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Signed double Roman k-domination in graphs

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Abstract

Let $G = (V, E)$ be a simple and finite graph with vertex set $V(G)$, and let $k \geq 1$ be an integer. A signed double Roman k-dominating function (SDRkDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2, 3\}$ such that (i) every vertex v with $f(v) = -1$ is adjacent to at least two vertices assigned with 2 or to at least one vertex w with $f(w) = 3$, (ii) every vertex v with $f(v) = 1$ is adjacent to at least one vertex w with $f(w) \geq 2$ and (iii) $\sum_{u \in N[v]} f(u) \geq k$ holds for any vertex v. The *weight* of an SDRkDF f is $\sum_{u \in V(G)} f(u)$, and the minimum weight of an SDRkDF is the signed double Roman k-domination number $\gamma^k_{sdR}(G)$ of G. In this paper, we initiate the study of the signed double Roman k-domination number in graphs and we present lower and upper bounds for $\gamma^k_{sdR}(T)$. In addition we determine this parameter for some classes of graphs.

1 Introduction

Throughout this paper, G denotes a simple graph, with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. Denote by K_n the complete graph and by C_n the cycle of order n. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of G has degree r, then G is said to be r-regular. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S}N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices of G. The complement of a graph G is denoted by $\overline{G}$. A *leaf* of G is a vertex with degree one and a *support vertex* is a vertex adjacent to a leaf. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $DS_{p,q}$. A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y; such a partition (X, Y) is called a *bipartition* of the graph. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$.

A set $S \subseteq V$ in a graph G is a *dominating set* if every vertex of G is either in S or adjacent to a vertex of S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [9, 10].

For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f : V(G) \to \mathbb{R}$, we define $f(S) = \sum_{x\in S} f(x)$. For a vertex v, we denoted $f(N[v])$ by $f[v]$ for notional convenience.

A *double Roman dominating function*(DRDF) is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or one neighbor with $f(w) = 3$, and if $f(v) = 1$, then vertex v must have at least one neighbor with $f(w) \geq 2$. The weight of a double Roman dominating function f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The *double Roman dominating number* of G is the minimum weight of a double Roman dominating function on G. The double Roman domination was introduced by Beeler et al. [7] and has been studied by several authors [1, 2, 4, 6, 16, 17, 21].

A signed Roman k-dominating function (SRkDF) on a graph G is a function $f : V \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N[v]} f(x) \geq k$ for each vertex $v \in V$, and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an SRkDF is the sum of its function values over all vertices. The signed Roman k-domination number of G, denoted $\gamma^k_{sR}(G)$, is the minimum weight of an SRkDF in G. The signed Roman k-domination number was introduced by Henning and Volkman in [11] and has been studied in

[12, 13, 18, 19, 20]. The special case $k = 1$ was introduced and investigated in [5] and has been studied in [14, 15].

In this paper, we continue the study of the double Roman dominating functions on graphs. Inspired by the previous research on the signed Roman k-dominating function [11, 12], we define the signed double Roman k-dominating function as follows.

Let $k \geq 1$ be an integer. A function $f : V(G) \to \{-1, 1, 2, 3\}$ is *a signed double Roman k-dominating function* (SDRkDF) on a graph G if the following conditions are fulfilled:

(i) $\sum_{x \in N[v]} f(v) \geq k$ for every vertex $v \in V(G)$;

(ii) if $f(v) = -1$, then vertex v must have at least two neighbors with label 2 or one neighbor with label 3;

(iii) if $f(v) = 1$, then vertex v must have at least one neighbor with $f(w) \geq 2$.

The weight of an SDRkDF is the sum of its function values over all vertices. The signed double Roman k-domination number of G, denoted $\gamma^k_{sdR}(G)$, is the minimum weight of an SDRkDF in G. A signed double Roman k-dominating function of G of weight $\gamma^k_{sdR}(G)$ is called a $\gamma^k_{sdR}(G)$*-function* or γ^k_{sdR}*-function* of G. The special case $k = 1$ has been studied by Ahangar et al. [3]. If f is a signed double Roman k-dominating function of G and $v \in V(G)$, then by definition we must have $k \leq \sum_{x \in N[v]} f(v) \leq \sum_{x \in N[v]} 3 = 3(\deg(v) + 1)$ yielding $\deg(v) \geq k/3 - 1$ and so $\delta(G) \geq k/3 - 1$. As the assumption $\delta(G) \geq k/3 - 1$ is necessary, we always assume that when we discuss $\gamma^k_{sdR}(G)$, all graphs involved satisfy $\delta(G) \geq k/3 - 1$ and thus $n(G) \geq k/3$.

An SDRkDF f can be represented by the ordered quadruple (V_{-1}, V_1, V_2, V_3) of $V(G)$ where $V_i = \{v \in V(G) \mid f(v) = i\}$ for $i \in \{-1, 1, 2, 3\}$. In this paper we initiate the study of signed double Roman k-domination numbers in graphs and investigate their basic properties. In particular, we establish some sharp bounds on signed double Roman k-domination. In addition, we determine the signed double Roman k-domination number of some classes of graphs.

We make use of the following results in this paper.

Observation 1.1. *If $f = (V_{-1}, V_1, V_2, V_3)$ is an SDRkDF on a graph G of order n, then*

(a) $|V_{-1}| + |V_1| + |V_2| + |V_3| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.

Observation 1.2. *If $k \geq 1$ is an integer and G is a graph of order n with $\delta(G) \geq \lfloor \frac{k-1}{2} \rfloor$, then $\gamma^k_{sdR}(G) \leq 2n$.*

Proof. Clearly, the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(x) = 2$ for $x \in V(G)$ is an SDRkDF on G of weight $2n$ and thus $\gamma^k_{sdR}(G) \leq 2n$. □

Let $k \geq 2$ and $n \geq \lfloor \frac{k}{2} \rfloor$ be integers and let V be a set of size n. Define $f^k_n : V \to \{-1, 1, 2, 3\}$ as follows. If $n + k \equiv 0 \pmod 3$, then let f^k_n assign 2 to $\frac{n+k}{3}$ elements of V and -1 to the remaining elements, if $n + k \equiv 1 \pmod 3$, then let f^k_n assign 3 to one element of V, 2 to $\frac{n+k-4}{3}$ elements of V and -1 to the remaining elements, and if $n + k \equiv 2 \pmod 3$, then let f^k_n assign 3 to two elements of V, 2 to $\frac{n+k-8}{3}$ elements of V and -1 to the remaining elements.

Observation 1.3. *Let $k \geq 2$ and $n \geq \lfloor \frac{k}{2} \rfloor$ be integers. If $n + k \geq 6$, then $\gamma^k_{sdR}(K_n) = k$.*

Proof. For any vertex $v \in V(K_n)$, we have $\gamma^k_{sdR}(G) = f[v] \geq k$. Now we show that $\gamma^k_{sdR}(K_n) \leq k$. If k is an even number and $n = k/2$, then the result follows from Observation 1.2. If k is an odd number or $n \geq \lfloor \frac{k}{2} \rfloor + 1$, then it is easy to verify that the function f^k_n defined above, is an SDRkDF on K_n of weight k, and so $\gamma^k_{sdR}(G) \leq k$. Thus $\gamma^k_{sdR}(G) = k$. □

2 Basic properties and bounds

In this section we present basic properties of the signed double Roman k-dominating function.

Proposition 2.1. *Let $f = (V_{-1}, V_1, V_2, V_3)$ be an SDRkDF on a graph G of order n. If $\delta \geq k - 1$, then*

(i) $(3\Delta + 3 - k)|V_3| + (2\Delta + 2 - k)|V_2| + (\Delta + 1 - k)|V_1| \geq (\delta + k + 1)|V_{-1}|$.

(ii) $(3\Delta + \delta + 4)|V_3| + (2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + k + 1)n$.

(iii) $(\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2| + 2(\delta - \Delta)|V_3|$.

(iv) $\omega(f) \geq (\delta - 3\Delta + 2k - 2)n/(3\Delta + \delta + 4) + |V_2| + 2|V_3|$.

Proof. (i) It follows from Observation 1.1(a) that

$$\begin{aligned} k(|V_{-1}| + |V_1| + |V_2| + |V_3|) &= kn \\ &\leq \sum_{v \in V(G)} f[v] \\ &= \sum_{v \in V(G)} (d(v) + 1) f(v) \\ &= \sum_{v \in V_3} 3(d(v) + 1) + \sum_{v \in V_2} 2(d(v) + 1) + \sum_{v \in V_1} (d(v) + 1) \\ &\quad - \sum_{v \in V_{-1}} (d(v) + 1) \\ &\leq 3(\Delta+1)|V_3| + 2(\Delta+1)|V_2| + (\Delta+1)|V_1| - (\delta+1)|V_{-1}|. \end{aligned}$$

The inequality chain leads to the desired bound in (i).

(ii) By Observation 1.1(a), we have $|V_{-1}| = n - |V_1| - |V_2| - |V_3|$. By this identity and Part (i) of Proposition 2.1, we reach (ii).

(iii) According to Observation 1.1 and Part (ii) of Proposition 2.1, we obtain Part (iii) of Proposition 2.1 as follows.

$$\begin{aligned} (\Delta + \delta + 2)\omega(f) &= (\Delta + \delta + 2)(2|V_1| + 3|V_2| + 4|V_3| - n) \\ &\geq 2(\delta + k + 1)n - 2(3\Delta + \delta + 4)|V_3| - 2(2\Delta + \delta + 3)|V_2| \\ &\quad + (\Delta + \delta + 2)(3|V_2| + 4|V_3| - n) \\ &= (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2| + 2(\delta - \Delta)|V_3|. \end{aligned}$$

(iv) The inequality chain in the proof of Part (i) and Observation 1.1(a) show that

$$\begin{aligned} kn &\leq 3(\Delta+1)|V_3| + 2(\Delta+1)|V_2| + (\Delta+1)|V_1| - (\delta+1)|V_{-1}| \\ &\leq 3(\Delta+1)|V_1 \cup V_2 \cup V_3| - (\delta+1)|V_{-1}| \\ &= 3(\Delta+1)|V_1 \cup V_2 \cup V_3| - (\delta+1)(n - |V_1 \cup V_2 \cup V_3|) \\ &= (3\Delta+\delta+4)|V_1 \cup V_2 \cup V_3| - (\delta+1)n \end{aligned}$$

and so

$$|V_1 \cup V_2 \cup V_3| \geq \frac{n(\delta+k+1)}{3\Delta+\delta+4}.$$

Using this equality and Observation 1.1, we obtain

$$\begin{aligned} \omega(f) &= 2|V_1 \cup V_2 \cup V_3| - n + |V_2| + 2|V_3| \\ &\geq \frac{n(\delta - 3\Delta + 2k - 2)}{3\Delta+\delta+4} + |V_2| + 2|V_3|. \end{aligned}$$

This is the bound in Part (iv), and the proof is complete. □

Proposition 2.2. *Let r be a non-negative integer with $r \geq \lceil \frac{k-3}{3} \rceil$. If G is an r-regular graph of order n, then*

$$\gamma_{sdR}^k(G) \geq \frac{kn}{r+1}.$$

The equality holds for the complete graph K_n when $n + k \geq 6$.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}^k(G)$-function. We have

$$(r+1)\gamma_{sdR}^k(G) = (r+1)\sum_{v \in V(G)} f(v) = \sum_{v \in V(G)} (r+1)f(v) = \sum_{v \in V(G)} f[v] \geq kn$$

and this leads to the desired bound. By Observation 1.3, the equality holds for the complete graph K_n if $n + k \geq 6$. □

Corollary 2.1. *If G is a graph of order n, minimum degree $\delta \geq k-1$ and maximum degree Δ, then*

$$\gamma_{sdR}^k(G) \geq \Big(\frac{-3\Delta^2 + 3\Delta\delta + 4k\Delta - 3\Delta + 3\delta + 4k}{(\Delta+1)(3\Delta+\delta+4)}\Big)n.$$

Proof. If $\delta < \Delta$, multiplying both sides of the inequality in Proposition 2.1 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2.1 (iii), we obtain the desired lower bound.

If $\delta = \Delta = r$, the desired inequality can be simplified to $\gamma_{sdR}^k(G) \geq \frac{kn}{r+1}$. It obviously holds according to Proposition 2.2. □

Proposition 2.3. *If G is a graph of order n with $\delta(G) \geq \lfloor\frac{k-3}{3}\rfloor$, then*

$$\gamma_{sdR}^k(G) \geq \Delta(G) + k + 1 - n.$$

Proof. Let $u \in V(G)$ be a vertex of maximum degree, and let f be a $\gamma_{sdR}^k(G)$-function. By the definitions we have

$$\gamma_{sdR}^k(G) = f[u] + \sum_{x\in V(G)-N[u]} f(x) \geq k - (n - \Delta(G) - 1) = \Delta(G) + k + 1 - n,$$

and the proof is complete. □

A set $S \subseteq V(G)$ is a 2-*packing* of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The 2-*packing* number $\rho(G)$ of G is defined by

$$\rho(G) = \max\{|S| : S \text{ is a 2-packing of } G\}.$$

Clearly, for all graphs G, $\rho(G) \leq \gamma(G)$.

Proposition 2.4. *If G is a graph of order n with $\delta(G) \geq \lfloor\frac{k-3}{3}\rfloor$, then*

$$\gamma_{sdR}^k(G) \geq \rho(G)(\delta(G) + k + 1) - n.$$

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a 2-packing of G, and let f be a $\gamma_{sdR}^k(G)$-function. Suppose $A = \bigcup_{i=1}^{\rho(G)} N[v_i]$. We have $|A| = \sum\limits_{i=1}^{\rho(G)} (d(v_i) + 1) \geq \rho(G)(\delta(G) + 1)$ and hence

$$\begin{aligned}
\gamma_{sdR}^k(G) &= \sum_{i=1}^{\rho(G)} f[v_i] + \sum_{x\in V(G)-A} f(x) \\
&\geq k\rho(G) + \sum_{x\in V(G)-A} f(x) \geq k\rho(G) - n + |A| \\
&\geq k\rho(G) - n + \rho(G)(\delta(G) + 1) = \rho(G)(\delta(G) + k + 1) - n.
\end{aligned}$$

□

Corollary 2.2. *If G is a graph of order n with $\delta(G) \geq \lfloor\frac{k-1}{2}\rfloor$ and $\rho(G) = \gamma(G)$, then*

$$\gamma_{sdR}^k(G) \geq (\delta(G) + k + 1)\gamma(G) - n.$$

If G is the graph obtained from a graph H by adding a pendant edge at each vertex of H, then clearly $\gamma_{sdR}^k(G) = 3n(G)/2$ and hence the bound in Corollary 2.2 is sharp.

Since for any connected graph G, we have $\rho(G) \geq 1 + \lfloor\frac{\mathrm{diam}(G)}{3}\rfloor$, the next result is an immediate consequence of Proposition 2.4.

Corollary 2.3. *If G is a graph of order n with $\delta(G) \geq \lfloor\frac{k-1}{2}\rfloor$, then*

$$\gamma_{sdR}^k(G) \geq \big(1 + \lfloor\frac{\mathrm{diam}(G)}{3}\rfloor\big)(\delta(G) + k + 1) - n.$$

Next we present a so-called Nordhous-Gaddum type inequality for the signed double Roman k-domination number of regular graphs.

Theorem 2.1. *If G is an r-regular graph of order n such that $r \geq \lceil \frac{k-3}{3} \rceil$ and $n-r-1 \geq \lceil \frac{k-3}{3} \rceil$, then*

$$\gamma^k_{sdR}(G) + \gamma^k_{sdR}(\overline{G}) \geq \frac{4kn}{n+1}.$$

If n is even, then $\gamma^k_{sdR}(G) + \gamma^k_{sdR}(\overline{G}) \geq \frac{4k(n+1)}{n+2}$.

Proof. Since G is an r-regular, the complement $\overline{G}$ is $(n-r-1)$-regular. By Proposition 2.2

$$\gamma^k_{sdR}(G) + \gamma^k_{sdR}(\overline{G}) \geq kn(\frac{1}{r+1} + \frac{1}{n-r}).$$

By assumptions $\lceil \frac{k-3}{3} \rceil \leq r \leq n-1-\lceil \frac{k-3}{3} \rceil$ and since the function $g(x) = 1/(x+1)+1/(n-x)$ takes its minimum at $x = (n-1)/2$ when $\lceil \frac{k-3}{3} \rceil \leq x \leq n-1-\lceil \frac{k-3}{3} \rceil$, we obtain

$$\gamma^k_{sdR}(G) + \gamma^k_{sdR}(\overline{G}) \geq kn(\frac{2}{n+1} + \frac{2}{n+1}) = \frac{4kn}{n+1},$$

and this is the desired bound. If n is even, then the function g takes its minimum at $r = x = (n-2)/2$ or $r = x = n/2$, since r is an integer. This implies that

$$\gamma^k_{sdR}(G) + \gamma^k_{sdR}(\overline{G}) \geq kn(\frac{1}{r+1} + \frac{1}{n-r}) \geq kn(\frac{2}{n} + \frac{2}{n+2}) = \frac{4k(n+1)}{n+2},$$

and the proof is complete. □

Next we establish bounds on the signed double Roman k-domination number in terms of order and domination number.

Proposition 2.5. *Let G be a connected graph of order n. If G has a $\gamma^k_{sdR}(G)$-function $f = (V_{-1}, V_1, V_2, V_3)$ with $V_{-1} = \emptyset$, then $\gamma^k_{sdR}(G) \geq n + \gamma(G)$.*

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^k_{sdR}(G)$-function such that $V_{-1} = \emptyset$. Since each vertex in V_1 must be adjacent to a vertex in $V_2 \cup V_3$, we deduce that $V_2 \cup V_3$ is a dominating set of G. It follows from Observation 1.1 that

$$\gamma^k_{sdR}(G) = \omega(f) = |V_1| + 2|V_2| + 3|V_3| \geq |V_1| + 2|V_2| + 2|V_3| \geq n + \gamma(G).$$

□

Proposition 2.6. *Let G be a graph of order n with $\delta(G) \geq 1$. If $k \in \{2, 3\}$, then*

$$\gamma^k_{sdR}(G) \leq n + \gamma(G).$$

Furthermore, this bound is sharp.

Proof. Let S be a $\gamma(G)$-set and define $f : V(G) \to \{-1, 1, 2, 3\}$ by $f(x) = 2$ for $x \in S$ and $f(x) = 1$ for $x \in V(G) - S$. Clearly, f is an SDRkDF on G of weight $n + \gamma(G)$ for $k \in \{2, 3\}$ and this implies that $\gamma^k_{sdR}(G) \le n + \gamma(G)$ for $k \in \{2, 3\}$.

To prove the sharpness, let $G_2 = mK_1$ and let G_3 be the graph obtained from a graph H by adding a pendant edge at each vertex of H. Clearly, for any $\gamma^k_{sdR}(G_k)$-function $f = (V_{-1}, V_1, V_2, V_3)$ we have $V_{-1} = \emptyset$. It follows from Proposition 2.5 that $\gamma^k_{sdR}(G_k) \ge n(G_k) + \gamma(G_k)$ yielding $\gamma^k_{sdR}(G_k) = n(G_k) + \gamma(G_k)$. □

The proof of next result is similar to the proof of Proposition 2.6 and therefore it is omitted.

Proposition 2.7. *For any graph G of order n with $\delta(G) \ge 1$,*

$$\gamma^4_{sdR}(G) \le n + 2\gamma(G).$$

The bound is sharp for the graph obtained from a graph H by adding at least two pendant edges at each vertex of H.

3 Signed double Roman 2-domination

In this section, we present bounds on the signed double Roman 2-domination of G. For convenience, we introduce some notation. For an SDR2DF $f = (V_{-1}, V_1, V_2, V_3)$ of G, we let $V'_{-1} = \{v \in V_{-1} \mid N(v) \cap V_3 \ne \emptyset\}$ and $V''_{-1} = V_{-1} - V'_{-1}$. For a subset $S \subseteq V$, we let $d_S(v)$ denote the number of vertices in S that are adjacent to v. In particular, $d_V(v) = d(v)$. For disjoint subsets U and W of vertices, we let $[U, W]$ denote the set of edges between U and W. For notational convenience, we let $V_{12} = V_1 \cup V_2, V_{13} = V_1 \cup V_3, V_{123} = V_1 \cup V_2 \cup V_3$ and let $|V_{12}| = n_{12}, |V_{13}| = n_{13}, |V_{123}| = n_{123}$, and let $|V_1| = n_1, |V_2| = n_2$ and $|V_3| = n_3$. Then, $n_{123} = n_1 + n_2 + n_3$. Further, we let $|V_{-1}| = n_{-1}$, and so $n_{-1} = n - n_{123}$. Let $G_{123} = G[V_{123}]$ be the subgraph induced by the set V_{123} and let G_{123} have size m_{123}. For $i = 1, 2, 3$, if $V_i \ne \emptyset$, let $G_i = G[V_i]$ be the subgraph induced by the set V_i and let G_i have size m_i. Hence, $m_{123} = m_1 + m_2 + m_3 + |[V_1, V_2]| + |[V_1, V_3]| + |[V_2, V_3]|$.

For $t \ge 1$, let L_t be the graph obtained from a graph H of order t by adding $3d_H(v) + 1$ pendant edges to each vertex v of H. Let $\mathcal{H} = \{L_t \mid t \ge 1\}$.

Theorem 3.1. *Let G be a graph of order n and size m without isolated vertex. Then*

$$\gamma^2_{sdR}(G) \ge \frac{5n - 6m}{2},$$

with equality if and only if $G \in \mathcal{H}$.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^2_{sdR}(G)$-function. If $V_{-1} = \emptyset$, then $\gamma^2_{sdR}(G) > n \ge \frac{5n-6m}{2}$ since G has no isolated vertex. Hence $V_{-1} \ne \emptyset$. We consider the following cases.

Case 1. $V_3 \ne \emptyset$.
Now, we consider the following subcases.

Subcase 1.1. $V_2 \neq \emptyset$.
By the definition of an SDR2DF, each vertex in V_{-1} is adjacent to at least one vertex in V_3 or to at least two vertices in V_2, and so

$$|[V_{-1}, V_{123}]| \geq |[V_{-1}, V_3]| + |[V_{-1}, V_2]| \geq |V'_{-1}| + 2|V''_{-1}| \geq n_{-1}.$$

Furthermore we have

$$2n_{-1} \leq 2|[V_{-1}, V_3]| + |[V_{-1}, V_2]| = 2\sum_{v\in V_3} d_{V_{-1}}(v) + \sum_{v\in V_2} d_{V_{-1}}(v).$$

For each vertex $v \in V_3$, we have that $f(v) + 3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 2$, and so $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) + 1$. Similarly, for each vertex $v \in V_2$, we have that $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v)$. Now, we have

$$\begin{aligned}
2n_{-1} &\leq 2\sum_{v\in V_3} d_{V_{-1}}(v) + \sum_{v\in V_2} d_{V_{-1}}(v)\\
&\leq 2\sum_{v\in V_3}(3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v) + 1) + \sum_{v\in V_2}(3d_{V_3}(v) + 2d_{V_2}(v) + d_{V_1}(v))\\
&= (12m_3 + 4|[V_2, V_3]| + 2|[V_1, V_3]| + 2n_3) + (3|[V_2, V_3]| + 4m_2 + |[V_1, V_2]|)\\
&= 12m_3 + 4m_2 + 7|[V_2, V_3]| + 2|[V_1, V_3]| + |[V_1, V_2]| + 2n_3\\
&= 12m_{123} - 12m_1 - 8m_2 - 5|[V_2, V_3]| - 10|[V_1, V_3]| - 11|[V_1, V_2]| + 2n_3,
\end{aligned}$$

which implies that

$$m_{123} \geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| - 2n_3).$$

Hence,

$$\begin{aligned}
m &\geq m_{123} + |[V_{-1}, V_{123}]| + m_{-1}\\
&\geq m_{123} + |[V_{-1}, V_{123}]|\\
&\geq \frac{1}{12}(2n_{-1} + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]|\\
&\quad + 11|[V_1, V_2]| - 2n_3) + n_{-1}\\
&= \frac{1}{12}(14n_{-1} - 2n_{123} + 2n_1 + 2n_2\\
&\quad + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]|)\\
&= \frac{1}{12}(14n - 16n_{123} + 2n_1 + 2n_2\\
&\quad + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]|)
\end{aligned}$$

and so

$$n_{123} \geq \frac{1}{16}(-12m+14n+2n_1+2n_2+12m_1+8m_2+5|[V_2, V_3]|+10|[V_1, V_3]|+11|[V_1, V_2]|).$$

Now, we have

$$\begin{aligned}
\gamma_{sdR}^2(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\
&= 4n_3 + 3n_2 + 2n_1 - n \\
&= 4n_{123} - n - n_2 - 2n_1 \\
&\geq \frac{1}{4}(-12m + 14n + 2n_1 + 2n_2 + 12m_1 + 8m_2 \\
&+ 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]|) - n - n_2 - 2n_1 \\
&= \frac{1}{4}(-12m + 14n - 4n) + \frac{1}{4}(2n_1 + 2n_2 + 12m_1 + 8m_2 \\
&+ 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| - 4n_2 - 8n_1) \\
&= \frac{5n - 6m}{2} + \frac{1}{4}(-6n_1 - 2n_2 + 12m_1 + 8m_2 \\
&+ 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]|).
\end{aligned}$$

Let $\Theta = -6n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]|$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = -2n_2 + 8m_2 + 5|[V_2, V_3]|$. If $v \in V_2$ and $d_{V_{23}}(v) = 0$, then since G has no isolated vertex, we have that every neighbor of v belongs to V_{-1}. But then $f[v] \leq 1$, a contradiction. Hence if $v \in V_2$, then we have that $d_{V_{23}}(v) \geq 1$. Then

$$\begin{aligned}
\Theta &= -2n_2 + 8m_2 + 5|[V_2, V_3]| \\
&= 4\sum_{v \in V_2} d_{V_2}(v) + 4\sum_{v \in V_2} d_{V_3}(v) + (|[V_2, V_3]| - 2n_2) \\
&= 4\sum_{v \in V_2} d_{V_{23}}(v) + (|[V_2, V_3]| - 2n_2) \\
&\geq 4n_2 - 2n_2 + |[V_2, V_3]| \\
&> 0.
\end{aligned}$$

If $n_1 \geq 1$, then $\Theta = -6n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]|$. By the definition of an SDR2DF of G we have $d_{V_{123}}(v) \geq 1$ for each $v \in V_1$. Then

$$\begin{aligned}
\Theta &= -6n_1 - 2n_2 + 12m_1 + 8m_2 + 5|[V_2, V_3]| + 10|[V_1, V_3]| + 11|[V_1, V_2]| \\
&= 6\sum_{v \in V_1} d_{V_1}(v) + 6\sum_{v \in V_1} d_{V_2}(v) + 6\sum_{v \in V_1} d_{V_3}(v) + 4\sum_{v \in V_2} d_{V_1}(v) + 4\sum_{v \in V_2} d_{V_2}(v) \\
&+ 4\sum_{v \in V_2} d_{V_3}(v) + (-6n_1 - 2n_2 + |[V_2, V_3]| + 4|[V_1, V_3]| + |[V_1, V_2]|) \\
&= 6\sum_{v \in V_1} d_{V_{123}}(v) + 4\sum_{v \in V_2} d_{V_{123}}(v) + (-6n_1 - 2n_2 + |[V_2, V_3]| + 4|[V_1, V_3]| + |[V_1, V_2]|) \\
&\geq 6n_1 + 4n_2 - 6n_1 - 2n_2 + |[V_2, V_3]| + 4|[V_1, V_3]| + |[V_1, V_2]| \\
&= 2n_2 + |[V_2, V_3]| + 3|[V_1, V_3]| \\
&> 0.
\end{aligned}$$

Therefore $\gamma^2_{sdR}(G) > \frac{5n-6m}{2}$.

Subcase 1.2. $V_2 = \emptyset$.

By the definition of an SDR2DF, each vertex in V_{-1} is adjacent to one vertex in V_3, and so

$$|[V_{-1}, V_{13}]| \geq |[V_{-1}, V_3]| \geq |V_{-1}| = n_{-1}. \tag{1}$$

Furthermore, we have

$$n_{-1} \leq |[V_{-1}, V_3]| = \sum_{v \in V_3} d_{V_{-1}}(v).$$

For each vertex $v \in V_3$, we have that $f(v) + 3d_{V_3}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 2$, and so $d_{V_{-1}}(v) \leq 3d_{V_3}(v) + d_{V_1}(v) + 1$. Now, we have

$$\begin{aligned} n_{-1} &\leq \sum_{v \in V_3} d_{V_{-1}}(v) \\ &\leq \sum_{v \in V_3} (3d_{V_3}(v) + d_{V_1}(v) + 1) \\ &= 6m_3 + |[V_1, V_3]| + n_3 \\ &= 6m_{13} - 6m_1 - 5|[V_1, V_3]| + n_3, \end{aligned} \tag{2}$$

which implies that

$$m_{13} \geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_2]| - n_3).$$

Hence,

$$\begin{aligned} m &\geq m_{13} + |[V_{-1}, V_{13}]| + m_{-1} \\ &\geq m_{13} + |[V_{-1}, V_{13}]| \\ &\geq \frac{1}{6}(n_{-1} + 6m_1 + 5|[V_1, V_2]| - n_3) + n_{-1} \\ &= \frac{1}{6}(7n_{-1} - n_{13} + n_1 + 6m_1 + 5|[V_1, V_3]|) \\ &= \frac{1}{6}(7n - 8n_{13} + n_1 + 6m_1 + 5|[V_1, V_3]|) \end{aligned}$$

and so

$$n_{13} \geq \frac{1}{8}(-6m + 7n + n_1 + 6m_1 + 5|[V_1, V_3]|).$$

Now, we have

$$\begin{aligned}\gamma^2_{sdR}(G) &= 3n_3 + n_1 - n_{-1}\\ &= 4n_3 + 2n_1 - n\\ &= 4n_{13} - n - 2n_1\\ &\geq \frac{1}{2}(-6m + 7n + n_1 + 6m_1 + 5|[V_1, V_3]|) - n - 2n_1 \qquad (3)\\ &= \frac{1}{2}(-6m + 7n - 2n) + \frac{1}{2}(n_1 + 6m_1 + 5|[V_1, V_3]| - 4n_1)\\ &= \frac{5n - 6m}{2} + \frac{1}{2}(6m_1 + 5|[V_1, V_3]| - 3n_1).\end{aligned}$$

Let $\Theta = 6m_1 + 5|[V_1, V_3]| - 3n_1$. We show that $\Theta \geq 0$. If $n_1 = 0$, then $\Theta = 0$. Suppose that $n_1 \geq 1$. By the definition of an SDR2DF, for each $v \in V_1$ we have $d_{V_{13}}(v) \geq 1$. Then

$$\begin{aligned}\Theta &= 6m_1 + 5|[V_1, V_3]| - 3n_1\\ &= 3\sum_{v\in V_1} d_{V_1}(v) + 3\sum_{v\in V_1} d_{V_3}(v) + (2|[V_1, V_3]| - 3n_1)\\ &= 3\sum_{v\in V_1} d_{V_{13}}(v) + (2|[V_1, V_3]| - 3n_1)\\ &\geq 3n_1 + 2|[V_1, V_3]| - 3n_1\\ &> 0.\end{aligned}$$

Therefore $\gamma^2_{sdR}(G) \geq \frac{5n-6m}{2}$.

Case 2. $V_3 = \emptyset$.

Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. By the definition of an SDR2DF, each vertex in V_{-1} is adjacent to at least two vertices in V_2, and so

$$|[V_{-1}, V_{12}]| \geq |[V_{-1}, V_2]| \geq 2|V_{-1}| = 2n_{-1}.$$

Furthermore we have

$$2n_{-1} \leq |[V_{-1}, V_2]| = \sum_{v\in V_2} d_{V_{-1}}(v).$$

For each vertex $v \in V_2$, we have that $f(v) + 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v) = f[v] \geq 2$, and so $d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v)$. Now, we have

$$\begin{aligned}2n_{-1} &\leq \sum_{v\in V_2} d_{V_{-1}}(v)\\ &\leq \sum_{v\in V_2} (2d_{V_2}(v) + d_{V_1}(v))\\ &= 4m_2 + |[V_1, V_2]|\\ &= 4m_{12} - 4m_1 - 3|[V_1, V_2]|,\end{aligned}$$

which implies that

$$m_{12} \geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]|).$$

Hence,

$$\begin{aligned} m &\geq m_{12} + |[V_{-1}, V_{12}]| + m_{-1} \\ &\geq m_{12} + |[V_{-1}, V_{12}]| \\ &\geq \frac{1}{4}(2n_{-1} + 4m_1 + 3|[V_1, V_2]|) + 2n_{-1} \\ &= \frac{1}{4}(10n_{-1} + 4m_1 + 3|[V_1, V_2]|) \\ &= \frac{1}{4}(10n - 10n_{12} + 4m_1 + 3|[V_1, V_2]|) \end{aligned}$$

and so

$$n_{12} \geq \frac{1}{10}(-4m + 10n + 4m_1 + 3|[V_1, V_2]|).$$

Now, we have

$$\begin{aligned} \gamma^2_{sdR}(G) &= 2n_2 + n_1 - n_{-1} \\ &= 3n_2 + 2n_1 - n \\ &= 3n_{12} - n - n_1 \\ &\geq \frac{3}{10}(-4m + 10n + 4m_1 + 3|[V_1, V_2]|) - n - n_1 \\ &= \frac{3}{10}(-4m + 10n - \frac{10}{3}n - \frac{1}{3}n) + \frac{3}{10}(4m_1 + 3|[V_1, V_2]| - \frac{10}{3}n_1 + \frac{1}{3}n) \\ &= \frac{19n - 12m}{10} + \frac{3}{10}(4m_1 + 3|[V_1, V_2]| - 3n_1 + \frac{1}{3}(n_{-1} + n_2)) \end{aligned}$$

Let $\Theta = 4m_1 + 3|[V_1, V_2]| - 3n_1 + \frac{1}{3}(n_{-1} + n_2)$. We show that $\Theta > 0$. If $n_1 = 0$, then $\Theta = \frac{1}{3}(n_2 + n_{-1}) > 0$. Suppose that $n_1 \geq 1$. By the definition of an SDR2DF of G, we have $d_{V_2}(v) \geq 1$ for each $v \in V_1$. Then

$$\begin{aligned} \Theta &= 4m_1 + 3|[V_1, V_2]| - 3n_1 + \frac{1}{3}(n_{-1} + n_2) \\ &= 3\sum_{v \in V_1} d_{V_2}(v) + (\frac{1}{3}(n_2 + n_{-1}) - 3n_1 + 4m_1) \\ &\geq 3n_1 + \frac{1}{3}(n_2 + n_{-1}) - 3n_1 + 4m_1 \\ &> 0 \end{aligned}$$

Therefore $\gamma^2_{sdR}(G) > \frac{19n-12m}{10}$. Since G has no isolated vertex, we have

$$\gamma^2_{sdR}(G) > \frac{19n - 12m}{10} > \frac{5n - 6m}{2}.$$

Let $\gamma^2_{sdR}(G) = \frac{5n-6m}{2}$. Then all inequalities (1), (2) and (3) must be equalities. In particular, $n_1 = 0$ and $n_3 = n_{13}$, and so $V_{13} = V_3$ and $V = V_3 \cup V_{-1}$. Furthermore, $m = m_3 + |[V_{-1}, V_3]|$, $|[V_{-1}, V_3]| = n_{-1}$ and $m_3 = \frac{1}{6}(n_{-1} - n_3)$. This implies that for each vertex $v \in V_{-1}$ we have $d_{V_{-1}}(v) = 0$ and $d_{V_3}(v) = 1$, and so each vertex of V_{-1} is a leaf in G. Moreover for each vertex $v \in V_3$ we have $d_{V_{-1}}(v) = 3d_{V_3}(v) + 1$. Hence $G \in \mathcal{H}$.

On the other hand, let $G \in \mathcal{H}$. Then $G = L_t$ for some $t \geq 1$. Thus, G is obtained from a graph H of order t by adding $3d_H(v) + 1$ pendant edges to each vertex v of H. Let G have order n and size m. Then,

$$n = \sum_{v \in V(H)} (3d_H(v) + 2) = 6m(H) + 2n(H)$$

and

$$m = m(H) + \sum_{v \in V(H)} (3d_H(v) + 1) = 7m(H) + n(H).$$

Assigning to every vertex in $V(H)$ the weight 3 and to every vertex in $V(G) - V(H)$ the weight -1 produces an SDR2DF f of weight $\omega(f) = 3n(H) - (6m(H) + n(H)) = 2n(H) - 6m(H) = \frac{5n-6m}{2}$. Hence $\gamma^2_{sdR}(G) \leq \frac{5n-6m}{2}$. It follows that $\gamma^2_{sdR}(G) = \frac{5n-6m}{2}$ and this completes the proof. □

Theorem 3.2. *Let G be a graph of order $n \geq 3$. Then*

$$\gamma^2_{sdR}(G) \geq 4\sqrt{\frac{n+2}{3}} - n.$$

This bound is sharp for $DS_{4,4}$.

Proof. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^2_{sdR}(G)$-function. If $|V_{-1}| = 0$, then $\gamma^2_{sdR}(G) > n \geq 4\sqrt{\frac{n+2}{3}} - n$. Hence $V_{-1} \neq \emptyset$. We consider the following cases.

Case 1. $V_3 \neq \emptyset$.

Since each vertex in V'_{-1} is adjacent to at least one vertex in V_3, we conclude that at least one vertex v of V_3 is adjacent to at least $\frac{n'_{-1}}{n_3}$ vertices of V'_{-1}. Also, since each vertex in V''_{-1} is adjacent to at least two vertices in V_2, we conclude that at least one vertex u of V_2 is adjacent to at least $\frac{2n''_{-1}}{n_2}$ vertices of V''_{-1}. Then $2 \leq f[v] \leq 3n_3 + 2n_2 + n_1 - \frac{n'_{-1}}{n_3}$ which implies that

$$0 \leq 3n_3^2 + 2n_2n_3 + n_1n_3 - n'_{-1} - 2n_3. \tag{4}$$

Similarly, we have

$$0 \leq 3n_3n_2 + 2n_2^2 + n_1n_2 - 2n''_{-1} - 2n_2. \tag{5}$$

By multiplying the inequality (4) by 2 and summing it with the inequality (5), we obtain

$$0 \leq 6n_3^2 + 2n_2^2 + 7n_2n_3 + 2n_1n_3 + n_1n_2 - 2n'_{-1} - 2n''_{-1} - 2n_2 - 4n_3.$$

Since $n = n_3 + n_2 + n_1 + n_{-1}$, we have

$$0 \leq 6n_3^2 + 2n_2^2 + 7n_2n_3 + 2n_1n_3 + n_1n_2 + 2n_1 - 2n_3 - 2n,$$

equivalently

$$\begin{aligned}
0 &\leq 16n_3^2 + \frac{16}{3}n_2^2 + \frac{56}{3}n_2n_3 + \frac{16}{3}n_1n_3 + \frac{16}{6}n_1n_2 + \frac{16}{3}n_1 - \frac{16}{3}(n+n_3) \\
&\leq 16n_3^2 + 9n_2^2 + 4n_1^2 + 24n_2n_3 + 16n_1n_3 + 12n_1n_2 - \frac{16}{3}(n+n_3) \\
&= (4n_3 + 3n_2 + 2n_1)^2 - \frac{16}{3}(n+n_3)
\end{aligned}$$

which implies that $4\sqrt{\frac{n+n_3}{3}} \leq 4n_3 + 3n_2 + 2n_1$. If $n_3 \geq 2$, then $4\sqrt{\frac{n+2}{3}} \leq 4n_3 + 3n_2 + 2n_1$. Hence let $n_3 = 1$. Then

$$\begin{aligned}
0 &\leq 16 + \frac{16}{3}n_2^2 + \frac{56}{3}n_2 + \frac{32}{3}n_1 + \frac{16}{6}n_1n_2 - \frac{16}{3}(n+1) \\
&\leq 16 + 9n_2^2 + 4n_1^2 + 24n_2 + 16n_1 + 12n_1n_2 - \frac{16}{3}(n+1) - \frac{16}{3}(n_1+n_2) \\
&= (4n_3 + 3n_2 + 2n_1)^2 - \frac{16}{3}(n + n_1 + n_2 + 1)
\end{aligned}$$

which implies that $4\sqrt{\frac{n+1+n_1+n_2}{3}} \leq 4n_3 + 3n_2 + 2n_1$. Since $n \geq 3$, we conclude that $n_1 + n_2 \geq 1$, and so

$$4\sqrt{\frac{n+2}{3}} \leq 4n_3 + 3n_2 + 2n_1.$$

Therefore

$$\begin{aligned}
\gamma_{sdR}(G) &= 3n_3 + 2n_2 + n_1 - n_{-1} \\
&= 4n_3 + 3n_2 + 2n_1 - n \\
&\geq 4\sqrt{\frac{n+2}{3}} - n.
\end{aligned}$$

Case 2. $V_3 = \emptyset$.
Since $V_{-1} \neq \emptyset$, we conclude that $V_2 \neq \emptyset$. As in Case 1, at least one vertex u of V_2 is adjacent to at least $\frac{2n_{-1}}{n_2}$ vertices of V_{-1}. Then $2 \leq f[u] \leq 2n_2 + n_1 - \frac{2n_{-1}}{n_2}$ which implies that

$$0 \leq 2n_2^2 + n_1n_2 - 2n_{-1} - 2n_2.$$

Since $n = n_2 + n_1 + n_{-1}$, we have

$$0 \leq 2n_2^2 + n_1n_2 + 2n_1 - 2n,$$

equivalently

$$\begin{aligned}
0 &\leq \frac{16}{3}n_2^2 + \frac{16}{6}n_1n_2 + \frac{16}{3}n_1 - \frac{16}{3}n \\
&\leq 9n_2^2 + 4n_1^2 + 12n_1n_2 - \frac{16}{3}n - 3n_2^2 \\
&= (3n_2 + 2n_1)^2 - \frac{16}{3}n - 3n_2^2.
\end{aligned}$$

Since $n_2 \geq 2$, we have $4\sqrt{\frac{n+2}{3}} < 3n_2 + 2n_1$. Therefore

$$\begin{aligned}\gamma^2_{sdR}(G) &= 2n_2 + n_1 - n_{-1}\\ &= 3n_2 + 2n_1 - n\\ &\geq 4\sqrt{\frac{n+2}{3}} - n,\end{aligned}$$

and this complete the proof. □

Next we establish lower and upper bounds on $\gamma^k_{sdR}(G)$ where G is a cubic graph and $k \leq 5$. We shall need the following result due to Favaron [8].

Theorem 3.3. *If G is a connected cubic graph G of order n, then $\rho(G) \geq \frac{n}{8}$, unless G is the Petersen graph in which case $\rho(G) = \frac{(n-2)}{8} = 1$.*

Theorem 3.4. *Let G be a connected cubic graph of order n. For $k \leq 5$,*

$$\frac{kn}{4} \leq \gamma^k_{sdR} \leq \frac{13n}{8}.$$

Proof. The lower bound follow from Proposition 2.2. Now, we prove the upper bound. Let S be a maximum 2-packing in G, and so $|S| = \rho(G)$. If G is the Petersen graph, then $n = 10$ and $\rho(G) = 1$. Consider the labeling of the Petersen graph in Figure 1.

Then the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(x_i) = 2$ for $1 \leq i \leq 5$ and $f(x_i) = 1$ for $6 \leq i \leq 10$, is an SDRkDF on G of weight 15. Hence $\gamma^k_{sdR}(G) \leq 15 < \frac{13n}{8}$.

Now, assume that G is not a Petersen graph. Then the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(x) = -1$ for $x \in S$ and $f(x) = 2$ otherwise. Since for each vertex $v \in V(G)$ we have $|N[v] \cap S| \leq 1$, we conclude that $f[v] \geq 5$. Hence f is an SDRkDF on G of weight

$$\omega(f) = 2(n - |S|) - |S| = 2n - 3|S| = 2n - 3\rho.$$

By Theorem 3.3, we have

$$\gamma^k_{sdR}(G) \leq 2n - 3\rho \leq 2n - 3\frac{n}{8} = \frac{13n}{8}.$$

□

To see that the lower bound presented in Theorem 3.4 is sharp, consider a cycle $C_{3t} : v_1v_2 \ldots v_{3t}v_1$, where $t \geq 1$, add t new vertices $x_1, x_2, \ldots, x_t$ and join x_i to the three vertices $v_{3i-2}, v_{3i-1}, v_{3i}$ for $i = 1, 2, \ldots, t$. Let G denote the resulting cubic graph of order $n = 4t$. We have following sharp examples.

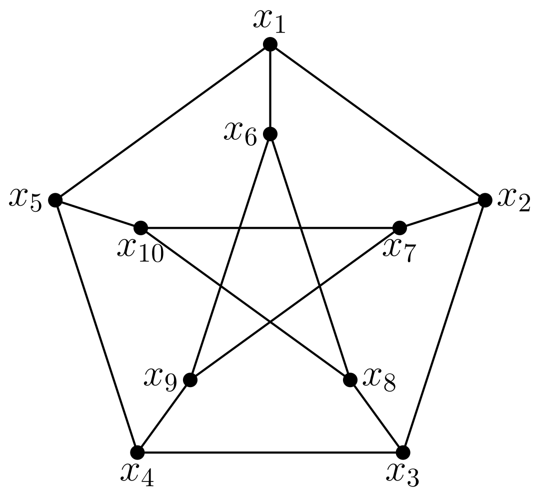

Figure 1: A labeling of the Petersen graph

- If $k = 2$, then the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(x_i) = f(v_{3i-2}) = -1$ for $1 \leq i \leq t$ and $f(x) = 2$ otherwise, is an SDR2DF on G of weight $2t = \frac{n}{2}$ and so $\gamma^2_{sdR}(G) \leq \frac{n}{2}$. Consequently, $\gamma^2_{sdR}(G) = \frac{n}{2}$.
- If $k = 3$, then the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(v_{3i}) = 3, f(v_{3i-2}) = 2$ and $f(v_{3i-1}) = f(x_i) = -1$ for $1 \leq i \leq t$, is an SDR3DF on G of weight $3t = \frac{3n}{4}$ and so $\gamma^3_{sdR}(G) \leq \frac{3n}{4}$. Consequently, $\gamma^3_{sdR}(G) = \frac{3n}{4}$.
- If $k = 4$, then the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(v_{3i-1}) = 3$, $f(x_i) = -1$ for $1 \leq i \leq t$ and $f(x) = 1$ otherwise, is an SDR4DF on G of weight $4t = n$ and so $\gamma^4_{sdR}(G) \leq n$. Consequently, $\gamma^2_{sdR}(G) = n$.
- If $k = 5$, then the function $f : V(G) \to \{-1, 1, 2, 3\}$ defined by $f(v_{3i-1}) = 3, f(v_{3i}) = 2, f(v_{3i-2}) = 1$ and $f(x_i) = -1$ for $1 \leq i \leq t$, is an SDR5DF on G of weight $5t = \frac{5n}{4}$ and so $\gamma^3_{sdR}(G) \leq \frac{5n}{4}$. Consequently, $\gamma^3_{sdR}(G) = \frac{5n}{4}$.

We believe the upper bound of Theorem 3.4 is not best possible and pose the following problem.

Problem 3.1. *Is it true that if G is a cubic graph of order n, then $\gamma^2_{sdR}(G) \leq n$.*

If this problem is true, then the bound is achieved, for example, by $K_{3,3}$.

4 Some classes of graphs

Ahangar et al. [3] determined the signed double domination number for complete bipartite graphs and cycles. In this section, we determine the signed double k-Roman domination number of some classes of graphs including complete bipartite graphs and cycles.

4.1 Complete bipartite graphs

If $k \geq 2$, $n \geq \lfloor \frac{k}{2} \rfloor$ are integers and V is a set of size n, then let $f_n^k : V \to \{-1, 1, 2, 3\}$ be the function defined in the end of section 1.

Proposition 4.1. *For $2 \leq m \leq n$,*

$$\gamma_{sdR}^2(K_{m,n}) = \begin{cases} 4 & \text{if } m = 2 \\ 6 & \text{if } m \geq 3. \end{cases}$$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{m,n}$. The result is immediate for $m = n = 2$. Assume that $n \geq 3$.

First let $m = 2$. Define the function $f : V(K_{2,n}) \to \{-1, 1, 2, 3\}$ by $f(x_1) = f(x_2) = 2$, $f(y_1) = 2, f(y_2) = -1$ and $f(y_i) = (-1)^i$ for $3 \leq i \leq n$, when n is odd, and by $f(x_1) = f(x_2) = 2$ and $f(y_i) = (-1)^i$ for $1 \leq i \leq n$ when n is even. It is clear that f is an SDR2DF of $K_{2,n}$ with weight 4 and hence $\gamma_{sdR}^2(K_{2,n}) \leq 4$.
Now, we show that $\gamma_{sdR}^2(K_{2,n}) \geq 4$. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}^2(K_{2,n})$-function. Assume without loss of generality that $f(x_2) \geq f(x_1)$. Since $f[y_1] \geq 2$, we must have $f(x_2) \geq 1$. If $f(x_1) + f(x_2) \leq 2$, then $f(y_i) \geq 1$ for each i and since each vertex with label -1 must have a neighbor with label 3 or two neighbors with label 2, we have $\gamma_{sdR}^2(K_{2,n}) \geq n+2 > 4$. Suppose $f(x_1) + f(x_2) \geq 3$. It follows that $f(x_2) \geq 2$. Hence $\gamma_{sdR}^2(K_{2,n}) = f(x_2) + f[x_1] \geq 2 + f[x_1] \geq 4$ as desired. Therefore $\gamma_{sdR}^2(K_{2,n}) = 4$.

Now let $m = 3$. Define the function $f : V(K_{3,n}) \to \{-1, 1, 2, 3\}$ by $f(x_1) = f(x_2) = f(x_3) = 2$, $f(y_1) = 2, f(y_2) = -1$ and $f(y_i) = (-1)^i$ for $3 \leq i \leq n$, when n is odd, and by $f(x_1) = f(x_2) = f(x_3) = 2$ and $f(y_i) = (-1)^i$ for $1 \leq i \leq n$ when n is even. Clearly, f is an SDR2DF of $K_{2,n}$ of weight 6 and hence $\gamma_{sdR}^2(K_{2,n}) \leq 6$.
To prove the inverse inequality, assume $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma_{sdR}^2(K_{3,n})$-function. Suppose, without loss of generality, that $f(x_3) \geq f(x_2) \geq f(x_1)$. Since $f[y_1] \geq 2$, we deduce that $\sum_{i=1}^3 f(x_i) \geq -1$. If $\sum_{i=1}^3 f(x_i) = -1$, then $f(y_i) = 3$ for each i and this implies that $\gamma_{sdR}^2(K_{3,n}) = \sum_{i=1}^3 f(x_i) + \sum_{i=1}^n f(y_i) \geq 3n - 1 > 6$. If $\sum_{i=1}^3 f(x_i) = 0$, then $f(y_i) \geq 2$ for each i and so $\gamma_{sdR}^2(K_{3,n}) = \sum_{i=1}^3 f(x_i) + \sum_{i=1}^n f(y_i) \geq 2n \geq 6$. If $\sum_{i=1}^3 f(x_i) = 1$ or 2, then $f(y_i) \geq 1$ for all i and $f(x_1) = -1$. Since any vertex with label -1 must have a neighbor with label 3 or two neighbors with label 2, we conclude that either $f(y_i) = 3$ for some i or $f(y_i), f(y_j) = 2$ for some i, j. Thus $\gamma_{sdR}^2(K_{3,n}) = \sum_{i=1}^3 f(x_i) + \sum_{i=1}^n f(y_i) \geq n + 3 \geq 6$. Let $f(x_1) + f(x_2) + f(x_3) \geq 3$. If $f(x_1) = -1$, then $f[x_1] \geq 2$ yields $\sum_{i=1}^n f(y_i) \geq 3$ and so $\gamma_{sdR}^2(K_{3,n}) = \sum_{i=1}^3 f(x_i) + \sum_{i=1}^n f(y_i) \geq 6$. Let $f(x_1) \geq 1$. If $f(x_2) + f(x_3) \geq 4$, then we have $\gamma_{sdR}^2(K_{3,n}) \geq f(x_2) + f(x_3) + f[x_1] \geq 6$. Suppose $f(x_2) + f(x_3) \leq 3$. It follows that $f(x_1) = f(x_2) = 1$ and $f(x_3) \leq 2$. Since any vertex with label -1 must have a neighbor with label 3 or two neighbors with label 2, we deduce that $f(y_i) \geq 1$ for each i and so $\gamma_{sdR}^2(K_{3,n}) \geq n + 3 \geq 6$. Thus $\gamma_{sdR}^2(K_{3,n}) = 6$.

Finally, let $m \geq 4$. To show that $\gamma_{sdR}^2(K_{m,n}) \geq 6$, let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}^2(K_{m,n})$-function. Assume without loss of generality that $f(x_m) \geq \cdots \geq f(x_1)$. If $V_{-1} = \emptyset$, then the result is trivial. Let $V_{-1} \neq \emptyset$. If $V_{-1} \cap X = \emptyset$ (the case $V_{-1} \cap Y = \emptyset$ is similar), then since any vertex with label -1 must have a neighbor with label 3 or two neighbors with label 2, we have $f(x_m) = 3$ or $f(x_m) = f(x_{m-1}) = 2$ and

so $\gamma_{sdR}(K_{m,n}) = \sum_{i=1}^{m-1} f(x_i) + f[x_m] \geq m + 2 \geq 6$. Assume that $x_1 \in V_{-1} \cap X$ and $y_1 \in V_{-1} \cap Y$. It follows from $f[x_1] \geq 2$ and $f[y_1] \geq 2$ that $\sum_{i=1}^{n} f(y_i) \geq 3$ and $\sum_{i=1}^{m} f(x_i) \geq 3$. Hence $\gamma^2_{sdR}(K_{m,n}) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 6$. Thus $\gamma^2_{sdR}(K_{m,n}) \geq 6$.
To prove the inverse inequality, define the functions $f, g, h : V(K_{m,n}) \to \{-1, 1, 2, 3\}$ as follows:
If m, n are odd, then let $f(x_1) = f(y_1) = 3$, $f(x_i) = (-1)^{i+1}$ for $2 \leq i \leq m$ and $f(y_i) = (-1)^{i+1}$ for $2 \leq i \leq n$. If m, n are even, then let $g(x_1) = g(y_1) = 3$, $g(x_2) = g(y_2) = 2$, $g(x_i) = g(y_i) = -1$ for $3 \leq i \leq 4$, $g(x_i) = (-1)^i$ for $5 \leq i \leq m$ and $g(y_i) = (-1)^i$ for $5 \leq i \leq n$. If m is odd and n is even (the case m is even and n is odd), then let $h(x) = f(x)$ if $x \in X$ and $h(x) = g(x)$ if $x \in Y$. Clearly, these function are SDR2DF of weight 6, and so $\gamma^2_{sdR}(K_{m,n}) \leq 6$. Thus $\gamma^2_{sdR}(K_{m,n}) = 6$ and the proof is complete. □

Proposition 4.2. *Let $k \geq 3$ and $n \geq m \geq k + 1$ be integers. Then $\gamma^k_{sdR}(K_{m,n}) = 2k + 2$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{m,n}$. First we show that $\gamma^k_{sdR}(K_{m,n}) \geq 2k + 2$. Let $f : V(K_{m,n}) \to \{-1, 1, 2, 3\}$ be an SDRkDF. Assume without loss of generality that $f(x_m) \geq \ldots \geq f(x_1)$. If $f(u) \geq 1$ for each $u \in V(K_{m,n})$, then $\gamma^k_{sdR}(K_{m,n}) = \omega(f) \geq 2m \geq 2k + 2$. Suppose $V_{-1} \neq \emptyset$. If $x_i \in V_{-1} \cap X$ and $y_j \in V_{-1} \cap Y$, then it follows from $f[x_i] \geq k$ and $f[y_j] \geq k$ that $\sum_{i=1}^{n} f(y_i) \geq k + 1$ and $\sum_{i=1}^{m} f(x_i) \geq k + 1$ yielding $\gamma^k_{sdR}(K_{m,n}) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 2k + 2$. Let $X \cap V_{-1} = \emptyset$ (the case $Y \cap V_{-1} = \emptyset$ is similar). Since any vertex with label -1 must have a neighbor with label 3 or two neighbors with label 2, we must have $f(x_i) = 3$ for some i or $f(x_i) = f(x_j) = 2$ for some i, j implying that $\sum_{i=1}^{m} f(x_i) \geq m + 2$. Since $f[x_1] \geq k$, we have $\sum_{i=1}^{n} f(y_i) \geq k - f(x_1)$. If $f(x_1) = 1$, then $\gamma^k_{sdR}(K_{m,n}) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq m + 2 + k - 1 \geq 2k + 2$. If $f(x_1) \geq 2$, then $\sum_{i=1}^{m} f(x_i) \geq 2m$ and so $\gamma^k_{sdR}(K_{m,n}) = \sum_{i=1}^{m} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 2m + k - 3 \geq 2k + 2$. Thus $\gamma^k_{sdR}(K_{m,n}) \geq 2k + 2$.

To prove the converse inequality, define the function $f : V(K_{m,n}) \to \{-1, 1, 2, 3\}$ as follows: $f(x) = f^{k+1}_m(x)$ for $x \in X$ and $f(x) = f^{k+1}_n(x)$ for $x \in Y$. Clearly f is an SDRkDF on $K_{m,n}$ of weight $2k + 2$ and so $\gamma^k_{sdR}(K_{m,n}) \leq 2k + 2$. Thus $\gamma^k_{sdR}(K_{m,n}) = 2k + 2$. □

Proposition 4.3. *Let $k \geq 4$ and $n \geq k - 1$ be integers. Then $\gamma^k_{sdR}(K_{k-1,n}) = 2k$*

Proof. Let $X = \{x_1, x_2, \ldots, x_{k-1}\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartite sets of $K_{k-1,n}$. First we show that $\gamma^k_{sdR}(K_{k-1,n}) \leq 2k$. Define the function $f : V(K_{k-1,n}) \to \{-1, 1, 2, 3\}$ as follows: if $n = k - 1$, then let f assign 2 to x_1, y_1 and $+1$ to the remaining vertices, and if $n \geq k$, then let f assign 3 to x_1, $+1$ to the remaining vertices of X and $f(y) = f^{k-1}_n(y)$ for $y \in Y$. Clearly, f is an SDRkDF on $K_{k-1,n}$ of weight $2k$ and so $\gamma^k_{sdR}(K_{k-1,n}) \leq 2k$.

Next, we show that $\gamma^k_{sdR}(K_{k-1,n}) \geq 2k$. Assume $f = (V_{-1}, V_1, V_2, V_3)$ is a $\gamma^k_{sdR}(K_{k-1,n})$-function. If $V_{-1} = \emptyset$, then f must assign 2 to at least one vertex in X and one vertex in Y and this implies that $\gamma^k_{sdR}(K_{k-1,n}) \geq k + n + 1 \geq 2k$. Let $V_{-1} \neq \emptyset$.

Assume without loss of generality that $f(x_{k-1}) \geq \cdots \geq f(x_1)$. If $x_i \in V_{-1} \cap X$ and $y_j \in V_{-1} \cap Y$, then it follows from $f[x_i] \geq k$ and $f[y_j] \geq k$ that $\sum_{i=1}^{n} f(y_i) \geq k+1$ and $\sum_{i=1}^{k-1} f(x_i) \geq k+1$ implying that $\gamma_{sdR}^k(K_{k-1,n}) = \sum_{i=1}^{k-1} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 2k+2$. Let $X \cap V_{-1} = \emptyset$ (the case $Y \cap V_{-1} = \emptyset$ is similar). Since any vertex with label -1 must have a neighbor with label 3 or two neighbors with label 2, we must have $f(x_i) = 3$ for some i or $f(x_i) = f(x_j) = 2$ for some i, j implying that $\sum_{i=1}^{k-1} f(x_i) \geq k+1$. Since $f[x_1] \geq k$, we have $\sum_{i=1}^{n} f(y_i) \geq k - f(x_1)$. If $f(x_1) = 1$, then $\gamma_{sdR}^k(K_{k-1,n}) = \sum_{i=1}^{k-1} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq k+1+k-1 \geq 2k$. If $f(x_1) = 2$, then $\sum_{i=1}^{k-1} f(x_i) \geq 2k-2$ and so $\gamma_{sdR}^k(K_{k-1,n}) = \sum_{i=1}^{k-1} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 2k-2+k-2 > 2k$. Finally, if $f(x_1) = 3$, then $\sum_{i=1}^{k-1} f(x_i) \geq 3k-3$ and so $\gamma_{sdR}^k(K_{k-1,n}) = \sum_{i=1}^{k-1} f(x_i) + \sum_{i=1}^{n} f(y_i) \geq 4k-6 > 2k$. Thus $\gamma_{sdR}^k(K_{k-1,n}) \geq 2k$ implying that $\gamma_{sdR}^k(K_{k-1,n}) = 2k$. $\square$

The proof of next result is similar to the proof of Propositions 4.2 and 4.3 and therefore it is omitted.

Proposition 4.4. *(i) For $n \geq 2$, $\gamma_{sdR}^3(K_{2,n}) = 5$.*
(ii) For $n \geq k \geq 3$, $\gamma_{sdR}^k(K_{k,n}) = 2k+1$.

4.2 Cycles

Ahangar et al. [3] determined the signed double Roman domination number of cycles. In this section, we determine the signed double Roman k-domination number of cycles for $k = 2, 3, 4$.

Theorem 4.1. *For $n \geq 3$, $\gamma_{sdR}^2(C_n) = n$.*

Proof. Let $C_n = (v_1 v_2 \ldots v_n)$. Define $f : V(C_n) \to \{-1, 1, 2, 3\}$ by $f(v_{3i+1}) = f(v_{3i+3}) = 2$, $f(v_{3i+2}) = -1$ for $0 \leq i \leq \lfloor n/3 \rfloor - 1$ and $f(x) = 1$ otherwise. It is easy to see that f is an SDR2DF on C_n of weight n yielding $\gamma_{sdR}^2(C_n) \leq n$.

To prove the inverse inequality, we proceed by induction on n. The result is clear for $n = 3, 4, 5$. Let $n \geq 6$ and suppose the statement holds for all cycles of order less than n. Let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma_{sdR}^2(C_n)$-function. If $V_{-1} = \emptyset$, then clearly $\gamma_{sdR}^2(C_n) \geq n$. Let $V_{-1} \neq \emptyset$ and let $v_i \in V_{-1}$. By the definition, v_i must have at least two neighbors in V_2 or one neighbor in V_3.

Suppose first v_i has a neighbor in V_3. Assume without loss of generality that $f(v_{i+1}) = 3$. Since $f[v_i] \geq 2$ and $f[v_{i+1}] \geq 2$, we must have $f(v_{i-1}) \geq 1$ and $f(v_{i+2}) \geq 1$. Let $C_{n-3} = (C_n - \{v_i, v_{i+1}, v_{i+2}\}) + v_{i-1}v_{i+3}$. Clearly, the function $g : V(C_{n-3}) \to \{-1, 1, 2, 3\}$ defined by $g(v_{i-1}) = \max\{f(v_{i-1}), f(v_{i+2})\}$ and $g(x) = f(x)$ otherwise, is an SDR2DF of C_{n-3} of weight at most $\gamma_{sdR}^2(C_n) - 3$ and by the induction hypothesis we have

$$\gamma_{sdR}^2(C_n) = 3 + \omega(g) \geq 3 + (n-3) = n.$$

Now let v_i have two neighbors in V_2. Then $f(v_{i-1}) = f(v_{i+1}) = 2$. Since $f[v_{i-1}] \geq 2$ and $f[v_{i+1}] \geq 2$, we must have $f(v_{i-2}) \geq 1$ and $f(v_{i+2}) \geq 1$. If $f(v_{i-2}) \geq 2$ or

$f(v_{i+2}) \geq 2$, then let $C_{n-3} = (C_n - \{v_{i-1}, v_i, v_{i+1}\}) + v_{i-2}v_{i+2}$. Clearly, the function f, restricted to C_{n-3} is an SDR2DF of C_{n-3} of weight $\gamma^2_{sdR}(C_n) - 3$ and by the induction hypothesis we have

$$\gamma^2_{sdR}(C_n) = 3 + \omega(f|_{C_{n-3}}) \geq 3 + (n-3) = n + 2.$$

Assume that $f(v_{i-2}) = f(v_{i+2}) = 1$. If $f(v_{i+3}) = -1$, then we must have $f(v_{i+4}) = 3$ and the results follows as above. Let $f(v_{i+3}) \geq 1$ and let $C_{n-1} = (C_n - \{v_{i+2}\}) + v_{i+1}v_{i+3}$. Clearly, the function f, restricted to C_{n-1} is clearly an SDR2DF of C_{n-1} of weight $\gamma^2_{sdR}(C_n) - 1$ and by the induction hypothesis we have

$$\gamma^2_{sdR}(C_n) = 1 + \omega(f|_{C_{n-1}}) \geq 1 + (n-1) = n$$

and the proof is complete. □

Theorem 4.2. *For $n \geq 3$,*

$$\gamma^3_{sdR}(C_n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3, \\ n+1 & \text{if} \quad n \equiv 1 \pmod 3, \\ n+2 & \text{if} \quad n \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $C_n = (v_1v_2 \ldots v_n)$. Define $f : V(C_n) \to \{-1, 1, 2, 3\}$ by $f(v_{3i+1}) = f(v_{3i+2}) = 2, f(v_{3i+3}) = -1$ for $0 \leq i \leq \lfloor n/3 \rfloor - 1$ and $f(x) = 2$ otherwise. Clearly, f is an SDR3DF on C_n of weight $n + r$ where $n \equiv r \pmod 3$ and this implies that

$$\gamma^3_{sdR}(C_n) \leq \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3, \\ n+1 & \text{if} \quad n \equiv 1 \pmod 3, \\ n+2 & \text{if} \quad n \equiv 2 \pmod 3 \end{cases}$$

If $n \equiv 0 \pmod 3$, then it follows from Proposition 2.2 that $\gamma^3_{sdR}(C_n) = n$ in this case.

Let $n \equiv 1 \pmod 3$. To prove the inverse inequality, let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^3_{sdR}(C_n)$-function. Since $f[v_i] \geq 3$ for each i, we conclude that there is no i with $f(v_i) = f(v_{i+1}) = -1$ or $f(v_i) = f(v_{i+2}) = -1$. It follows that there are three consecutive vertices, say v_1, v_2, v_3, with positive weight. If $f(v_2) \geq 2$, then clearly $f[v_2] \geq 4$, and if $f(v_2) = 1$, then v_2 must have a neighbor in $V_2 \cup V_3$ yielding $f[v_2] \geq 4$. Therefore

$$3\gamma^3_{sdR}(C_n) = \sum_{i=1}^{n} \sum_{j=0}^{2} f(v_{i+j}) \geq 4 + \sum_{i=2}^{n} \sum_{j=0}^{2} f(v_{i+j}) \geq 3n + 1.$$

Since $\gamma^3_{sdR}(C_n)$ is an integer, we obtain $\gamma^3_{sdR}(C_n) \geq \lceil \frac{3n+1}{3} \rceil = n + 1$. Thus $\gamma^3_{sdR}(C_n) = n + 1$ in this case.

Let $n = 3t+2$ for some $t \geq 1$. To prove $\gamma^3_{sdR}(C_n) \geq n+2$, we proceed by induction on t. The result is clear for $t = 1$. Let $t \geq 2$ and assume that the statement is true for all cycles of order $3t' + 2$ where $t' < t$. Suppose $C_n = (v_1v_2 \ldots v_n)$ and let $f = (V_{-1}, V_1, V_2, V_3)$ be a $\gamma^3_{sdR}(C_n)$-function. If $V_{-1} = \emptyset$, then clearly $|V_2| + |V_3| \geq 2$

implying that $\gamma^3_{sdR}(C_n) \geq n+2$ as desired. Let $V_{-1} \neq \emptyset$ and let $v_i \in V_{-1}$. By the definition, v_i must have at least two neighbors in V_2 or one neighbor in V_3. If v_i has a neighbor in V_3, then as in the proof of Theorem 4.1, we can see that $\gamma^3_{sdR}(C_n) \geq n+2$.

Let v_i have two neighbors in V_2. Then $f(v_{i-1}) = f(v_{i+1}) = 2$. Since $f[v_{i-1}] \geq 3$ and $f[v_{i+1}] \geq 3$, we must have $f(v_{i-2}) \geq 2$ and $f(v_{i+1}) \geq 2$. Let $C_{n-3} = (C_n - \{v_{i-1}, v_i, v_{i+1}\}) + v_{i-2}v_{i+2}$. Clearly, the function f, restricted to C_{n-3} is an SDR3DF of weight at most $\gamma^3_{sdR}(C_n) - 3$ and by the induction hypothesis we obtain

$$\gamma^3_{sdR}(C_n) = 3 + \omega(f|_{C_{n-3}}) \geq 3 + (n-3+2) = n+2$$

and the proof is complete. □

Theorem 4.3. *For $n \geq 3$, $\gamma^4_{sdR}(C_n) = \lceil (4n)/3 \rceil$.*

Proof. Let $C_n = (v_1 v_2 \ldots v_n)$. Define $f : V(C_n) \to \{-1, 1, 2, 3\}$ by $f(v_{3i+1}) = 2, f(v_{3i+2}) = f(v_{3i+3}) = 1$ for $0 \leq i \leq \lfloor n/3 \rfloor - 1$ when $n \equiv 0 \pmod 3$, by $f(v_n) = 2, f(v_{3i+1}) = 2$, $f(v_{3i+2}) = f(v_{3i+3}) = 1$ for $0 \leq i \leq \lfloor n/3 \rfloor - 1$ when $n \equiv 1 \pmod 3$, and by $f(v_n) = 1, f(v_{n-1}) = 2$, $f(v_{3i+1}) = 2$, $f(v_{3i+2}) = f(v_{3i+3}) = 1$ for $0 \leq i \leq \lfloor n/3 \rfloor - 1$ when $n \equiv 2 \pmod 3$. Clearly, f is an SDR4DF on C_n of weight $\lceil (4n)/3 \rceil$ and so $\gamma^4_{sdR}(C_n) \leq \lceil (4n)/3 \rceil$.

On the other hand, since $\gamma^3_{sdR}(C_n)$ is an integer, we deduce from Proposition 2.2 that $\gamma^3_{sdR}(C_n) = \lceil (4n)/3 \rceil$. □

We conclude this paper with some open problems.

Problem 4.1. *Find upper bounds on $\gamma^k_{sdR}(G)$ in terms of order of G and k.*

Problem 4.2. *What can one say about the minimum and maximum values of $|V_{-1}|$, $|V_1|$, $|V_2|$ and $|V_3|$ for a γ^k_{sdR}-function $f = (V_{-1}, V_1, V_2, V_3)$ of a graph G ?*

The cartesian product $G = G_1 \times G_2$ of two disjoint graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2 v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1 v_1 \in E(G_1)$. The cartesian product of two paths is called a grid, the cartesian product a cycle and a path is called a cylinder and the cartesian product of two cycles is called a torus.

Problem 4.3. *Can one determine the signed double Roman k-domination of grids, cylinders or tori?*

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