt{-g}\cdot\sum_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x_1}\cdot\frac{dx_\mu}{ds}\cdot\frac{dx_\nu}{ds}.$$

We note that

$$\Theta_{\mu\nu} = \rho_0\frac{dx_\mu}{ds}\cdot\frac{dx_\nu}{ds}$$ [21]

is a second-rank contravariant tensor with respect to arbitrary substitutions. From the foregoing one surmises that the momentum-energy law will have the form

(10) $$\sum_{\mu\nu}\frac{\partial}{\partial x_\nu}(\sqrt{-g}\cdot g_{\sigma\mu}\Theta_{\mu\nu}) - \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\Theta_{\mu\nu} = 0\,. \qquad (\sigma = 1,2,3,4)$$ [22]

The first three of these equations ($\sigma = 1,2,3$) express the momentum law, and the last one ($\sigma = 4$) the energy law. It turns out that these equations are in fact covariant with respect to arbitrary substitutions.[9] Also, the equations of motion of the material point from which we started out can be rederived from these equations by integrating over the thread of flow.

We call the tensor $\Theta_{\mu\nu}$ the (contravariant) *stress-energy tensor of the material flow*. We ascribe to equation (10) a validity range that goes far beyond the special case of the flow of incoherent masses. The equation represents in general the energy [23] balance between the gravitational field and an arbitrary material process; one has only to substitute for $\Theta_{\mu\nu}$ the stress-energy tensor corresponding to the material system under consideration. The first sum in the equation contains the space derivatives of the stresses or of the density of the energy flow, and the time derivatives of the momentum density or of the energy density; the second sum is an expression for the effects exerted by the gravitational field on the material process.

§5. The Differential Equations of the Gravitational Field

Having established the momentum-energy equation for material processes (mechanical, electrical, and other processes) in relation to the gravitational field, there remains for us only the following task. Let the tensor $\Theta_{\mu\nu}$ for the material process be given.

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[9]Cf. Part II, §4, No. 1.

What differential equations permit us to determine the quantities g_{ik}, i.e., the gravitational field? In other words, we seek the generalization of Poisson's equation

$$\Delta\varphi = 4\pi k\rho.$$

We have not found a method for the solution of this problem as thoroughly compelling as that for the solution of the problem discussed previously. It would be necessary to introduce several assumptions whose correctness seems plausible but not evident.

The generalization that we seek would likely have the form

$$\kappa\cdot\Theta_{\mu\nu} = \Gamma_{\mu\nu}, \qquad (11)$$

where κ is a constant and $\Gamma_{\mu\nu}$ a second-rank contravariant tensor derived from the fundamental tensor $g_{\mu\nu}$ by differentiatial operations. In line with the Newton-Poisson law one would be inclined to require that these equations (11) be *second order*. But it must be stressed that, given this assumption, it proves impossible to find a differential expression $\Gamma_{\mu\nu}$ that is a generalization of $\Delta\varphi$ and that proves to be a *tensor* with respect to *arbitrary* transformations.[10] To be sure, it cannot be negated a priori that the final, exact equations of gravitation could be of higher than second order. Therefore there still exists the possibility that the perfectly exact differential equations of gravitation could be covariant with respect to *arbitrary* substitutions. But given the present state of our knowledge of the physical properties of the gravitational field, the attempt to discuss such possibilities would be premature. For that reason we have to confine ourselves to the second order, and we must therefore forgo setting up gravitational equations that are covariant with respect to arbitrary transformations. Besides, it should be emphasized that we have no basis whatsoever
[24] for assuming a general covariance of the gravitational equations.[11]

The Laplacian scalar $\Delta\varphi$ is obtained from the scalar φ if one forms the expansion (the gradient) of the latter and then the inner operator (the divergence) of this. Both operations can be generalized in such a way that one can carry them out on every tensor of arbitrarily high rank, namely while permitting arbitrary substitutions of the basic variables.[12] But these operations degenerate if they are carried out on the fundamental tensor $g_{\mu\nu}$.[13] From this it seems to follow that the equations sought will be covariant only with respect to a particular group of transformations,
[25] which group, however, is as yet unknown to us.

[10] Cf. Part II, §4, No. 2.
[11] Cf. also the arguments given at the beginning of §6.
[12] Part II, §2.
[13] Cf. the remark on p. 28 in Part II, §2.

Given this state of affairs, and in view of the old theory of relativity, it seems natural to assume that *the transformation group we are seeking also includes the linear transformations.* Hence we require that $\Gamma_{\mu\nu}$ be a tensor with respect to arbitrary linear transformations.

Now it is easy to prove (by carrying out the transformation) the following theorems: [26]

1. If $\Theta_{\alpha\beta\ldots\lambda}$ is a contravariant tensor of rank n with respect to linear transformations, then

$$\sum_{\mu} \gamma_{\mu\nu} \frac{\partial \Theta_{\alpha\beta\ldots\lambda}}{\partial x_{\mu}}$$

is a contravariant tensor of rank $n + 1$ with respect to linear transformations (expansion).[14]

2. If $\Theta_{\alpha\beta\ldots\lambda}$ is a contravariant tensor of rank n with respect to linear transformations, then

$$\sum_{\lambda} \frac{\partial \Theta_{\alpha\beta\ldots\lambda}}{\partial x_{\lambda}}$$

is a contravariant tensor of rank $n - 1$ with respect to linear transformations (divergence).

If one carries out these two operations on a tensor in succession, one obtains a tensor of the same rank as the original one (operation Δ, carried out on a tensor). For the fundamental tensor $\gamma_{\mu\nu}$ one obtains

(a) $$\sum_{\alpha\beta} \frac{\partial}{\partial x_{\alpha}}\left(\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_{\beta}}\right).$$ [27]

One can also see from the following argument that this operator is related to the Laplacian operator. In the theory of relativity (absence of gravitational field) one would have to set

$$g_{11} = g_{22} = g_{33} = -1, \quad g_{44} = c^2, \quad g_{\mu\nu} = 0, \quad \text{for } \mu \neq \nu;$$

hence $$\gamma_{11} = \gamma_{22} = \gamma_{33} = -1, \quad \gamma_{44} = \frac{1}{c^2}, \quad \gamma_{\mu\nu} = 0, \text{ for } \mu \neq \nu.$$

If a gravitational field is present that is sufficiently weak, i.e., if the $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ differ only infinitesimally from the values just given, then one obtains instead of the expression (a), neglecting the second-order terms,

[14] $\gamma_{\mu\nu}$ is the contravariant tensor reciprocal to $g_{\mu\nu}$ (Part II, §1).

$$-\left(\frac{\partial^2\gamma_{\mu\nu}}{\partial x_1^2}+\frac{\partial^2\gamma_{\mu\nu}}{\partial x_2^2}+\frac{\partial^2\gamma_{\mu\nu}}{\partial x_3^2}-\frac{1}{c^2}\frac{\partial^2\gamma_{\mu\nu}}{\partial x_4^2}\right).$$

If the field is static and only $g_{\mu\nu}$ is variable, we thus arrive at the case of the Newtonian theory of gravitation if we take the expression obtained for the quantity
[28] $\Gamma_{\mu\nu}$ up to a constant.

Hence one might think that, up to a constant factor, the expression (a) must already be the generalization of $\Delta\varphi$ that we are seeking. But this would be a mistake; for alongside this expression, in a generalization of this kind there could also appear terms that are themselves tensors and that vanish when we neglect the kinds of terms just indicated. This always occurs when two first derivatives of the $g_{\mu\nu}$ or $\gamma_{\mu\nu}$ are multiplied by each other. Thus, for example,

$$\sum_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x_\mu}\cdot\frac{\partial\gamma_{\alpha\beta}}{\partial x_\nu}$$

is a covariant tensor of the second rank (with respect to linear transformations); it becomes infinitesimally small to the second order if the quantities $g_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ deviate from constant values only infinitesimally to the first order. We must therefore allow still other terms in $\Gamma_{\mu\nu}$, in addition to (a), which terms, for now, must satisfy only the condition that, taken together, they must possess the character of a tensor with respect to linear transformations.

We make use of the momentum-energy law to find these terms. To make myself
[29] clear about the method used, I will first apply it to a generally known example.

In *electrostatics* $-\frac{\partial\varphi}{\partial x_\nu}\rho$ is the νth component of the momentum transferred to the matter per unit volume, if φ denotes the electrostatic potential and ρ the electric density. We seek a differential equation for φ of such kind that the law of the conservation of momentum is always satisfied. It is well known that the equation

$$\sum_\nu\frac{\partial^2\varphi}{\partial x_\nu^2}=\rho$$

solves the problem. The fact that the momentum law is satisfied follows from the identity

$$\sum_\mu\frac{\partial}{\partial x_\mu}\left(\frac{\partial\varphi}{\partial x_\nu}\frac{\partial\varphi}{\partial x_\mu}\right)-\frac{\partial}{\partial x_\nu}\left(\frac{1}{2}\sum_\mu\left(\frac{\partial\varphi}{\partial x_\mu}\right)^2\right)=\frac{\partial\varphi}{\partial x_\nu}\sum_\mu\frac{\partial^2\varphi}{\partial x_\mu^2}\left(=-\frac{\partial\varphi}{\partial x_\nu}\cdot\rho\right).$$

Thus, if the momentum law is satisfied, then an identity of the following construction must exist for every ν: On the right side, $-\frac{\partial\varphi}{\partial x_\nu}$ is multiplied by the left side of the differential equation; on the left side of the identity there is a sum of the differential quotients.

If the differential equation for φ were not yet known, the problem of finding it would be reduced to that of finding this identity. What is essential for us to realize is that this identity can be derived *if one of the terms occurring in it is known.* All one has to do is to apply repeatedly the product differentiation rule in the forms

$$\frac{\partial}{\partial x_\nu}(uv) = \frac{\partial u}{\partial x_\nu}v + \frac{\partial v}{\partial x_\nu}u$$

and

$$u\frac{\partial v}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}(uv) - \frac{\partial u}{\partial x_\nu}v,$$

and then finally to put the terms that are differential quotients on the left side and the rest of the terms on the right side. For example, if one starts with the first term of the above identity, one obtains, one after another,

$$\sum_\mu \frac{\partial}{\partial x_\mu}\left(\frac{\partial \varphi}{\partial x_\nu}\frac{\partial \varphi}{\partial x_\mu}\right) = \sum_\mu \frac{\partial \varphi}{\partial x_\nu}\cdot\frac{\partial^2 \varphi}{\partial x_\mu^2} + \sum_\mu \frac{\partial \varphi}{\partial x_\mu}\cdot\frac{\partial^2 \varphi}{\partial x_\nu \partial x_\mu}$$
$$= \frac{\partial \varphi}{\partial x_\nu}\cdot\sum_\mu \frac{\partial^2 \varphi}{\partial x_\mu^2} + \frac{\partial}{\partial x_\nu}\left\{\frac{1}{2}\sum_\mu\left(\frac{\partial \varphi}{\partial x_\mu}\right)^2\right\},$$

from which we obtain the above identity upon rearrangement.

Now we turn again to our problem. It follows from equation (10) that

$$\frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\Theta_{\mu\nu} \qquad (\sigma = 1,2,3,4)$$

is the momentum (or energy) imparted by the gravitational field to the matter per unit volume. For the energy-momentum law to be satisfied, the differential expressions $\Gamma_{\mu\nu}$ of the fundamental quantities $\gamma_{\mu\nu}$ that enter the gravitational equations

$$\kappa\cdot\Theta_{\mu\nu} = \Gamma_{\mu\nu}$$

must be chosen such that

$$\frac{1}{2\kappa}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\Gamma_{\mu\nu}$$

can be rewritten in such a way that it appears as the sum of differential quotients. On the other hand, we know that the term (a) appears in the expression sought for $\Gamma_{\mu\nu}$. Hence the identity that is being sought has the following form:

Sum of differential quotients

$$= \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\left\{\sum_{\alpha\beta}\frac{\partial}{\partial x_\alpha}\left(\gamma_{ab}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right)\right.$$

+ the other terms, which vanish with the first approximation.}

The identity that is being sought is thereby uniquely determined; if one [30] constructs it according to the procedure indicated,[15] one obtains

$$
(12)\left\{\begin{aligned}
&\sum_{\alpha\beta\tau\varrho}\frac{\partial}{\partial x_\alpha}\left(\sqrt{-g}\cdot\gamma_{\alpha\beta}\frac{\partial\gamma_{\tau\varrho}}{\partial x_\beta}\cdot\frac{\partial g_{\tau\varrho}}{\partial x_\sigma}\right)-\frac{1}{2}\cdot\sum_{\alpha\beta\tau\varrho}\frac{\partial}{\partial x_\sigma}\left(\sqrt{-g}\cdot\gamma_{\alpha\beta}\frac{\partial\gamma_{\tau\varrho}}{\partial x_\alpha}\frac{\partial g_{\tau\varrho}}{\partial x_\beta}\right)\\
&=\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\left\{\sum_{\alpha\beta}\frac{1}{\sqrt{-g}}\cdot\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\cdot\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right)-\sum_{\alpha\beta\tau\varrho}\gamma_{\alpha\beta}g_{\tau\varrho}\frac{\partial\gamma_{\mu\tau}}{\partial x_\alpha}\frac{\partial\gamma_{\nu\varrho}}{\partial x_\beta}\right.\\
&\qquad\left.+\frac{1}{2}\sum_{\alpha\beta\tau\varrho}\gamma_{\alpha\mu}\gamma_{\beta\nu}\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\varrho}}{\partial x_\beta}-\frac{1}{4}\sum_{\alpha\beta\tau\varrho}\gamma_{\mu\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\varrho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\varrho}}{\partial x_\beta}\right\}.
\end{aligned}\right.
$$

Thus, the expression for $\Gamma_{\mu\nu}$ that is enclosed between the curly brackets on the right-hand side is the tensor that is being sought that enters into the gravitational equations

$$\kappa\Theta_{\mu\nu} = \Gamma_{\mu\nu}.$$

To make these equations more comprehensible, we introduce the following abbreviations:

$$(13)\qquad -2\kappa\cdot\vartheta_{\mu\nu} = \sum_{\alpha\beta\tau\rho}\left(\gamma_{\alpha\mu}\gamma_{\beta\nu}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\cdot\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta} - \frac{1}{2}\gamma_{\mu\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}\right).$$

We will designate $\vartheta_{\mu\nu}$ as the "*contravariant stress-energy tensor of the* [31] *gravitational field.*" The covariant tensor reciprocal to it will be denoted by $t_{\mu\nu}$; then we have

$$(14)\qquad -2\kappa\cdot t_{\mu\nu} = \sum_{\alpha\beta\tau\rho}\left(\frac{\partial g_{\tau\rho}}{\partial x_\mu}\cdot\frac{\partial\gamma_{\tau\rho}}{\partial x_\nu} - \frac{1}{2}g_{\mu\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x_\alpha}\frac{\partial\gamma_{\tau\rho}}{\partial x_\beta}\right).$$

Likewise, for the sake of brevity, we introduce the following notations for differential operations carried out on the fundamental tensors γ and g:

$$(15)\qquad \Delta_{\mu\nu}(\gamma) = \sum_{\alpha\beta}\frac{1}{\sqrt{-g}}\cdot\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\cdot\frac{\partial\gamma_{\mu\nu}}{\partial x_\beta}\right) - \sum_{\alpha\beta\tau\rho}\gamma_{\alpha\beta}g_{\tau\rho}\frac{\partial\gamma_{\mu\tau}}{\partial x_\alpha}\frac{\partial\gamma_{\nu\rho}}{\partial x_\beta},$$

and

$$(16)\qquad D_{\mu\nu}(\gamma) = \sum_{\alpha\beta}\frac{1}{\sqrt{-g}}\cdot\frac{\partial}{\partial x_\alpha}\left(\gamma_{\alpha\beta}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\beta}\right) - \sum_{\alpha\beta\tau\rho}\gamma_{\alpha\beta}\gamma_{\tau\rho}\frac{\partial g_{\mu\tau}}{\partial x_\alpha}\frac{\partial g_{\nu\rho}}{\partial x_\beta}.$$

[15] Cf. Part II, §4, No. 8.

Each of these operators yields again a tensor of the same kind (w. resp. to linear transformations).

With the application of these abbreviations the identity (12) assumes the form

(12a) $$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot g_{\sigma\mu}\cdot\kappa\vartheta_{\mu\nu}\right\} = \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\{-\Delta_{\mu\nu}(\gamma) + \kappa\vartheta_{\mu\nu}\},$$

or also

(12b) $$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot \gamma_{\mu\nu}\cdot\kappa t_{\mu\sigma}\right\} = \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma}\{-D_{\mu\nu}(g) + \kappa\cdot t_{\mu\nu}\}.$$

If we write the conservation law (10) for matter and the conservation law (12a) for the gravitational field in the form

(10) $$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot g_{\sigma\mu}\cdot\Theta_{\mu\nu}\right\} - \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\cdot\Theta_{\mu\nu} = 0$$

(12c) $$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left(\sqrt{-g}\cdot g_{\sigma\mu}\cdot\vartheta_{\mu\nu}\right) - \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\mu}\cdot\vartheta_{\mu\nu} = -\frac{1}{2\kappa}\cdot\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\cdot\Delta_{\mu\nu}(\gamma),$$

then one recognizes that the stress-energy tensor $\vartheta_{\mu\nu}$ of the gravitational field enters the conservation law for the gravitational field in exactly the same way as the tensor $\Theta_{\mu\nu}$ of the material process enters the conservation law for this process; this is a noteworthy circumstance considering the difference in the derivation of the two laws.

From equation (12a) follows the expression for the differential tensor entering into the gravitational equations

(17) $$\Gamma_{\mu\nu} = \Delta_{\mu\nu}(\gamma) - \kappa\cdot\vartheta_{\mu\nu}.$$

Thus, the gravitational equations (11) are of the form

(18) $$\Delta_{\mu\nu}(\gamma) = \kappa(\Theta_{\mu\nu} + \vartheta_{\mu\nu}).$$ [32]

These equations satisfy a requirement that, in our opinion, must be imposed on a relativity theory of gravitation; that is to say, they show that the tensor $\vartheta_{\mu\nu}$ of the gravitational field acts as a field generator in the same way as the tensor $\Theta_{\mu\nu}$ of the material processes. An exceptional position of gravitational energy in comparison [33] with all other kinds of energies would lead to untenable consequences.

Adding equations (10) and (12a) while taking into account equation (18), one finds

[34] (19) $$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot g_{\sigma\mu}(\Theta_{\mu\nu} + \vartheta_{\mu\nu})\right\} = 0 \qquad (\sigma = 1,2,3,4)$$

This shows that the conservation laws hold for the matter and the gravitational field taken together.

In the foregoing we have given preference to the contravariant tensors, because the contravariant stress-energy tensor of the flow of incoherent masses can be expressed in an especially simple manner. However, we can express the fundamental relations that we have obtained just as simply by using covariant tensors. Instead of $\Theta_{\mu\nu}$, we must then take $T_{\mu\nu} = \sum_{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} \Theta_{\alpha\beta}$ as the stress-energy tensor of the material process. Instead of equation (10), we obtain through term-by-term reformulation

(20) $$\sum_{\mu\nu} \frac{\partial}{\partial x_\nu}(\sqrt{-g}\cdot\gamma_{\mu\nu} T_{\mu\sigma}) + \frac{1}{2}\sum_{\mu\nu}\sqrt{-g}\cdot\frac{\partial \gamma_{\mu\nu}}{\partial x_\sigma}\cdot T_{\mu\nu} = 0.$$

It follows from this equation and equation (16) that the equations of the gravitational field can also be written in the form

(21) $$-D_{\mu\nu}(g) = \kappa(t_{\mu\nu} + T_{\mu\nu});$$

these equations can also be derived directly from (18). The equation that corresponds to (19) reads

(22) $$\sum_{\nu} \frac{\partial}{\partial x_\nu}\left\{\sqrt{-g}\cdot\gamma_{\sigma\mu}(T_{\mu\nu} + t_{\mu\nu})\right\} = 0.$$

§6. Influence of the Gravitational Field on Physical Processes, Especially on the Electromagnetic Processes

[35]

Since momentum and energy play a role in every physical process and, for their part, also determine the gravitational field and are influenced by it, the quantities $g_{\mu\nu}$ that determine the gravitational field must appear in all systems of physical equations. Thus, we have seen that the motion of the material point is determined by the equation

$$\delta\left\{\int ds\right\} = 0,$$

where

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu .$$

ds is an invariant with respect to arbitrary substitutions. The equations to be sought, which determine the course of some physical process or other, must be so constructed that the invariance of ds will entail the covariance of the equation system in question.

But in the pursuit of solutions to these general problems, we at first encounter a fundamental difficulty. We do not know with respect to which group of transformations the equations we are seeking must be covariant. At first it seems most natural to demand that the systems of equations should be covariant with respect to *arbitrary* transformations. But opposed to this is the fact that the equations of the gravitational field that we have set up do not possess this property. For the equations of gravitation we have only been able to prove that they are covariant with respect to arbitrary *linear* transformations; but we do not know whether there exists a general group of transformations with respect to which the equations are covariant. The question as to the existence of such a group for the system of equations (18) and (21) is the most important question connected with the considerations presented here. At any rate, given the present state of the theory, it is not justifiable for us to demand a covariance of physical equations with respect to arbitrary substitutions. [36]

But on the other hand we have seen that for material processes it is indeed possible to set up an energy-momentum balance equation that does permit arbitrary transformations (§4, equation 10). Therefore it nevertheless seems natural to assume that all systems of physical equations, with the exception of the gravitational equations, should be formulated in such a way that they are covariant with respect to arbitrary substitutions. This exceptional position that the gravitational equations occupy in this respect, as compared with all of the other systems, has to do, in my opinion, with the fact that only the former can contain second derivatives of the components of the fundamental tensor.

The construction of such systems of equations requires the resources of generalized vector analysis as it is presented in Part II.

Here we confine ourselves to indicating how one obtains the electromagnetic field equations for the vacuum in this way.[16] We start from the assumption that the electrical charge is to be viewed as something unchangeable. Suppose that an infinitesimally small, arbitrarily moving body has the charge e and the volume dV_0

[16]On this point, cf. also the article by Kottler, §3, cited on p. 23. [37]

with respect to a comoving body (rest volume). We define $\frac{e}{dV_0} = \rho_0$ as the true density of the electricity; this is a scalar by definition. Hence

[38] $$\rho_0 \frac{dx_\nu}{ds} \qquad (\nu = 1,2,3,4)$$

is a contravariant four-vector, which we reformulate by defining the density ρ of the electricity, referred to a coordinate system, by the equation

$$\rho_0 dv_0 = \rho dV.$$

Using the equation

$$dV_0 ds = \sqrt{-g} \cdot dV \cdot dt$$

from §4, we obtain

$$\rho_0 \frac{dx_\nu}{ds} = \frac{1}{\sqrt{-g}} \rho \frac{dx_\nu}{dt},$$

i.e., the contravariant vector of the electric current.

We reduce the electromagnetic field to a special, contravariant tensor of second
[39] rank $\varphi_{\mu\nu}$ (a six-vector), and form the "dual" contravariant tensor of second rank $\varphi^*_{\mu\nu}$ by the method explained in Part II, §3 (formula 42). According to formula 40 in §3 of Part II, the divergence of a special contravariant tensor of second rank is

$$\frac{1}{\sqrt{-g}} \sum_\nu \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \varphi_{\mu\nu}\right).$$

[40] As a generalization of the Maxwell-Lorentz field equations, we set up the equations

$$\sum_\nu \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \varphi_{\mu\nu}\right) = \rho \frac{dx_\mu}{dt}, \qquad (dt = dx_4) \tag{23}$$

$$\sum_\nu \frac{\partial}{\partial x_\nu} \left(\sqrt{-g} \cdot \varphi^*_{\mu\nu}\right) = 0, \tag{24}$$

the covariance of which is self-evident. If we set

$$\sqrt{-g} \cdot \varphi_{23} = \mathfrak{H}_x, \; \sqrt{-g} \cdot \varphi_{31} = \mathfrak{H}_y, \; \sqrt{-g} \cdot \varphi_{12} = \mathfrak{H}_z;$$
$$\sqrt{-g} \cdot \varphi_{14} = -\mathfrak{E}_x, \; \sqrt{-g} \cdot \varphi_{24} = -\mathfrak{E}_y, \; \sqrt{-g} \cdot \varphi_{34} = -\mathfrak{E}_z,$$

and

$$\rho \frac{dx_\mu}{dt} = u_\mu,$$

then the system of equations (23), written out in a more detailed manner, takes the form

$$\frac{\partial \mathfrak{H}_z}{\partial y} - \frac{\partial \mathfrak{H}_y}{\partial z} - \frac{\partial \mathfrak{E}_x}{dt} = u_x$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\frac{\partial \mathfrak{E}_x}{\partial x} + \frac{\partial \mathfrak{E}_y}{\partial y} + \frac{\partial \mathfrak{E}_z}{\partial z} = \varrho\,,$$

Up to the choice of the units, these equations coincide with Maxwell's first system. In constructing the second system, one has first to bear in mind that to the components

$$\mathfrak{H}_x,\ \mathfrak{H}_y,\ \mathfrak{H}_z,\ -\mathfrak{E}_x,\ -\mathfrak{E}_y,\ -\mathfrak{E}_z$$

of

$$\sqrt{-g}\cdot\varphi_{\mu\nu}$$

there correspond the components

$$-\mathfrak{E}_x,\ -\mathfrak{E}_y,\ -\mathfrak{E}_z,\ \mathfrak{H}_x,\ \mathfrak{H}_y,\ \mathfrak{H}_z$$

of the complement $f_{\mu\nu}$ (Part II, §3, formulas 41a). For the case where no gravitational field is present, this yields the second system, i.e., equation (24) in the form

$$-\frac{\partial \mathfrak{E}_z}{\partial x} + \frac{\partial \mathfrak{E}_y}{\partial z} - \frac{1}{c^2}\frac{\partial \mathfrak{H}_x}{\partial t} = 0$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$-\frac{1}{c^2}\frac{\partial \mathfrak{H}_x}{\partial x} - \frac{1}{c^2}\frac{\partial \mathfrak{H}_y}{\partial t} - \frac{1}{c^2}\frac{\partial \mathfrak{H}_z}{\partial z} = 0.$$ [41]

This proves that the equations we have set up really constitute a generalization of the equations of the ordinary theory of relativity.

§7. Can the Gravitational Field Be Reduced to a Scalar? [42]

In view of the undeniable complexity of the theory of gravitation propounded here, we must ask ourselves in earnest whether the conception that has, until now, been the only one advanced, according to which the gravitational field is reduced to a scalar Φ, is the only one that is reasonable and justified. I will briefly explain why we think that this question must be answered in the negative.

When one characterizes the gravitational feld by a scalar, a path presents itself that is completely analogous with that which we followed in the foregoing. One sets up the equation of motion of the material point in Hamiltonian form

$$\delta\left\{\int \Phi\, ds\right\} = 0,$$

where ds is the four-dimensional line element from the ordinary theory of relativity and Φ is a scalar, and then proceeds wholly by analogy with the foregoing, without having to give up the ordinary theory of relativity.

Here, too, the material process of an arbitrary kind is characterized by a stress-energy tensor $T_{\mu\nu}$. But with this conception it is a *scalar* that determines the interaction between the gravitational field and the material process. As Mr. Laue pointed out to me, this scalar can only be

$$\sum_{\mu} T_{\mu\mu} = P,$$

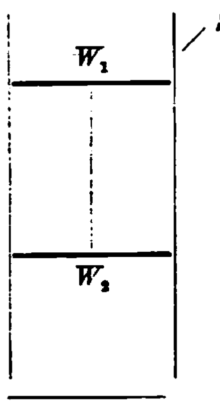
[43] which I will call the "Laue scalar."[17] Here too one can then do justice to the law of the equivalence of inertial and gravitational mass up to a certain degree. For Mr. Laue drew my attention to the fact that for a closed system

$$\int P dV = \int T_{44} d\tau.$$

From this, one can see that according to this conception too the gravity of a closed
[44] system is determined by its total energy.

But the gravity of systems that are not closed would depend on the orthogonal stresses T_{11} etc. to which the system is subjected. This leads to consequences that seem to me unacceptable, as shall be demonstrated with the example of cavity radiation.

As we know, for radiation in a vacuum, the scalar P vanishes. If the radiation is enclosed in a massless reflecting box, then its walls experience tensile stresses, as the result of which the system—taken as a whole—possesses a gravitational mass $\int P d\tau$ corresponding to the energy E of the radiation.



But instead of enclosing the radiation in a hollow box, I now imagine that it is bounded

1. by the reflecting walls of a firmly fixed shaft S,
2. by two reflecting walls W_1 and W_2 that can be displaced vertically and that are rigidly tied to each other by a rod.

In that case, the gravitational mass $\int P d\tau$ of the movable system amounts only to one-third of the value obtained in the case of a box moving as a whole. Thus, in order to lift the radiation against a gravitational field, one would have to apply only one-third of the work that one would have to apply in the previously considered case of the radiation enclosed in a box. This seems unacceptable to me.

Of course, I must admit that, for me, the most effective argument for the rejection of such a theory rests on the conviction that relativity holds not only with respect to

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[17] Cf. Part II, §1, last formula.

orthogonal linear substitutions but also with respect to a much wider group of substitutions. But already the mere fact that we were not able to find the (most general) group of substitutions associated with our gravitational equations makes it unjustifiable for us to press this argument.

II
Mathematical Part
by Marcel Grossman

The mathematical tools for developing the vector analysis of a gravitational field, which is characterized by the invariance of the line element

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx_\mu dx_\nu,$$

derive from Christoffel's fundamental paper on the transformation of quadratic differential forms.[1] Taking Christoffel's results as their starting point, Ricci and Levi-Cività[2] developed their methods of the absolute differential calculus—i.e., a differential calculus that is independent of the coordinate system—which permit our
[47] giving an invariant form to the differential equations of mathematical physics. But since the vector analysis of a Euclidean space referred to arbitrary curvilinear coordinates is formally identical with the vector analysis of an arbitrary manifold specified by its line element, the extension of the vector-analytical conceptions that
[48] Minkowski, Sommerfeld, Laue, et al. worked out for the theory of relativity in recent years to the general theory of Einstein's expounded above does not present any difficulty.

With some practice, *the general vector analysis* obtained in this way is as simple to handle as the special vector analysis of three- or four-dimensional Euclidean space; in fact, the greater generality of its conceptions lends it a clarity that is lacking often enough in the special case.

The theory of special tensors (§3) has been treated to the full in a paper by Kottler,[3] published while this work was in progress; the treatment is based on the theory of integral forms, something that is not possible in the general case.

Since more detailed mathematical investigations will have to be done in connection with Einstein's theory of gravitation, and especially in connection with the [50]
problem of the differential equations of the gravitational field, a systematic presentation of the general vector analysis might be in order. I have purposely not employed geometrical aids because, in my opinion, they contribute very little to an intuitive understanding of the conceptions of vector analysis.

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[1]Christoffel, "Über die Transformation der homogenen Differentialausdrücke zweiten
[45] Grades," *J. f. Math.* 70 (1869): 46.

[2]Ricci et Levi-Cività, "Méthodes de calcul différentiel absolu et leurs application."
[46] *Math. Ann.* 54 (1901): 125.

[49] [3]Kottler, "Über die Raumzeitlinien der Minkowskischen Welt." *Wien. Ber.* 121 (1912).

$$T_\sigma = \sum_\mu g_{\sigma\mu}\Theta_\mu = \sum_{\mu\nu k}\left(\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\nu}(\sqrt{g}\cdot g_{\sigma\mu}\cdot\Theta_{\mu\nu}) - \frac{\partial g_{\sigma\mu}}{\partial x_\nu}\cdot\Theta_{\mu\nu} + g_{\sigma\mu}\begin{Bmatrix}\nu k\\ \mu\end{Bmatrix}\cdot\Theta_{\nu k}\right).$$

But the last term of this sum is equal to

$$\sum_{\nu k}\begin{bmatrix}\nu k\\ \sigma\end{bmatrix}\Theta_{\nu k} = \sum_{\mu\nu}\frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right)\cdot\Theta_{\mu\nu}.$$

Hence, we end up with

$$T_\sigma = \sum_{\mu\nu}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\nu}(\sqrt{g}\cdot g_{\sigma\mu}\Theta_{\mu\nu}) - \frac{1}{2}\sum_{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\cdot\Theta_{\mu\nu},$$

i.e., the left side of the investigated equation, up to the factor $\frac{1}{\sqrt{g}}$. Thus, if that equation is divided by $\sqrt{g}$, then its left side represents the σ-component of a covariant vector, and is, therefore, in fact, covariant. For that reason, the content of those four equations can also be expressed thus:

The divergence of the (contravariant) stress-energy tensor of the material flow or of the physical process vanishes.

2. *Differential Tensors of a Manifold Given by Its Line Element*

The problem of constructing the differential equations of a gravitational field (Part I, §5) draws one's attention to the *differential invariants* and *differential covariants* of the quadratic differential form

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}dx_\mu dx_\nu.$$

In the sense of our general vector analysis, the theory of these differential covariants leads to the *differential tensors* that are given with a gravitational field. The complete system of these differential tensors (with respect to arbitrary transformations) goes back to a covariant differential tensor of fourth rank found by Riemann[12] and, independently of him, by Christoffel,[13] which we shall call the *Riemann differential tensor*, and which reads as follows:

$$R_{iklm} = (ik,lm) = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x_k\partial x_l} + \frac{\partial^2 g_{kl}}{\partial x_i\partial x_m} - \frac{\partial^2 g_{il}}{\partial x_k\partial x_m} - \frac{\partial^2 g_{mk}}{\partial x_l\partial x_i}\right) \tag{43}$$

[66] [12] Riemann, *Ges. Werke*, p. 270.

[67] [13] Christoffel, l.c., p. 54.

$$+ \sum_{\rho\sigma} \gamma_{\rho\sigma} \left(\begin{bmatrix} im \\ \rho \end{bmatrix} \begin{bmatrix} kl \\ \sigma \end{bmatrix} - \begin{bmatrix} il \\ \rho \end{bmatrix} \begin{bmatrix} km \\ \sigma \end{bmatrix} \right).$$ [68]

By means of covariant algebraic and differential operations we obtain the complete system of differential tensors (thus also the differential invariants) of the manifold from the Riemann differential tensor and the discriminant tensor (§3, formula 38). [69]

The (ik, lm) are also called the *Christoffel four-index symbols of the first kind.* In addition to these, of importance are also the *four-index symbols of the second kind*

$$(44) \qquad \{ik,\ lm\} = \frac{\partial \begin{Bmatrix} il \\ k \end{Bmatrix}}{\partial x_m} - \frac{\partial \begin{Bmatrix} im \\ k \end{Bmatrix}}{\partial x_l} + \sum_{\rho} \left(\begin{Bmatrix} il \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho m \\ k \end{Bmatrix} - \begin{Bmatrix} im \\ \rho \end{Bmatrix} \begin{Bmatrix} \rho l \\ k \end{Bmatrix} \right),$$

which are related to the former in the following way:

$$(45) \qquad \begin{cases} \{i\rho,\ lm\} = \sum_k \gamma_{\rho k}(ik,\ lm), \text{ or, when solved,} \\ \\ (ik,\ lm) = \sum_\rho g_{k\rho}\ \{i\rho,\ lm\}. \end{cases}$$

In general vector analysis, the four-index symbols of the second kind take on the meaning of the components of a *mixed tensor* that is covariant of third rank and contravariant of first rank.[14]

The extraordinary importance of these conceptions for the *differential geometry*[15] of a manifold that is given by its line element makes it a priori probable that these general differential tensors may also be of importance for the problem of the differential equations of a gravitational field. To begin with, it is, in fact, possible [71] to specify a covariant differential tensor of second rank and second order G_{im} that could enter into those equations, namely,

$$(46) \qquad G_{im} = \sum_{kl} \gamma_{kl}\,(ik,\ lm) = \sum_k \{ik,\ km\}.$$

It turns out, however, that in the special case of the infinitely weak, static gravitational field this tensor does *not* reduce to the expression $\Delta\varphi$. We must [72] therefore leave open the question to what extent the general theory of the differential tensors associated with a gravitational field is connected with the problem of the

[14]This follows from the first of equations 45.

[15]The identical vanishing of the tensor R_{iklm} constitutes a necessary and sufficient condition for the differential form's being transformable to the form $\sum_i dx_i^2$. [70]

gravitational equations. Such a connection would have to exist insofar as the gravitational equations are to permit *arbitrary* substitutions; but in that case, it seems that it would be impossible to find *second-order* differential equations. On the other hand, if it were established that the gravitational equations permit only a particular group of transformations, then it would be understandable if one could not manage with the differential tensors yielded by the general theory. As has been explained in the physical part, we are not able to take a stand on these questions.—

3. *On the Derivation of the Gravitational Equations*

The derivation of the gravitational equations described by Einstein (Part I, §5) is carried out step by step in the following way:

We start out from the term that is definitely to be expected in the energy balance,

$$U = \sum_{\alpha\beta\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \frac{\partial}{\partial x_\alpha}\left(\sqrt{g}\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right), \tag{47}$$

and reformulate it by integrating by parts.[16] In this way we obtain

$$U = \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{g}\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right) - \sum_{\alpha\beta\mu\nu}\sqrt{g}\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\cdot\frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma \partial x_\alpha}.$$

The first sum on the right-hand side has the desired form of a sum of differential quotients and shall be denoted by A, so that we have

$$A = \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\alpha}\left(\sqrt{g}\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right). \tag{48}$$

We once again integrate by parts in the second sum on the right-hand side. The identity will then take the form

$$U = A - \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\sigma}\left(\sqrt{g}\cdot\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\cdot\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right) + \sum_{\alpha\beta\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\cdot\frac{\partial}{\partial x_\sigma}\left(\sqrt{g}\cdot\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\right).$$

The first of the sums obtained on the right-hand side can be written as a sum of differentials and shall be denoted by

$$B = \sum_{\alpha\beta\mu\nu} \frac{\partial}{\partial x_\sigma}\left(\sqrt{g}\gamma_{\alpha\beta}\frac{\partial \gamma_{\mu\nu}}{\partial x_\beta}\frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right). \tag{49}$$

We differentiate in the second sum. Then we get

[16] The derivation of the identity we are seeking becomes simpler, without affecting the result, if we put the factor $\sqrt{g}$ inside the differentiation sign.

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