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Excerpt from Economic and Financial Decisions Under Risk

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Risk Aversion

This chapter looks at a basic concept behind modeling individual preferences in the face of risk. As with any social science, we of course are fallible and susceptible to second-guessing in our theories. It is nearly impossible to model many natural human tendencies such as "playing a hunch" or "being superstitious." However, we can develop a systematic way to view choices made under uncertainty. Hopefully, our models can capture the basic human tendencies enough to be useful in understanding market behavior towards risk. In other words, even if we are not correct in predicting behavior under risk for every individual in every circumstance, we can still make general claims about such behavior and can still make market predictions, which after all are based on the "marginal consumer."

To use (vaguely) mathematical language, the understanding of this chapter is a *necessary* but not *sufficient* condition to go further into the analysis. Because of the importance of risk aversion in decision making under uncertainty, it is worthwhile to first take an "historical" perspective about its development and to indicate how economists and decision scientists progressively have elaborated upon the tools and concepts we now use to analyze risky choices. In addition, this "history" has some surprising aspects that are interesting in themselves. To this end, our first section in this chapter broadly covers these retrospective topics. Subsequent sections are more "modern" and they represent an intuitive introduction to the central contribution to our field, that of Pratt (1964).

1.1 An Historical Perspective on Risk Aversion

As it is now widely acknowledged, an important breakthrough in the analysis of decisions under risk was achieved when Daniel Bernoulli, a distinguished Swiss mathematician, wrote in St Petersburg in 1738 a paper in Latin entitled: "*Specimen theoriae novae de mensura sortis*," or "Exposition of a new theory on the measurement of risk." Bernoulli's paper, translated into English in Bernoulli (1954), is essentially nontechnical. Its main purpose is to show that two people facing the same lottery may value it differently because of a difference in their psychology. This idea was quite novel at the time, since famous scientists before Bernoulli (among them

Pascal and Fermat) had argued that the value of a lottery should be equal to its mathematical expectation and hence identical for all people, independent of their risk attitude.

In order to justify his ideas, Bernoulli uses three examples. One of them, the "St Petersburg paradox" is quite famous and it is still debated today in scientific circles. It is described in most recent texts of finance and microeconomics and for this reason we do not discuss it in detail here. Peter tosses a fair coin repetitively until the coin lands head for the first time. Peter agrees to give to Paul 1 ducat if head appears on the first toss, 2 ducats if head appears only on the second toss, 4 ducats if head appears for the first time on the third toss, and so on, in order to double the reward to Paul for each additional toss necessary to see the head for the first time. The question raised by Bernoulli is how much Paul would be ready to pay to Peter to accept to play this game.

Unfortunately, the celebrity of the paradox has overshadowed the other two examples given by Bernoulli that show that, most of the time, the value of a lottery is not equal to its mathematical expectation. One of these two examples, which presents the case of an individual named "Sempronius," wonderfully anticipates the central contributions that would be made to risk theory about 230 years later by Arrow, Pratt and others.

Let us quote Bernoulli:[1]

> Sempronius owns goods at home worth a total of 4000 ducats and in addition possesses 8000 ducats worth of commodities in foreign countries from where they can only be transported by sea. However, our daily experience teaches us that of [two] ships one perishes.

In modern-day language, we would say that Sempronius faces a risk on his wealth. This wealth may represented by a lottery $\tilde{x}$, which takes on a value of 4000 ducats with probability $\frac{1}{2}$ (if his ship is sunk), or 12 000 ducats with probability $\frac{1}{2}$. We will denote such a lottery $\tilde{x}$ as being distributed as $(4000, \frac{1}{2}; 12\,000, \frac{1}{2})$. Its mathematical expectation is given by:

$$E\tilde{x} \equiv \tfrac{1}{2}4000 + \tfrac{1}{2}12\,000 = 8000 \text{ ducats.}$$

Now Sempronius has an ingenious idea. Instead of "trusting all his 8000 ducats of goods to one ship," he now "trusts equal portions of these commodities to two ships." Assuming that the ships follow independent but equally dangerous routes, Sempronius now faces a more diversified lottery $\tilde{y}$ distributed as

$$(4000, \tfrac{1}{4}; 8000, \tfrac{1}{2}; 12\,000, \tfrac{1}{4}).$$

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[1]We altered Bernoulli's probabilities to simplify the computations. In particular, Bernoulli's original example had one ship in ten perish.

Indeed, if both ships perish, he would end up with his sure wealth of 4000 ducats. Because the two risks are independent, the probability of these joint events equals the product of the individual events, i.e. $(\frac{1}{2})^2 = \frac{1}{4}$. Similarly, both ships will succeed with probability $\frac{1}{4}$, in which case his final wealth amounts to 12 000 ducats. Finally, there is the possibility that only one ship succeeds in downloading the commodities safely, in which case only half of the profit is obtained. The final wealth of Sempronius would then just amount to 8000 ducats. The probability of this event is $\frac{1}{2}$ because it is the complement of the other two events which have each a probability of $\frac{1}{4}$.

Since common wisdom suggests that diversification is a good idea, we would expect that the value attached to $\tilde{y}$ exceeds that attributed to $\tilde{x}$. However, if we compute the expected profit, we obtain that

$$E\tilde{y} = \tfrac{1}{4}4000 + \tfrac{1}{2}8000 + \tfrac{1}{4}12\,000 = 8000 \text{ ducats,}$$

the same value as for $E\tilde{x}$! If Sempronius would measure his well-being *ex ante* by his expected future wealth, he should be indifferent about whether to diversify or not. In Bernoulli's example, we obtain the same expected future wealth for both lotteries, even though most people would find $\tilde{y}$ more attractive than $\tilde{x}$. Hence, according to Bernoulli and to modern risk theory, the mathematical expectation of a lottery is not an adequate measure of its value. Bernoulli suggests a way to express the fact that most people prefer $\tilde{y}$ to $\tilde{x}$: a lottery should be valued according to the "expected utility" that it provides. Instead of computing the expectation of the monetary outcomes, we should use the expectation of the utility of the wealth. Notice that most human beings do not extract utility from wealth. Rather, they extract utility from consuming goods that can be purchased with this wealth. The main insight of Bernoulli is to suggest that there is a nonlinear relationship between wealth and the utility of consuming this wealth.

What ultimately matters for the decision maker *ex post* is how much satisfaction he or she can achieve with the monetary outcome, rather than the monetary outcome itself. Of course, there must be a relationship between the monetary outcome and the degree of satisfaction. This relationship is characterized by a utility function u, which for every wealth level x tells us the level of "satisfaction" or "utility" $u(x)$ attained by the agent with this wealth. Of course, this level of satisfaction derives from the goods and services that the decision maker can purchase with a wealth level x. While the outcomes themselves are "objective," their utility is "subjective" and specific to each decision maker, depending upon his or her tastes and preferences. Although the function u transforms the objective result x into a perception $u(x)$ by the individual, this transformation is assumed to exhibit some basic properties of rational behavior. For example, a higher level of x (more wealth) should induce a higher level of utility: the function should be increasing in x. Even for someone

who is very altruistic, a higher x will allow them to be more philanthropic. Readers familiar with indirect utility functions from microeconomics (essentially utility over budget sets, rather than over bundles of goods and services) can think of $u(x)$ as essentially an indirect utility of wealth, where we assume that prices for goods and services are fixed. In other words, we may think of $u(x)$ as the highest achievable level of utility from bundles of goods that are affordable when our income is x.

Bernoulli argues that if the utility u is not only increasing but also concave in the outcome x, then the lottery $\tilde{y}$ will have a higher value than the lottery $\tilde{x}$, in accordance with intuition. A twice-differentiable function u is concave if and only if its second derivative is negative, i.e. if the marginal utility $u'(x)$ is decreasing in x.[2] In order to illustrate this point, let us consider a specific example of a utility function, such as $u(x) = \sqrt{x}$, which is an increasing and concave function of x. Using these preferences in Sempronius's problem, we can determine the expectation of $u(x)$:

$$Eu(\tilde{x}) = \tfrac{1}{2}\sqrt{4000} + \tfrac{1}{2}\sqrt{12\,000} = 86.4$$

$$Eu(\tilde{y}) = \tfrac{1}{4}\sqrt{4000} + \tfrac{1}{2}\sqrt{8000} + \tfrac{1}{4}\sqrt{12\,000} = 87.9.$$

Because lottery $\tilde{y}$ generates a larger expected utility than lottery $\tilde{x}$, the former is preferred by Sempronius. The reader can try using concave utility functions other than the square-root function to obtain the same type of result. In the next section, we formalize this result.

Notice that the concavity of the relationship between wealth x and satisfaction/utility u is quite a natural assumption. It simply implies that the marginal utility of wealth is decreasing with wealth: one values a one-ducat increase in wealth more when one is poorer than when one is richer. Observe that, in Bernoulli's example, diversification generates a mean-preserving transfer of wealth from the extreme events to the mean. Transferring some probability weight from $x = 4000$ to $x = 8000$ increases expected utility. Each probability unit transferred yields an increase in expected utility equaling $u(8000) - u(4000)$. On the contrary, transferring some probability weight from $x = 12\,000$ to $x = 8000$ reduces expected utility. Each probability unit transferred yields a reduction in expected utility equaling $u(12\,000) - u(8000)$. But the concavity of u implies that

$$u(8000) - u(4000) > u(12\,000) - u(8000), \tag{1.1}$$

i.e. that the positive effect of these combined transfers must dominate the negative effect. This is why all investors with a concave utility would support Sempronius's strategy to diversify risks.

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[2]For simplicity, we maintain the assumption that u is twice differentiable throughout the book. However, a function need not be differentiable to be concave. More generally, a function u is concave if and only if $\lambda u(a) + (1 - \lambda)u(b)$ is smaller than $u(\lambda a + (1 - \lambda)b)$ for all (a, b) in the domain of u and all scalars λ in [0, 1]. A function must, however, be continuous to be concave.
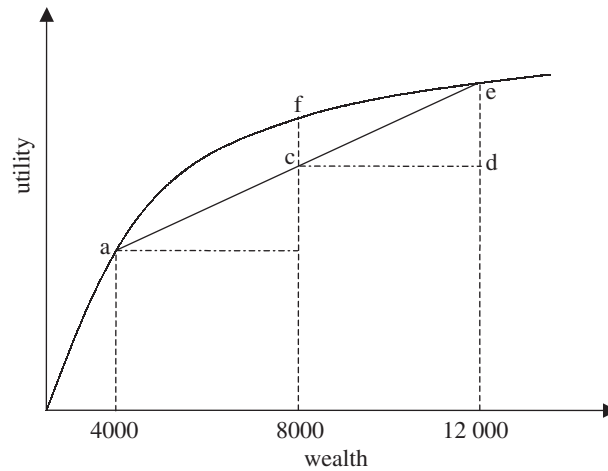
Figure 1.1. Measuring the expecting utility of final wealth $(4000, \frac{1}{2}; 12000, \frac{1}{2})$.

1.2 Definition and Characterization of Risk Aversion

We assume that the decision maker lives for only one period, which implies that he immediately uses all his final wealth to purchase and to consume goods and services. Later in this book, we will disentangle wealth and consumption by allowing the agent to live for more than one period. Final wealth comes from initial wealth w plus the outcome of any risk borne during the period.

Definition 1.1. An agent is risk-averse if, at any wealth level w, he or she dislikes every lottery with an expected payoff of zero: $\forall w$, $\forall \tilde{z}$ with $E\tilde{z} = 0$, $Eu(w + \tilde{z}) \leqslant u(w)$.

Observe that any lottery $\tilde{z}$ with a non-zero expected payoff can be decomposed into its expected payoff $E\tilde{z}$ and a zero-mean lottery $\tilde{z} - E\tilde{z}$. Thus, from our definition, a risk-averse agent always prefers receiving the expected outcome of a lottery with certainty, rather than the lottery itself. For an expected-utility maximizer with a utility function u, this implies that, for any lottery $\tilde{z}$ and for any initial wealth w,

$$Eu(w + \tilde{z}) \leqslant u(w + E\tilde{z}). \tag{1.2}$$

If we consider the simple example from Sempronius's problem, with only one ship the initial wealth w equals 4000, and the profit $\tilde{z}$ takes the value 8000 or 0 with equal probabilities. Because our intuition is that Sempronius must be risk averse, it must follow that

$$\tfrac{1}{2}u(12\,000) + \tfrac{1}{2}u(4000) \leqslant u(8000). \tag{1.3}$$

If Sempronius could find an insurance company that would offer full insurance at an actuarially fair price of $E\tilde{z} = 4000$ ducats, Sempronius would be better off by

purchasing the insurance policy. We can check whether inequality (1.3) is verified in Figure 1.1. The right-hand side of the inequality is represented by point 'f' on the utility curve u. The left-hand side of the inequality is represented by the middle point on the arc 'ae', i.e. by point 'c'. This can immediately be checked by observing that the two triangles 'abc' and 'cde' are equivalent, since they have the same base and the same angles. We observe that 'f' is above 'c': *ex ante*, the welfare derived from lottery $\tilde{z}$ is smaller than the welfare obtained if one were to receive its expected payoff $E\tilde{z}$ with certainty. In short, Sempronius is risk-averse. From this figure, we see that this is true whenever the utility function is concave. The intuition of the result is very simple: if marginal utility is decreasing, then the potential loss of 4000 reduces utility more than the increase in utility generated by the potential gain of 4000. Seen *ex ante*, the expected utility is reduced by these equally weighted potential outcomes.

It is noteworthy that Equations (1.1) and (1.3) are exactly the same. The preference for diversification is intrinsically equivalent to risk aversion, at least under the Bernoullian expected-utility model.

Using exactly the opposite argument, it can easily be shown that, if u is convex, the inequality in (1.2) will be reversed. Therefore, the decision maker prefers the lottery to its mathematical expectation and he reveals in this way his inclination for taking risk. Such individual behavior will be referred to as risk loving. Finally, if u is linear, then the welfare Eu is linear in the expected payoff of lotteries. Indeed, if $u(x) = a + bx$ for all x, then we have

$$Eu(w + \tilde{z}) = E[a + b(w + \tilde{z})] = a + b(w + E\tilde{z}) = u(w + E\tilde{z}),$$

which implies that the decision maker ranks lotteries according to their expected outcome. The behavior of this individual is called risk-neutral.

In the next proposition, we formally prove that inequality (1.2) holds for any lottery $\tilde{z}$ and any initial wealth w if and only if u is concave.

Proposition 1.2. *A decision maker with utility function u is risk-averse, i.e. inequality (1.2) holds for all w and $\tilde{z}$, if and only if u is concave.*

Proof. The proof of sufficiency is based on a second-order Taylor expansion of $u(w + z)$ around $w + E\tilde{z}$. For any z, this yields

$$u(w + z) = u(w + E\tilde{z}) + (z - E\tilde{z})u'(w + E\tilde{z}) + \tfrac{1}{2}(z - E\tilde{z})^2 u''(\xi(z))$$

for some $\xi(z)$ in between z and $E\tilde{z}$. Because this must be true for all z, it follows that the expectation of $u(w + \tilde{z})$ is equal to

$$Eu(w + \tilde{z}) = u(w + E\tilde{z}) + u'(w + E\tilde{z})E(\tilde{z} - E\tilde{z}) + \tfrac{1}{2}E[(\tilde{z} - E\tilde{z})^2 u''(\xi(\tilde{z}))].$$

Observe now that the second term of the right-hand side above is zero, since $E(\tilde{z} - E\tilde{z}) = E\tilde{z} - E\tilde{z} = 0$. In addition, if u'' is uniformly negative, then the third term takes the expectation of a random variable $(\tilde{z} - E\tilde{z})^2 u''(\xi(\tilde{z}))$ that is always negative, as it is the product of a squared scalar and negative u''. Hence, the sum of these three terms is less than $u(w + E\tilde{z})$. This proves sufficiency.

Necessity is proven by contradiction. Suppose that u is not concave. Then, there must exist some w and some $\delta > 0$ for which $u''(x)$ is positive in the interval $[w - \delta, w + \delta]$. Now take a small zero-mean risk $\tilde{\varepsilon}$ such that the support of final wealth $w + \tilde{\varepsilon}$ is entirely contained in $(w - \delta, w + \delta)$. Using the same Taylor expansion as above yields

$$Eu(w + \tilde{\varepsilon}) = u(w) + \tfrac{1}{2} E[\tilde{\varepsilon}^2 u''(\xi(\tilde{\varepsilon}))].$$

Because $\xi(\tilde{\varepsilon})$ has a support that is contained in $[w - \delta, w + \delta]$ where u is locally convex, $u''(\xi(\tilde{\varepsilon}))$ is positive for all realizations of $\tilde{\varepsilon}$. Consequently, it follows that $E[\tilde{\varepsilon}^2 u''(\xi(\tilde{\varepsilon}))]$ is positive, and $Eu(w + \tilde{\varepsilon})$ is larger than $u(w)$. Thus, accepting the zero-mean lottery $\tilde{\varepsilon}$ raises welfare and the decision maker is not risk-averse. This is a contradiction. □

The above proposition is in fact nothing more than a rewriting of the famous Jensen inequality. Consider any real-valued function ϕ. Jensen's inequality states that $E\phi(\tilde{y})$ is smaller than $\phi(E\tilde{y})$ for any random variable $\tilde{y}$ if and only if ϕ is a concave function. It builds a bridge between two alternative definitions of the concavity of u: the negativity of u'' and the property that any arc linking two points on curve u must lie below this curve. Figure 1.1 illustrates this point. It is intuitive that decreasing marginal utility ($u'' < 0$) means risk aversion. In a certain world, decreasing marginal utility means that an increase in wealth by 100 dollars has a positive effect on utility that is smaller than the effect of a reduction in wealth by 100 dollars. Then, in an uncertain world, introducing the risk to gain or to lose 100 dollars with equal probability will have a negative net impact on expected utility. In expectation, the benefit of the prospect of gaining 100 dollars is overweighted by the cost of the prospect of losing 100 dollars with the same probability. Over the last two decades, many prominent researchers in the field have challenged the idea that risk aversion comes only from decreasing marginal utility. Some even challenged the idea itself, that there should be any link between the two.[3]

1.3 Risk Premium and Certainty Equivalent

A risk-averse agent is an agent who dislikes zero-mean risks. The qualifier "zero-mean" is very important. A risk-averse agent may like risky lotteries if the expected

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[3]This question will be discussed in the last chapter of this book. Yaari (1987) provides a model that is dual to expected utility, where agents may be risk-averse in spite of the fact that their utility is linear in wealth.

payoffs that they yield are large enough. Risk-averse investors may want to purchase risky assets if their expected returns exceed the risk-free rate. Risk-averse agents may dislike purchasing insurance if it is too costly to acquire. In order to determine the optimal trade-off between the expected gain and the degree of risk, it is useful to quantify the effect of risk on welfare. This is particularly useful when the agent subrogates the risky decision to others, as is the case when we consider public safety policy or portfolio management by pension funds, for example. It is important to quantify the degree of risk aversion in order to help people to know themselves better, and to help them to make better decisions in the face of uncertainty. Most of this book is about precisely this problem. Clearly, people have different attitudes towards risks. Some are ready to spend more money than others to get rid of a specific risk. One way to measure the degree of risk aversion of an agent is to ask her how much she is ready to pay to get rid of a *zero-mean* risk $\tilde{z}$. The answer to this question will be referred to as the risk premium Π associated with that risk. For an agent with utility function u and initial wealth w, the risk premium must satisfy the following condition:

$$Eu(w + \tilde{z}) = u(w - \Pi). \tag{1.4}$$

The agent ends up with the same welfare either by accepting the risk or by paying the risk premium Π. When risk $\tilde{z}$ has an expectation that differs from zero, we usually use the concept of the certainty equivalent. The certainty equivalent e of risk $\tilde{z}$ is the sure increase in wealth that has the same effect on welfare as having to bear risk $\tilde{z}$, i.e.

$$Eu(w + \tilde{z}) = u(w + e). \tag{1.5}$$

When $\tilde{z}$ has a zero mean, comparing (1.4) and (1.5) implies that the certainty equivalent e of $\tilde{z}$ is equal to minus its its risk premium Π.

A direct consequence of Proposition 1.2 is that the risk premium Π is nonnegative when u is concave, i.e. when she is risk-averse. In Figure 1.2, we measure Π for the risk $(-4000, \frac{1}{2}; 4000, \frac{1}{2})$ for initial wealth $w = 8000$. Notice first that the risk premium is zero when u is linear, and it is nonpositive when u is convex.

One very convenient property of the risk premium is that it is measured in the same units as wealth, e.g. we can measure Sempronius's risk premium in ducats. Although the measure of satisfaction or utility is hard to compare between different individuals—what would it mean to say Sempronius was "happier" than Alexander?—the risk premium is not. We can easily determine whether Sempronius or Alexander is more affected by risk $\tilde{z}$ by comparing their two risk premia.

The risk premium is a complex function of the distribution of $\tilde{z}$, of initial wealth w and of the utility function u. We can estimate the amount that the agent is ready to pay for the elimination of this zero-mean risk by considering small risks. Assume
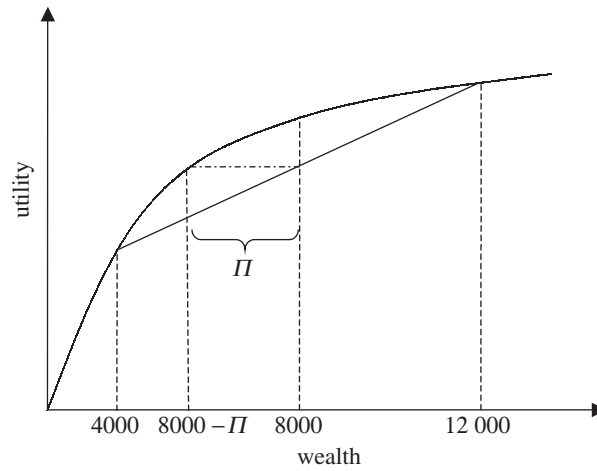
Figure 1.2. Measuring the risk premium P of risk $(-4000, \frac{1}{2}; 4000, \frac{1}{2})$ when initial wealth is $w = 8000$.

that $E\tilde{z} = 0$. Using a second-order and a first-order Taylor approximation for the left-hand side and the right-hand side of equation (1.4), respectively, we obtain that

$$u(w - \Pi) \simeq u(w) - \Pi u'(w)$$

and

$$
\begin{aligned}
Eu(w + \tilde{z}) &\simeq E[u(w) + zu'(w) + \tfrac{1}{2}\tilde{z}^2 u''(z)] \\
&= u(w) + u'(w)E\tilde{z} + \tfrac{1}{2}u''(w)E\tilde{z}^2 \\
&= u(w) + \tfrac{1}{2}\sigma^2 u''(w),
\end{aligned}
$$

where $E\tilde{z} = 0$ and $\sigma^2 = E\tilde{z}^2$ is the variance of the outcome of the lottery. Replacing these two approximations in equation (1.4) yields

$$\Pi \simeq \tfrac{1}{2}\sigma^2 A(w), \tag{1.6}$$

where the function A is defined as

$$A(w) = \frac{-u''(w)}{u'(w)}. \tag{1.7}$$

Under risk aversion, function A is positive. It would be zero or negative respectively for a risk-neutral or risk-loving agent. $A(\cdot)$ is hereafter referred to as the degree of absolute risk aversion of the agent. From (1.6), we see that the risk premium associated with risk $\tilde{\varepsilon}$ for an agent with wealth w is approximately equal to one-half the product of the variance of $\tilde{z}$ and the degree of absolute risk aversion of the agent evaluated at w. Equation (1.6) is known as the Arrow–Pratt approximation, as it was developed independently by Arrow (1963) and Pratt (1964).

The cost of risk, as measured by the risk premium, is approximately proportional to the variance of its payoffs. Thus, the variance might appear to be a good measure of the degree of riskiness of a lottery. This observation induced many authors to use a mean–variance decision criterion for modeling behavior under risk. In a mean–variance model, we assume that individual risk attitudes depend only upon the mean and the variance of the underlying risks. However, the validity of these models is dependent on the degree of accuracy of the approximation in (1.6), which can be considered accurate only when the risk is small or in very special cases. In such cases, the mean–variance approach for decisions under risk, which has historically played a very important role in the development of the theory of finance, can be seen as a special case of the expected-utility theory. In most cases however, the risk premium associated with any (large) risk will also depend upon the other moments of the distribution of the risk, not just its mean and variance. For example, it seems intuitive that whether or not $\tilde{x}$ is symmetrically distributed about its mean matters for determining the risk premium. The degree of skewness (i.e. third moment) might very well affect the desirability of a risk. Hence, two risks with the same mean and variance, but one with a distribution that is skewed to the right and the other with a distribution that is skewed to the left, should not be expected to necessarily have the same risk premium. A similar argument can be made about the kurtosis (fourth moment), which is linked to the probability mass in the tails of the distribution.

At this stage, it is worth noting that, at least for small risks, the risk premium increases with the size of the risk proportionately to the square of this size. To see this, let us assume that $\tilde{z} = k\tilde{\varepsilon}$, with $E\tilde{\varepsilon} = 0$. Parameter k can be interpreted as the size of the risk. When k tends to zero, the risk becomes very small. Of course, the risk premium is a function of the size of the risk. We may expect that this function $\Pi(k)$ is increasing in k. We are interested in describing the functional form linking the risk premium Π to the size k of the risk. Because the variance of $\tilde{z}$ equals k^2 times the variance of $\tilde{\varepsilon}$,[4] we obtain that

$$\Pi(k) \simeq \tfrac{1}{2}k^2\sigma_{\tilde{\varepsilon}}^2 A(w),$$

i.e. the risk premium is approximately proportional to the square of the size of the risk. From this observation, we can observe directly that, not only does $\Pi(k)$ approach zero as k approaches zero, but also $\Pi'(0) = 0$. This is an important property of expected-utility theory. At the margin, accepting a small zero-mean risk has *no* effect on the welfare of risk-averse agents! We say that risk aversion is a

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[4]The general formula is

$$\mathrm{var}(a\tilde{x} + b\tilde{y}) = a^2\,\mathrm{var}(\tilde{x}) + b^2\,\mathrm{var}(\tilde{y}) + 2ab\,\mathrm{cov}(\tilde{x}, \tilde{y}).$$

second-order phenomenon.[5] "In the small," we—the expected-utility maximizers—are all risk neutral.

Proposition 1.3. *If the utility function is differentiable, the risk premium tends to zero as the square of the size of the risk.*

Proof. In the following, we prove formally that $\Pi'(0) = 0$, as suggested by the Arrow–Pratt approximation in our comments above. The relationship between Π and k can be obtained by fully differentiating the equation $Eu(w + k\tilde{\varepsilon}) = u(w - \Pi(k))$ with respect to k. This yields

$$\Pi'(k) = \frac{-E\tilde{\varepsilon}u'(w + k\tilde{\varepsilon})}{u'(w - \Pi(k))}. \tag{1.8}$$

We directly infer that $\Pi'(0) = 0$, since by assumption $E\tilde{\varepsilon} = 0$. □

1.4 Degree of Risk Aversion

Let us consider the following simple decision problem. An agent is offered a take-it-or-leave-it offer to accept lottery $\tilde{z}$ with mean μ and variance σ^2. Of course, the optimal decision is to accept the lottery if

$$Eu(w + \tilde{z}) \geqslant u(w), \tag{1.9}$$

or, equivalently, if the certainty equivalent e of $\tilde{z}$ is positive. In the following, we examine how this decision is affected by a change in the utility function.

Notice at this stage that an increasing linear transformation of u has no effect on the decision maker's choice, and on certainty equivalents. Indeed, consider a function $v(\cdot)$ such that $v(x) = a + bu(x)$ for all x, for some pair of scalars a and b, where $b > 0$. Then, obviously $Ev(w + \tilde{z}) \geqslant v(w)$ yields exactly the same restrictions on the distribution of $\tilde{z}$ as condition (1.9). The same analysis can be done on equation (1.5) defining certainty equivalents. The neutrality of certainty equivalents to linear transformations of the utility function can be verified in the case of small risks by using the Arrow–Pratt approximation. If $v \equiv a + bu$, it is obvious that

$$A(x) = \frac{-v''(x)}{v'(x)} = \frac{-bu''(x)}{bu'(x)} = \frac{-u''(x)}{u'(x)}$$

for all x. Thus, by (1.6), risk premia for small risks are not affected by the linear transformation. Because the certainty equivalent equals the mean payoff of the risk minus the risk premium, the same neutrality property holds for certainty equivalents.

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[5]This property in general models, not restricted to expected utility, is called "second-order risk aversion." Within the expected-utility model, this property relies on the assumption that the utility function is differentiable.

Limiting the analysis to small risks, we see from this analysis that agents with a larger absolute risk aversion $A(w)$ will be more reluctant to accept small risks. The minimum expected payoff that makes the risk acceptable for them will be larger. This is why we say that A is a measure of the degree of risk aversion of the decision maker. From a more technical viewpoint, $A = -u''/u'$ is a measure of the degree of concavity of the utility function. It measures the speed at which marginal utility is decreasing.

We are now interested in extending these observations to any risk, not only small risks. We consider the following definition for comparative risk aversion.

Definition 1.4. Suppose that agents u and v have the same wealth w, which is arbitrary. An agent v is more risk-averse than another agent u with the same initial wealth if any risk that is undesirable for agent u is also undesirable for agent v. In other words, the risk premium of any risk is larger for agent v than for agent u.

This must be true independently of the common initial wealth level w of the two agents. If this definition were restricted to small risks, we know from the above analysis that this would be equivalent to requiring that

$$A_v(w) = \frac{-v''(w)}{v'(w)} \geqslant \frac{-u''(w)}{u'(w)} = A_u(w),$$

for all w. If limited to small risks, v is more risk-averse than u if function A_v is uniformly larger than A_u. We say in this case that v is more concave than u in the sense of Arrow–Pratt. It is important to observe that this is equivalent to the condition that v is a *concave* transformation of u, i.e. that there exists an increasing and concave function ϕ such that $v(w) = \phi(u(w))$ for all w. Indeed, we have that $v'(w) = \phi'(u(w))u'(w)$ and

$$v''(w) = \phi''(u(w))(u'(w))^2 + \phi'(u(w))u''(w),$$

which implies that

$$A_v(w) = A_u(w) + \frac{-\phi''(u(w))u'(w)}{\phi'(u(w))}.$$

Thus, A_v is uniformly larger than A_u if and only if ϕ is concave. This is equivalent to requiring that A_v be uniformly larger than A_u or that v be a concave transformation of u. It yields that agent v values small risks less than agent u. Do we need to impose more restrictions to guarantee that agent v values any risk less than agent u, i.e. that v is more risk-averse than u? The following proposition, which is due to Pratt (1964), indicates that no additional restriction is required.

Proposition 1.5. *The following three conditions are equivalent.*

   (a) *Agent v is more risk-averse than agent u, i.e. the risk premium of any risk is larger for agent v than for agent u.*

(b) *For all w, $A_v(w) \geqslant A_u(w)$.*

(c) *Function v is a concave transformation of function u : $\exists \phi(\cdot)$ with $\phi' > 0$ and $\phi'' \leqslant 0$ such that $v(w) = \phi(u(w))$ for all w.*

Proof. We have already shown that (b) and (c) are equivalent. That (a) implies (b) follows directly from the Arrow–Pratt approximation. We now prove that (c) implies (a). Consider any lottery $\tilde{z}$. Let Π_u and Π_v denote the risk premium for zero-mean lottery $\tilde{z}$ of agent u and agent v, respectively. By definition, we have that

$$v(w - \Pi_v) = Ev(w + \tilde{z}) = E\phi(u(w + \tilde{z})).$$

Define random variable $\tilde{y}$ as $\tilde{y} = u(w + \tilde{z})$. Because ϕ is concave, $E\phi(\tilde{y})$ is smaller than $\phi(E\tilde{y})$ by Jensen's inequality. It thus follows that

$$v(w - \Pi_v) \leqslant \phi(Eu(w + \tilde{z})) = \phi(u(w - \Pi_u)) = v(w - \Pi_u).$$

Because v is increasing, this implies that Π_v is larger than Π_u. $\qquad\square$

In the case of small risks, the only thing that we need to know to determine whether a risk is desirable is the degree of concavity of u locally at the current wealth level w. For larger risks, the proposition above shows that we need to know much more to take a decision. Namely, we need to know the degree of concavity of u at all wealth levels. The degree of concavity must be increased at all wealth levels to guarantee that a change in u makes the decision maker more reluctant to accept risks. If v is locally more concave at some wealth levels and is less concave at other wealth levels, the comparative analysis is intrinsically ambiguous.

To illustrate the proposition, let us go back to the example of Sempronius's single ship yielding outcome $\tilde{z} = (0, \frac{1}{2}; 8000, \frac{1}{2})$, with a initial wealth $w_0 = 4000$ ducats. If Sempronius's utility function is $u(w) = \sqrt{w}$, his certainty equivalent of $\tilde{z}$ equals $e_u = 3464.1$, since

$$\tfrac{1}{2}\sqrt{4000} + \tfrac{1}{2}\sqrt{12\,000} = 86.395 = \sqrt{7464.1}$$

Alternatively, suppose that Sempronius's utility function is $v(w) = \ln(w)$, which is also increasing and concave. It is easy to check that v is more concave than u in the sense of Arrow–Pratt. Indeed, these functions yield

$$A_v(w) = \frac{1}{w} \geqslant \frac{1}{2w} = A_u(w)$$

for all w. From the above proposition, this change in utility should reduce the certainty equivalent of any risk. In the case of $w_0 = 4000$ and $\tilde{z} \sim (0, \frac{1}{2}; 8000, \frac{1}{2})$, the certainty equivalent of $\tilde{z}$ under v equals $e_v = 2928.5$, since

$$\tfrac{1}{2}\ln(4000) + \tfrac{1}{2}\ln(12\,000) = 8.8434 = \ln(6928.5).$$

Thus, e_v is smaller than e_u. Notice that the risk premium $\Pi_v = 1071.5$ under v is approximately twice the risk premium $\Pi_u = 535.9$. This was predicted by the Arrow–Pratt approximation, since A_v is equal to $2A_u$.

1.5   Decreasing Absolute Risk Aversion and Prudence

We have seen that risk aversion is driven by the fact that one's marginal utility is decreasing with wealth. In this section, we examine another question related to increasing wealth. Namely, we are interested in determining how the risk premium for a given zero-mean risk $\tilde{z}$ is affected by a change in initial wealth w. Arrow argued that intuition implies that wealthier people are generally less willing to pay for the elimination of fixed risk. A lottery to gain or lose 100 with equal probability is potentially life-threatening for an agent with initial wealth $w = 101$, whereas it is essentially trivial for an agent with wealth $w = 1\,000\,000$. The former should be ready to pay more than the latter for the elimination of risk. We can check that this property holds for the square-root utility function, with $\Pi = 43.4$ when $w = 101$ and $\Pi = 0.0025$ when $w = 1\,000\,000$. If wealth is measured in euros, the individual would be willing to pay over 43 euros to avoid the risk when wealth is $w = 101$, whereas the same individual would not even pay one euro cent to get rid of this risk when wealth is one million euros! In the following, we characterize the set of utility functions that have this property.

The risk premium $\Pi = \pi(w)$ as a function of initial wealth w can be evaluated by solving

$$Eu(w + \tilde{z}) = u(w - \pi(w)) \tag{1.10}$$

for all w. Fully differentiating (1.10) with respect to w yields

$$Eu'(w + \tilde{z}) = (1 - \pi'(w))u'(w - \pi),$$

or, equivalently,

$$\pi'(w) = \frac{u'(w - \pi) - Eu'(w + \tilde{z})}{u'(w - \pi)}. \tag{1.11}$$

Thus, the risk premium is decreasing with wealth if and only if

$$Ev(w + \tilde{z}) \leqslant v(w - \pi(w)), \tag{1.12}$$

where function $v \equiv -u'$ is defined as minus the derivative of function u. Because the function v is increasing, we can also interpret it as *another* utility function. Condition (1.12) then just states that the risk premium of agent v is larger than the risk premium π of agent u. From Proposition 1.5, this is true if and only if v is more concave than u in the sense of Arrow–Pratt, that is, if $-u'$ is a concave transformation of u. For this utility v, the measure of absolute risk aversion is $A_v = A_{-u'} = -u'''/u''$. This measure has several uses, which will be made clearer

later in this book. For this reason, without justifying the terminology at this stage, we will define $P(w) = -u'''(w)/u''(w)$ as the degree of absolute prudence of the agent with utility u. It follows from (1.12) that $-u'$ is more concave than u if and only if

$$P(w) \geqslant A(w)$$

for all w. We conclude that condition $P \geqslant A$ uniformly is necessary and sufficient to guarantee that an increase in wealth reduces risk premia. Because

$$A'(w) = A(w)[A(w) - P(w)],$$

condition $P \geqslant A$ is equivalent to the condition $A' \leqslant 0$. We obtain the following proposition.

Proposition 1.6. *The risk premium associated to any risk $\tilde{z}$ is decreasing in wealth if and only if absolute risk aversion is decreasing; or equivalently if and only if prudence is uniformly larger than absolute risk aversion.*

Observe that the utility function $u(w) = \sqrt{w}$ satisfies this condition. Indeed, we have $A_u(w) = \frac{1}{2}w^{-1}$, which is decreasing. This can alternatively checked by observing that $v(w) = -\frac{1}{2}w^{-1/2}$ and $A_v(w) = P_u(w) = 1.5w^{-1}$, which is uniformly larger than $A_u(w)$. Notice that Decreasing Absolute Risk Aversion (DARA) requires that the third derivative of the utility function be positive. Otherwise, prudence would be negative, which would imply that $P < A$: a condition that implies that absolute risk aversion would be increasing in wealth. Thus, DARA, a very intuitive condition, requires the necessary (but not sufficient) condition that u''' be positive, or that marginal utility be convex.

1.6 Relative Risk Aversion

Absolute risk aversion is the rate of decay for marginal utility. More particularly, absolute risk aversion measures the rate at which marginal utility decreases when wealth is increased *by one euro*.[6] If the monetary unit were the dollar, absolute risk aversion would be a different number. In other words, the index of absolute risk aversion is not unit free, as it is measured per euro (per dollar, or per yen).

Economists often prefer unit-free measurements of sensitivity. To this end, define the index of *relative* risk aversion R as the rate at which marginal utility decreases

---

[6]In general, the *growth rate* for a function $f(x)$ is defined as

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \cdot \frac{1}{f(x)}.$$

Since marginal utility $u'(x)$ declines in wealth, its growth rate is negative. The absolute value of this negative growth rate, which is the measure of absolute risk aversion, is called the *decay rate*.

when wealth is increased *by one percent*. In terms of standard economic theory, this measure is simply the wealth-elasticity of marginal utility. It can be computed as

$$R(w) = -\frac{\mathrm{d}u'(w)/u'(w)}{\mathrm{d}w/w} = \frac{-wu''(w)}{u'(w)} = wA(w). \qquad (1.13)$$

Note that the measure of relative risk aversion is simply the product of wealth and absolute risk aversion.

The (absolute) risk premium and the index of absolute risk aversion are linked by the Arrow–Pratt approximation and by Propositions 1.5 and 1.6. We can develop analogous kinds of results for relative risk aversion. Suppose that your initial wealth w is invested in a portfolio whose return $\tilde{z}$ over the period is uncertain. Let us assume that $E\tilde{z} = 0$. Which share of your initial wealth are you ready to pay to get rid of this proportional risk? The solution to this problem is referred to as the relative risk premium $\hat{\Pi}$. This measure also is a unit-free measure, unlike the absolute risk premium, which is measured in euros. It is defined implicitly via the following equation:

$$Eu(w(1 + \tilde{z})) = u(w(1 - \hat{\Pi})). \qquad (1.14)$$

Obviously, the relative risk premium and the absolute risk premium are equal if we normalize initial wealth to unity. More generally, the relative risk premium for proportional risk $\tilde{z}$ equals the absolute risk premium for absolute risk $w\tilde{z}$, divided by initial wealth w: $\hat{\Pi}(\tilde{z}) = \Pi(w\tilde{z})/w$. From this observation, we obtain the fact that, if agent v is more risk-averse than agent u with the same initial wealth, then agent v will be ready to pay a larger share of his wealth than agent u to insure against a given proportional risk $\tilde{z}$. Moreover, if σ^2 denotes the variance of $\tilde{z}$, then the variance of $w\tilde{z}$ equals $w^2\sigma^2$. Using the Arrow–Pratt approximation thus yields

$$\hat{\Pi}(\tilde{z}) = \frac{\Pi(w\tilde{z})}{w} \simeq \frac{\frac{1}{2}w^2\sigma^2 A(w)}{w} = \tfrac{1}{2}\sigma^2 R(w). \qquad (1.15)$$

The relative risk premium is approximately equal to half of the variance of the proportional risk times the index of relative risk aversion. This can be used to establish a range for acceptable degrees of risk aversion. Suppose that one's wealth is subject to a risk of a gain or loss of 20% with equal probability. What is the range that one would find reasonable for the share of wealth Π that one would be ready to pay to get rid of this zero-mean risk? From our various experiments in class, we found that most people would be ready to pay between 2% and 8% of their wealth. Because risk $\tilde{z}$ in this experiment has a variance of $0.5(0.2)^2 + 0.5(-0.2)^2 = 0.04$, using approximation (1.15) yields a range for relative risk aversion between 1 and 4. This information will be useful later in this book.

There is no definitive argument for or against decreasing relative risk aversion. Arrow originally conjectured that relative risk aversion is likely to be constant, or

perhaps increasing, although he stated that the intuition was not as clear as was the intuition for decreasing absolute risk aversion. Since then, numerous empirical studies have offered conflicting results. We might also try to examine this question by introspection. If your wealth would increase, would you want to devote a larger or a smaller share of your wealth to get rid of a given zero-mean proportional risk? For example, what would you pay to avoid the risk of gaining or losing 20% of your wealth, each with an equal probability? If the share is decreasing with wealth, you have decreasing relative risk aversion. There are two contradictory effects here that need to be considered. On the one hand, under the intuitive DARA assumption, becoming wealthier also means becoming less risk-averse. This effect tends to reduce Π. But, on the other hand, becoming wealthier also means facing a larger absolute risk $w\tilde{z}$. This effect tends to raise Π. There is no clear intuition as to whether the first effect or the second effect will dominate. For example, many of the classic models in macroeconomics are based on relative risk aversion being constant over all wealth levels, which is implicitly assuming that our two effects exactly cancel each other out. Of course, there also is no *a priori* reason to believe that the dominant effect will not change over various wealth levels. For instance, some recent empirical evidence indicates a possible "U-shape" for relative risk aversion, with R decreasing at low wealth levels, then leveling off somewhat before increasing at higher wealth levels.

1.7 Some Classical Utility Functions

As already noted above, expected-utility (EU) theory has many proponents and many detractors. In Chapter 13, we examine some generalizations of the EU criterion that satisfy those who find expected utility too restrictive. But researchers in both economics and finance have long considered—and most of them still do—EU theory as an acceptable paradigm for decision making under uncertainty. Indeed, EU theory has a long and prominent place in the development of decision making under uncertainty. Even detractors of the theory use EU as a standard by which to compare alternative theories. Moreover, many of the models in which EU theory has been applied can be modified, often yielding better results.

Whereas the current trend is to generalize the EU model, researchers often *restrict* EU criterion by considering a specific subset of utility functions. This is done to obtain tractable solutions to many problems. It is important to note the implications that derive from the choice of a particular utility function. Some results in the literature may be robust enough to apply for all risk-averse preferences, while others might be restricted to applying only for a narrow class of preferences. In this section, we examine several particular types of utility functions that are often encountered in the economics and the finance literature. Remember that utility is unique only up to a linear transformation.

Historically, much of the theory of finance was developed during the 1960s by considering the subset of utility functions that are quadratic of the form

$$u(w) = aw - \tfrac{1}{2}w^2, \quad \text{for } w \leqslant a.$$

Note that the domain of wealth on which u is defined comes from the necessary requirement that u be nondecreasing, which is true only if w is smaller than a. This set of functions is useful because the EU generated by any distribution of final wealth is a function of only the first two moments of this distribution:

$$Eu(\tilde{w}) = aE\tilde{w} - \tfrac{1}{2}E\tilde{w}^2.$$

Therefore, in this case, the EU theory simplifies to a mean–variance approach to decision making under uncertainty. However, as already discussed, it is very hard to believe that preferences among different lotteries be determined only by the mean and variance of these lotteries.

Above wealth level a, marginal utility becomes negative. Since quadratic utility is decreasing in wealth for $w > a$, many people might feel this is not appropriate as a utility function. However, it is important to remember that we are trying to model human behavior with mathematical models. For example, if the quadratic utility function models your behavior quite well with $a = 100$ million euros, is it really a problem that this function declines for higher wealth levels? The point is that the quadratic utility might work well for more realistic wealth levels, and if it does, we should not be overly concerned about its properties at unrealistically high wealth levels. However, the quadratic utility function has another property that is more problematic. Namely, the quadratic utility functions exhibit increasing absolute risk aversion:

$$A(w) = \frac{1}{a - w} \Rightarrow A'(w) = \frac{1}{(a - w)^2} > 0.$$

For this reason, quadratic utility functions are not as in fashion anymore.

A second set of classical utility functions is the set of so-called constant-absolute-risk-aversion (CARA) utility functions, which are exponential functions characterized by

$$u(w) = -\frac{\exp(-aw)}{a},$$

where a is some positive scalar. The domain of these functions is the real line. The distinguishing feature of these utility functions is that they exhibit constant absolute risk aversion, with $A(w) = a$ for all w. It can be shown that the Arrow–Pratt approximation is exact when u is exponential and $\tilde{w}$ is normally distributed with

mean μ and variance σ^2. Indeed, we can take expectations to see that

$$
\begin{aligned}
Eu(\tilde{w}) \\
&= \frac{-1}{\sigma a\sqrt{2\pi}} \int \exp(-aw)\exp\left(-\frac{(w-\mu)^2}{2\sigma^2}\right)\mathrm{d}w \\
&= -\frac{1}{a}\exp(-a(\mu-\tfrac{1}{2}a\sigma^2))\left[\frac{1}{\sigma\sqrt{2\pi}}\int \exp\left(-\frac{(w-(\mu-\tfrac{1}{2}a\sigma^2))^2}{2\sigma^2}\right)\mathrm{d}w\right] \\
&= -\frac{1}{a}\exp(-a(\mu-\tfrac{1}{2}a\sigma^2)) = u(\mu-\tfrac{1}{2}a\sigma^2). \quad\quad (1.16)
\end{aligned}
$$

The third equality comes from the fact that the bracketed term is the integral of the density of the normal distribution $N(\mu - \tfrac{1}{2}a\sigma^2, \sigma)$, which must be equal to unity. Thus, the risk premium is indeed equal to $\tfrac{1}{2}\sigma^2 A(w)$. In this very specific case, we obtain that the Arrow–Pratt approximation is exact. The fact that risk aversion is constant is often useful in analyzing choices among several alternatives. As we will see later, this assumption eliminates the income effect when dealing with decisions to be made about a risk whose size is invariant to changes in wealth. However, this is often also the main criticism of the CARA utility, since absolute risk aversion is constant rather than decreasing.

Finally, one set of preferences that has been by far the most used in the literature is the set of power utility functions. Researchers in finance and in macroeconomics are so accustomed to this restriction that many of them do not even mention it anymore when they present their results. Suppose that

$$
u(w) = \frac{w^{1-\gamma}}{1-\gamma} \quad \text{for } w > 0.
$$

The scalar γ is chosen so that $\gamma > 0$, $\gamma \neq 1$. It is easy to show that γ equals the degree of relative risk aversion, since $A(w) = \gamma/w$ and $R(w) = \gamma$ for all w. Thus, this set exhibits decreasing absolute risk aversion and constant relative risk aversion, which are two reasonable assumptions. For this reason, these utility functions are called the constant-relative-risk-aversion (CRRA) class of preferences. Notice that our definition does not allow for $\gamma = 1$. However, it is straightforward to show that function $u(w) = \ln(w)$ satisfies the property that $R(w) = 1$ for all w. Thus, the set of all CRRA utility functions is completely defined by[7]

$$
u(w) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma} & \text{for } \gamma \geqslant 0,\ \gamma \neq 1, \\[2mm] \ln(w) & \text{for } \gamma = 1. \end{cases} \quad\quad (1.17)
$$

---

[7]We can also show that $u(w) = \ln(w)$ as a limiting case of the power utility function. To this end, rewrite the power utility function, using a linear transformation, as

$$
u(w) = \frac{1}{1-\gamma}(w^{1-\gamma} - 1).
$$

As we will see later in this book, this class of utility functions eliminates any income effects when making decisions about risks whose size is proportional to one's level of wealth. For example, the relative risk premium $\hat{\Pi}$ defined by equation (1.14) is independent of wealth w in this case. The assumption that relative risk aversion is constant enormously simplifies many of the problems often encountered in macroeconomics and finance.

1.8   Bibliographical References, Extensions and Exercises

The contribution by Pratt (1964) basically opened and closed the field covered in this chapter. It is, however, fair to mention that the measure of absolute risk aversion has been discovered independently by Arrow (1963) and de Finetti (1952). The paper by de Finetti was written in Italian and even today is not given the attention it deserves. The paper by Pratt is by far the most advanced in defining the notions of an increase in risk aversion and of decreasing absolute risk aversion. The orders of risk aversion are introduced by Segal and Spivak (1990).

Ross (1981) challenged the idea that $A = -u''/u'$ is a good measure of the degree of risk aversion of an agent. Kihlstrom, Romer and Williams (1981) and Nachman (1982) showed that if initial wealth is uncertain, it is not true that an agent v, who is more risk-averse than another agent u in the sense of Arrow–Pratt, will be ready to pay more to get rid of another risk. Ross (1981) characterized the conditions on u and v that imply that $\Pi_v \geqslant \Pi_u$ even when initial wealth is uncertain and potentially correlated with the risk under scrutiny. These conditions are of course stronger than $A_v \geqslant A_u$.

There is much contradictory empirical evidence on the shape of relative risk aversion as a function of wealth. Many authors have empirically estimated R, assuming that we have CRRA. Fewer authors have examined whether R might be increasing or decreasing in wealth. A good summary of many of these results appears in Ait-Sahalia and Lo (2000).

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Taking the limit as $\gamma \rightarrow 1$ and applying L'Hôpital's rule, we obtain

$$\lim_{\gamma \to 1} u(w) = \lim_{\gamma \to 1} \frac{-(w^{1-\gamma})\ln(w)}{-1} = \ln(w).$$

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Exercises

(1.1) An individual has the following utility function:

$$u(w) = w^{1/2}.$$

Her initial wealth is 10 and she faces the lottery $\tilde{X} : (-6, \frac{1}{2}; +6, \frac{1}{2})$.

   (a) Compute the exact value of the certainty equivalent and of the risk premium.

   (b) Apply Pratt's formula to obtain an approximation of the risk premium.

   (c) Show that with such a utility function absolute risk aversion is decreasing in wealth while relative risk aversion is constant.

   (d) If the utility function becomes

$$v(w) = w^{1/4},$$

   answer again part (a). Are you surprised by the changes in the certainty equivalent and in the risk premium? Relate this change to the notion of 'more risk averse' (i.e. express $v(w)$ as a concave transformation of $u(w)$).

   (e) If the risk becomes $\tilde{Y} : (-3, \frac{1}{2}; +3, \frac{1}{2})$, compute the new risk premium as approximated by Pratt's formula. Why is the approximated risk premium four times smaller than the risk premium for $\tilde{X}$?

(1.2) As in the previous exercise, consider an initial wealth of 10 and the lottery $\tilde{X}$. Assume now that the utility is:

$$u = \begin{cases} w & \text{for } w \leqslant 10, \\ \frac{1}{2}w + 5 & \text{for } w \geqslant 10. \end{cases} \qquad (1.18)$$

   (a) Draw the utility function. Is it globally concave?

   (b) Compute the certainty equivalent and the risk premium attached to $\tilde{X}$.

   (c) Can you apply the Arrow–Pratt approximation? Why?

   (d) Consider now the lottery $\tilde{Y}$ defined in exercise 1.1. Compute the risk premium attached to $\tilde{Y}$. Is it smaller than for $\tilde{X}$? Why?

   (e) Answer (b) and (d) above if the individual has an initial wealth of 20. How do the risk premia for $\tilde{X}$ and $\tilde{Y}$ compare?

(1.3) Let $u = w^2$ for $w \geqslant 0$.

    (a) Compute the exact risk premium if initial wealth is 4 and if a decision maker faces the lottery $(-2, \frac{1}{2}; +2, \frac{1}{2})$. Explain why the risk premium is negative.

    (b) If the utility function becomes $v = w^4$, what happens to the risk premium? Show that v is a convex transformation of u.

(1.4) Let $u = \ln w$.

    (a) Does this utility function exhibit the DARA property?

    (b) Compute $-u'''/u''$ and compare it with $-u''/u'$.

    (e) Prove that $-u'(w)$ is a concave transformation of $u(w)$ (hint: use Pratt's theorem).

(1.5) Consider the family of exponential utility functions

$$u = \frac{1 - \exp(-aw)}{a}.$$

    (a) Show that a is the degree of absolute risk aversion.

    (b) Show that u becomes linear in w when a tends to zero (hint: use L'Hôpital's rule).

    (c) Consider lottery $\tilde{x}$ with positive and negative payoffs. Determine the value of $Eu(\tilde{x})$ when a tends to infinity.