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t-partitions and s-complete t-partitions of a graph

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Abstract

A partition $\{V_1, V_2, \cdots, V_p\}$ of the vertex set of a graph $G = (V, E)$ is a *t-partition* if the number $e(V_i)$ of edges contained in the class V_i is at most t for $1 \leq i \leq p$. The minimum number of classes in a t-partition of G is the *t-chromatic number* $\chi_t(G)$. Since a 0-partition is a partition of V into independent sets, $\chi_0(G)$ equals the chromatic number $\chi(G)$. A t-partition is *s-complete* if the number $e(V_i, V_j)$ of edges between two parts V_i and V_j is at least s for all $1 \leq i < j \leq p$. The minimum number of classes in a s-complete t-partition of G, if any, is denoted $\chi_t^s(G)$.

We study some properties of $\chi_t(G)$ and $\chi_t^s(G)$, in particular bounds on $\chi_t(G)$, the complexity of $\chi_t(G)$ and conditions for the existence of $\chi_t^s(G)$.

1 Introduction

Let $G = (V, E)$ be a finite and simple graph with $|V| = n$ vertices and $|E| = e$ edges. If S and T are disjoint vertex subsets of G, we denote by $E(S)$ and $e(S)$ the set of the edges of G having both endvertices in S and their number, and by $E(S, T)$ and

$e(S,T)$ the set of the edges of G having one endvertex in S and one in T and their number, respectively. The induced subgraph $(S, E(S))$ is denoted by $G[S]$ and the bipartite subgraph $(S \cup T, E(S,T))$ by $G[S,T]$. We write $\omega(G)$ for the *clique number* of G, that is the maximum number of vertices in a complete subgraph of G.

A partition $\mathbf{P} = \{V_1, V_2, \cdots, V_p\}$ of the vertex set V is said to be a *t-partition* of G if $e(V_i) \leq t$ for $1 \leq i \leq p$ where t is a non-negative integer. Since the trivial partition into n classes containing one vertex each is a t-partition for any t, t-partitions exist for all values of t. The *t-chromatic number* $\chi_t(G)$ is defined as the minimum number of classes in a t-partition of G. Since a 0-partition is a partition into independent sets, $\chi_0(G)$ equals $\chi(G)$, the chromatic number of G. (Note that similar kinds of partitions have been considered before, e.g. that in which the maximum degree $\Delta(G[V_i])$ is at most t for $1 \leq i \leq p$ [1], [5]).

A t-partition $\{V_1, V_2, \cdots, V_p\}$ of G is said to be *s-complete* if $e(V_i, V_j) \geq s$ for $1 \leq i < j \leq p$. The *s-complete t-chromatic number* $\chi_t^s(G)$ of G is defined as the minimum number of classes in a s-complete t-partition of G, provided that such a partition exists.

Our aim in this paper is to study some properties of $\chi_t(G)$ and $\chi_t^s(G)$. For $\chi_t^s(G)$, the main question is that of the existence of s-complete t-partitions of G while for $\chi_t(G)$, which always exists, we are interested in good bounds and in complexity results.

The idea of t-partitions is not new but the problem has been presented until now under the following different form. Given a partition $\mathbf{P} = \{V_1, \cdots, V_p\}$ of V into p classes let $\gamma(\mathbf{P}) = \max \{e(V_i) \mid 1 \leq i \leq p\}$. Furthermore, let

$$\gamma_p(G) = \min \{\gamma(\mathbf{P}) \mid \mathbf{P} \text{ is a partition of } V \text{ into } p \text{ classes}\}.$$

The problem is to find a good upper bound on $\gamma_p(G)$. For $p = 2$, Erdős conjectured in 1988 that $\gamma_2(G) \leq e/4 + O(\sqrt{e})$ for every graph with e edges [4]. This was proved by Porter in [7]. Later, several authors worked on the generalization of the problem (see for instance [2], [3], [8]). The following sharp bound was established by Bollobás and Scott.

Theorem A [2] *Let p be a positive integer and G a graph with e edges. Then G has a vertex partition into p classes such that each of them has at most $\frac{e}{p^2} + \frac{p-1}{2p^2}\left(\sqrt{2e + \frac{1}{4}} - \frac{1}{2}\right)$ edges.*

2 Properties of χ_t

Since every t-partition of G is a t'-partition for all $t' \geq t$, the t-chromatic numbers of a graph G with e edges form an inequality chain

$$\chi(G) = \chi_0(G) \geq \chi_1(G) \geq \ldots \geq \chi_{e-1}(G) \geq \chi_e(G) = 1. \tag{2.1}$$

If $t < e$, then not all edges of G can belong to the same class and thus $\chi_t(G) \geq 2$. If $t = e - 1$, two classes always suffice and thus $\chi_{e-1}(G) = 2$.

For a given graph G, it would be interesting to know the smallest value of t for which $\chi_t(G) = 2$ and more generally to know an upper bound on $\chi_t(G)$. If G is bipartite, $\chi_0(G) = 2$ and if G is an odd cycle, then $\chi_0(G) = 3$ and $\chi_1(G) = 2$. For general graphs, Theorem A provides an answer to the previous question. It is easy to see that $\gamma_p(G) \leq t$ holds if and only if $\chi_t(G) \leq p$. Therefore Theorem A leads to the following corollary.

Theorem 2.1. *Let $p \geq 1$. Every graph G with e edges satisfies*

$$\chi_{\left\lfloor \frac{e}{p^2}+\frac{p-1}{2p^2}\left(\sqrt{2e+\frac{1}{4}}-\frac{1}{2}\right)\right\rfloor}(G) \leq p.$$

Before giving other bounds on $\chi_t(G)$, we show that the terms of the decreasing sequence (2.1) are not completely unrelated. We need the following lemma.

Lemma 2.2. *Every graph H with m edges satisfies $\chi(H) \leq m^*$ where m^* is the largest integer j such that $\binom{j}{2} \leq m$.*

Proof. $\{U_1, U_2, \ldots, U_q\}$ with $q = \chi(H)$ be a minimum 0-partition of H. Since for $i \neq j$, $U_i \cup U_j$ is not independent, H has at least one edge between each pair (U_i, U_j), and thus has at least $\binom{q}{2}$ edges. Hence $\binom{q}{2} \leq m$, which implies $m^* \geq q = \chi(H)$ by the definition of m^*. □

Theorem 2.3. *For all $t \geq 1$, every graph G satisfies $\chi(G) \leq t^*\chi_t(G)$ where t^* is the largest integer j such that $\binom{j}{2} \leq t$.*

Proof. Let $\{V_1, V_2, \ldots, V_p\}$ with $p = \chi_t(G)$ be a minimum t-partition of G. Each V_i can be partitioned into $\chi(G[V_i])$ independent sets which all together form a 0-partition of G. By Lemma 2.2, $\chi(G[V_i]) \leq (e(V_i))^* \leq t^*$. Hence $\chi(G) \leq pt^* = t^*\chi_t(G)$. □

The inequality in Theorem 2.3 is sharp as can be seen in the family $\mathbf{F}_t$ below.

Definition 2.4. A graph $G = (V, E)$ belongs to the family $\mathbf{F}_t$ if there exists a positive integer p such that

(i) V is the disjoint union $V_1 \cup V_2 \cdots \cup V_p$ where $e(G[V_i]) \leq t$ and $\omega(G[V_i]) = t^*$ and

(ii) $G[X_1 \cup \cdots \cup X_p]$ is complete where X_i is a clique K_{t^*} of $G[V_i]$.

Theorem 2.5. *Every graph G in $\mathbf{F}_t$ satisfies $\chi(G) = t^*\chi_t(G)$.*

Proof. Clearly, $V_1 \cup \ldots \cup V_p$ is a t-partition of G and thus $\chi_t(G) \leq p$. On the other hand, the graph G contains a clique with $|X_1 \cup \ldots \cup X_p| = pt^*$ vertices and thus $\chi(G) \geq pt^* \geq t^*\chi_t(G)$ vertices. By Theorem 2.3, $\chi(G) = t^*\chi_t(G)$. □

Remark 2.6. There exist graphs not in $\mathbf{F}_t$ satisfying $\chi(G) = t^*\chi_t(G)$. For instance let G be obtained from a complete p-partite graph $K_{5,5,\ldots,5}$ by adding five edges forming a C_5 in each part (note that G belongs to the family $\mathbf{G}_5$ defined below in Definition 2.10). Then $\chi_5(G) = p$ and $\chi(G) = 3p = 5^*\chi_5(G)$.

However when $t = 1$, the graphs in $\mathbf{F}_1$ are the only graphs for which $\chi(G) = 1^*\chi_1(G) = 2\chi_1(G)$. For, if $\chi(G) = 2\chi_1(G)$ then, from the proof of Theorem 2.3, every class V_i of a minimum 1-partition of G contains exactly one edge x_iy_i. If $G[\{x_1, y_1, \cdots, x_p, y_p\}]$ is not complete, let, say, $x_1x_2 \notin E(G)$, then $((V_1 - \{x_1\}), (V_2 - \{x_2\}), \{x_1, x_2\}, (V_3 - \{x_3\}), \{x_3\}, \cdots, (V_p - \{x_p\}), \{x_p\})$ is a 0-partition of G with $2p-1$ classes, a contradiction to $\chi(G) = 2\chi_1(G) = 2p$. Therefore $G[\{x_1, y_1, \cdots, x_p, y_p\}]$ is complete and $G \in \mathbf{F}_1$.

Remark 2.7. Similar proofs give the same kind of results relating $\chi_t(G)$ and $\chi_{t'}(G)$ for other values of t and t'. For instance, the consideration of all the possible configurations of three edges yields to $\chi_1(G) \leq 2\chi_3(G)$ for every graph G. This bound is sharp as shown by the following example. The graph constructed from a complete k-multipartite graph $K_{6,6,..,6}$ by adding the edges of a perfect matching in each class satisfies $\chi_3(G) = k$ and $\chi_1(G) = 2k = 2\chi_3(G)$.

We now give sharp bounds on $\chi_t(G)$ in terms of the numbers of vertices and edges of G.

Theorem 2.8. *Let t^* be the largest integer j such that $\binom{j}{2} \leq t$. Every graph G with n vertices satisfies $\chi_t(G) \leq \lceil \frac{n}{t^*} \rceil$ and for the clique, $\chi_t(K_n) = \lceil \frac{n}{t^*} \rceil$.*

Proof. We obtain a minimum t-partition of K_n by taking as many parts of order t^* as possible plus possibly one smaller part. Hence $\chi_t(K_n) = \lceil \frac{n}{t^*} \rceil$. If G is a subgraph of H, then any t-partition of H is a t-partition of G and thus $\chi_t(G) \leq \chi_t(H)$. Therefore, $\chi_t(G) \leq \chi_t(K_n) = \lceil \frac{n}{t^*} \rceil$. □

Corollary 2.9. *If G is a connected graph with maximum degree $\Delta \geq 1$, then $\chi_1(G) \leq \Delta$.*

Proof. By Brook's Theorem, $\chi_1(G) \leq \chi_0(G) \leq \Delta + 1$ and $\chi_0(G) = \Delta + 1$ if and only if G is either a clique K_n, in which case $\chi_1(G) = \lceil \frac{n}{1^*} \rceil = \lceil \frac{n}{2} \rceil \leq n - 1 = \Delta$, or an odd cycle, in which case $\chi_1(G) = 2 = \Delta$. □

Definition 2.10. A graph G belongs to the family $\mathbf{G}_t$ if its vertex set V admits a partition $\{V_1, V_2, \cdots, V_p\}$ such that $|V_1| = |V_2| = \cdots = |V_p|$, $G[V_i]$ has exactly t edges, and the graph $(V, E \setminus (E(V_1) \cup E(V_2) \cup \cdots \cup E(V_p)))$ is complete p-partite.

For $t = 0$, a lower bound on the usual chromatic number of a graph G in terms of its order n and size e is known, namely $\chi(G) \geq \frac{n^2}{n^2 - 2e}$ with equality if and only if $G \in \mathbf{G}_0$ [6]. We give an analogous result for any positive value of t in the following

Theorem 2.11. *For $t \geq 1$, every graph G of order n and size e satisfies*

$$\chi_t(G) \geq \frac{2e - n^2 + \sqrt{(n^2 - 2e)^2 + 8tn^2}}{4t}$$

with equality if and only if $G \in \mathbf{G}_t$.

Proof. Let $\{V_1, V_2, \cdots, V_p\}$ with $p = \chi_t(G)$ be a minimum t-partition of G. The complement $\overline{G}$ of G has at least $\binom{n_i}{2} - t$ edges in the part V_i where $n_i = |V_i|$. Hence

$$e(\overline{G}) \geq \sum_{i=1}^{p} \frac{n_i(n_i-1)}{2} - tp = \frac{1}{2}\sum_{i=1}^{p} n_i^2 - \frac{n}{2} - tp$$

with equality if and only if G contains all the edges between two different parts and $e(V_i) = t$ for all i. By Schwarz's inequality, $p\sum_{i=1}^{p} n_i^2 \geq \left(\sum_{i=1}^{p} n_i\right)^2$ with equality if and only if all the n_i's are equal. Hence $e(\overline{G}) \geq \frac{n^2}{2p} - \frac{n}{2} - tp$ with equality if and only if $G \in \mathbf{G}_t$. Therefore

$$e(G) = \frac{n(n-1)}{2} - e(\overline{G}) \leq \frac{n^2}{2} - \frac{n^2}{2p} + tp$$

that is

$$2tp^2 + (n^2 - 2e)p - n^2 \geq 0,$$

and p is at least equal to the positive root of the equation $2tx^2 + (n^2 - 2e)x - n^2 = 0$. Whence

$$\chi_t(G) \geq \frac{2e - n^2 + \sqrt{(n^2 - 2e)^2 + 8tn^2}}{4t}$$

and equality holds if and only if $G \in \mathbf{G}_t$. □

3 Complexity issues

For integers $t \geq 0$ and $k \geq 1$ we consider the following problem.

Problem t-CN $\leq k$

Instance: A graph G.

Question: Is $\chi_t(G) \leq k$?

For $t = 0$, the problem of the well-known colorability problem which is known to be polynomial for $k = 2$ and NP-complete for $k \geq 3$. In this section we show that the same result holds for every value of t.

Theorem 3.1. *Let $t \geq 0$ and $k \geq 1$ be given integers.*
(i) The problem t-CN $\leq k$ is solvable in polynomial time for $k = 2$.
(ii) The problem t-CN $\leq k$ is NP-complete for $k \geq 3$.

Proof. (i) The graph G satisfies $\chi_t(G) \leq 2$ if and only if we can find two disjoint sets F_1 and F_2 of at most t edges of G such that the graphs $(V(F_1), F_1)$ and $(V(F_2), F_2)$ are induced in G, and the two vertex sets $V(F_1)$ and $V(F_2)$ are disjoint and can be extended to a bipartition $A_1 \cup A_2$ of the graph $(V, E \setminus (F_1 \cup F_2))$ such that $V(F_1) \subseteq A_1$ and $V(F_2) \subseteq A_2$. Since the value of t is fixed, the choice of F_1 and F_2, the examination of the graphs $(V(F_i), F_i)$, and the test of the extension of (V_1, V_2) to a bipartition of $G - (F_1 \cup F_2)$ are polynomial.

(ii) For $k \geq 3$, the problem is clearly in NP.

Claim If the graph H consists of $tn+1$ disjoint copies of a graph G of order n, then $\chi_t(H) = \chi(G)$.

Proof of the claim: Clearly, $\chi_t(H) \leq \chi(H) = \chi(G) \leq n$.

Now let $\mathbf{P} = \{V_1, V_2, \ldots, V_p\}$ be a t-partition of H with $p = \chi_t(H)$. We have $\sum_{i=1}^{p} e(V_i) \leq pt \leq nt$. If for each copy C of G in H there exists one V_i such that V_i contains at least one edge of C, then $\sum_{i=1}^{p} e(V_i) \geq tn+1$ which is a contradiction. Hence there is one copy of G in H such that the restriction of $\mathbf{P}$ to it yields a p-coloring of G. Thus $\chi(G) \leq \chi_t(H)$ and the proof of the claim is complete. □

Now the desired result follows from the NP-completeness of the ordinary colorability problem ($t = 0$). □

4 Existence of the s-complete t-chromatic numbers

Usually a partition is called "minimal" if no two different classes can be gathered into one without creating a violation. This immediately implies that minimal 0-partitions are 1-complete, χ_0^1 always exists and equals χ_0. Already the complete graph shows that χ_0^2 does not always exist.

In this section we prove results on the existence of s-complete t-partitions. Since every s-complete partition is s'-complete for all $s' \leq s$, we are interested in the maximum value of s for which s-complete t-partitions exist. For a given partition $\mathbf{P} = \{V_1, \ldots, V_p\}$ we define the *weight* of $\mathbf{P}$ as $f(\mathbf{P}) := \sum_{i=1}^{p} e(V_i)$.

Theorem 4.1. *For every graph G, $\chi_1^2(G)$ exists and $\chi_1^2(G) = \chi_1(G)$. For $t \geq 2$, $\chi_t^{\lfloor \frac{t}{2} \rfloor + 1}(G)$ exists and $\chi_t^{\lfloor \frac{t}{2} \rfloor + 1}(G) = \chi_t(G)$.*

Proof. Let $\mathbf{P} = \{V_1, V_2, ..., V_p\}$ be a t-partition of G such that p is minimum and subject to this condition $f(\mathbf{P})$ is minimum. Clearly, $p = \chi_t(G)$ and

$$e(V_i) \leq t \text{ for all } 1 \leq i \leq p. \tag{4.1}$$

Since $\mathbf{P}$ is minimal

$$e(V_i) + e(V_j) + e(V_i, V_j) \geq t + 1 \text{ for all } 1 \leq i \neq j \leq p. \tag{4.2}$$

If the partition $\mathbf{P}$ is not s-complete for some integer s, $1 \leq s \leq t+3$, there exist two indices $i \neq j$ such that

$$e(V_i, V_j) \leq s - 1. \tag{4.3}$$

For these indices i and j, we have, by (4.1) and (4.3),

$$e(V_i) + e(V_j) + e(V_i, V_j) \leq 2t + s - 1. \tag{4.4}$$

Let $\{V_i', V_j'\}$ define a partition of $V_i \cup V_j$ such that $e(V_i', V_j')$ is maximum. Since $G[V_i' \cup V_j'] = G[V_i \cup V_j]$, inequality (4.2) implies

$$e(V_i', V_j') + e(V_i') + e(V_j') = e(V_i, V_j) + e(V_i) + e(V_j) \geq t + 1. \tag{4.5}$$

Since $e(V_i', V_j')$ is maximum, $|N_G(u) \cap V_j'| \geq |N_G(u) \cap V_i'|$ for all $u \in V_i'$ and $|N_G(v) \cap V_i'| \geq |N_G(v) \cap V_j'|$ for all $v \in V_j'$. Adding these inequalities gives

$$e(V_i', V_j') \geq 2e(V_i') \text{ and } e(V_i', V_j') \geq 2e(V_j') \tag{4.6}$$

and thus

$$e(V_i') + e(V_j') \leq e(V_i', V_j') \tag{4.7}$$

(note that the inequalitie (4.7) expresses the property of every graph to contain a bipartite subgraph with at least $|E|/2$ edges.

If $e(V_i') = e(V_j') = 0$ then, by (4.5), $e(V_i', V_j') \geq t + 1 \geq 2$ while if $e(V_i') \geq 1$ or $e(V_j') \geq 1$ then, by (4.6), $e(V_i', V_j') \geq 2$. On the other hand, by (4.7) and (4.5), $e(V_i', V_j') \geq \lceil \frac{t+1}{2} \rceil$. Therefore

$$e(V_i', V_j') \geq \max\left\{\left\lceil \frac{t+1}{2} \right\rceil, 2\right\}. \tag{4.8}$$

Let $\mathbf{P}'$ be the partition $(\mathbf{P} - \{V_i, V_j\}) \cup \{V_i', V_j'\}$ of G. If $e(V_i') \geq t+1$ then, by (4.6) and (4.4),

$$\begin{aligned} 3(t+1) &\leq 3e(V_i') + e(V_j') \\ &\leq e(V_i', V_j') + e(V_i') + e(V_j') \\ &= e(V_i, V_j) + e(V_i) + e(V_j) \\ &\leq 2t + s - 1 \end{aligned}$$

which contradicts $s \leq t + 3$. Hence $e(V_i') \leq t$. Similarly $e(V_j') \leq t$ and thus $\mathbf{P}'$ is a minimum t-partition of G. If $s - 1 < \max\left\{\lceil \frac{t+1}{2} \rceil, 2\right\}$ then, by (4.3) and (4.8), $e(V_i', V_j') > e(V_i, V_j)$ and by (4.5), $e(V_i) + e(V_j) > e(V_i') + e(V_j')$. This implies $f(\mathbf{P}') < f(\mathbf{P})$ and contradicts the choice of $\mathbf{P}$. This proves that if the t-partition $\mathbf{P}$ is not s-complete, with $s \leq t + 3$, then $s \geq \max\left\{\lceil \frac{t+1}{2} \rceil, 2\right\} + 1$. In other words, for $s = \max\left\{\lceil \frac{t+1}{2} \rceil, 2\right\} = \max\left\{\lfloor \frac{t}{2} \rfloor + 1, 2\right\}$, $\mathbf{P}$ is a s-complete t-partition of G with $\chi_t(G)$ parts. Hence $\chi_t^s(G)$ exists for this value of s and is at most $\chi_t(G)$. Since the inverse inequality always holds, $\chi_t^s(G) = \chi_t(G)$. □

For $t = 1$ and $t = 2$, Theorem 2 says that $\chi_1^2(G) = \chi_1(G)$ and $\chi_2^2(G) = \chi_2(G)$ for every graph G. These results are sharp since the 1-partitions and 2-partitions of K_{2k+1} consist of parts isomorphic to K_1 (at least one) and possibly K_2 and are never 3-complete. They are probably not sharp for large values of t.

Theorem 4.2. *If the graph G has no odd cycle of length at most s, then $\chi_t^s(G)$ exists and $\chi_t^s(G) = \chi_t(G)$ for all integer $t \geq s - 1$.*

Proof. We suppose $t \geq s - 1$ and as in the proof of Theorem 4.1, we consider a minimum t-partition $\mathbf{P} = \{V_1, V_2, ..., V_p\}$ of G of minimum weight $f(\mathbf{P})$. If $\mathbf{P}$ is not s-complete, the inequalities (4.1) to (4.3) are still valid and for the two particular indices i and j, we get from (4.2) and (4.3)

$$e(V_i) + e(V_j) \geq t - s + 2 \geq 1\,. \tag{4.9}$$

Let U_i (U_j respectively) be the set of the vertices in V_i (V_j respectively) that are incident with an edge in $E(V_i, V_j)$. Suppose $e(U_i) > 0$ and let u_1, u_2 be two vertices of U_i such that u_1u_2 is an edge of G. Since $e(V_i, V_j) \leq s - 1$ and G has no odd cycle of length at most s, the graph $G' = (U_i \cup U_j, E(V_i, V_j) \cup \{u_1u_2\})$ is bipartite. Let $\{U_i', U_j'\}$ be a bipartition of G'. There is no edge between $U_i' \cap V_j$ and $V_i \setminus U_i$ by the definition of U_i, and no edge between $U_i' \cap V_j$ and $U_i' \cap V_i$ since U_i' is a class of the bipartition. Hence $E((V_i \setminus U_i) \cup U_i') \subseteq E(V_i)$. Similarly, $E(V_j \setminus U_j) \cup U_j') \subseteq E(V_j)$. Therefore $\mathbf{P}' = (\mathbf{P} - \{V_i, V_j\}) \cup \{(V_i \setminus U_i) \cup U_i', V_j \setminus U_j) \cup U_j'\}$ is a t-partition of G. Moreover, $f(\mathbf{P}') < f(\mathbf{P})$ since the edge u_1u_2 of the class V_i of $\mathbf{P}$ joins different classes of $\mathbf{P}'$. This contradicts the choice of $\mathbf{P}$. Therefore $e(U_i) = 0$ and by symmetry $e(U_j) = 0$. By (4.9) we can assume without loss of generality that $e(V_i) \neq 0$. Since $e(U_i) = 0$, there exists a vertex $v \in V_i - U_i$ with $|N_G(v) \cap V_i| > 0$. Then $\mathbf{P}'' = (\mathbf{P} - \{V_i, V_j\}) \cup \{V_i \setminus \{v\}, V_j \cup \{v\}\}$ is a t-partition of G with $f(\mathbf{P}'') < f(\mathbf{P})$, a contradiction. Hence the t-partition $\mathbf{P}$ is s-complete. □

Finally, we can remark that the notion of minimality of a t-partition, meaning that $e(V_i) + e(V_j) + e(V_i, V_j) \geq t + 1$ for all $i \neq j$, can be reduced to a notion of completeness of the partition only when $t = 0$.

References

[1] C. Berge, Graphes et hypergraphes, Ed. Dunod, Paris 1973.

[2] B. Bollobás and A.D. Scott, Exact bounds for judicious partitions of graphs, *Combinatorica* 19(4) (1999), 473–486.

[3] B. Bollobás and A.D. Scott, Problems and results on judicious partitions, *Random Struct. Algo.* 21 (3–4) (2002), 414–430.

[4] P. Erdős, Sixth International Conference on the Theory and Applications of Graphs, 1988 (Kalamazoo, MI), mentioned during his invited talk.

[5] G. Hopkins and W. Staton, Vertex partitions and k-small subsets of graphs, *Ars Combin.* 22 (1986), 19–24.

[6] B.R. Myers and R. Liu, A lower bound on the chromatic number of a graph, *Networks* 1 (1971), 273–277.

[7] T.D. Porter, On a bottleneck bipartition conjecture of Erdős, *Combinatorica* 12(3) (1992), 317–321.

[8] T.D. Porter and B. Yang, Graph partitions II, *J. Combin. Math. Combin. Comput.* 37 (2001), 159–171.

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