STATMAN'S HIERARCHY THEOREM

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Abstract. In the Simply Typed λ-calculus [Hin97, BDS13] Statman investigates the reducibility relation $\leq_{\beta\eta}$ between types: for $A, B \in \mathbb{T}^0$, types freely generated using $\to$ and a single ground type 0, define $A \leq_{\beta\eta} B$ if there exists a λ-definable injection from the closed terms of type A into those of type B. Unexpectedly, the induced partial order is the *(linear) well-ordering (of order type)* $\omega + 4$, see [Sta80a, Sta80b, Sta81, BDS13].

In the proof a finer relation $\leq_h$ is used, where the above injection is required to be a Böhm transformation ([Bar84]), and an (a posteriori) coarser relation $\leq_{h^+}$, requiring a finite family of Böhm transformations that is jointly injective.

We present this result in a self-contained, syntactic, constructive and simplified manner. *En route* similar results for $\leq_h$ (order type $\omega + 5$) and $\leq_{h^+}$ (order type 8) are obtained. Five of the equivalence classes of $\leq_{h^+}$ correspond to canonical term models of Statman, one to the trivial term model collapsing all elements of the same type, and one does not even form a model by the lack of closed terms of many types, [BDS13].

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1. Hierarchy of types

We work in simply typed lambda calculus over a single base type 0. The set of open terms of (simple) type A is written $\Lambda(A)$, while the set of closed terms of type A is denoted by $\Lambda^\varepsilon(A)$ (for reasons which become clear in Section 2).

For types A, B one defines $A \leq_{\beta\eta} B$ if there is a closed term $\Phi\colon A \to B$ that is an injection on closed terms modulo $\beta\eta$-equality.

To get some feeling for the relation $\leq_{\beta\eta}$ we begin by observing

$$\begin{aligned}
B \to (A \to C) &\leq_{\beta\eta} A \to (B \to C) && \text{via } \lambda mab.\ mba \text{ (see Corollary 4.17);}\\
A \to C &\leq_{\beta\eta} A \to B \to C && \text{via } \lambda mab.\ ma \text{ (see Lemma 4.16);}\\
A &\leq_{\beta\eta} (A \to 0) \to 0 && \text{via } \lambda mf.\ fm \text{ (see Lemma 4.22);}\\
[0,0] \triangleq 0 \to 0 \to 0 &\not\leq_{\beta\eta} 0 \to 0 && \text{by counting closed inhabitants.}
\end{aligned}$$

Less intuitively clear is that for all simple types A over 0 one has

$$A \leq_{\beta\eta} [0,0] \to 0 \to 0 \qquad \text{(see Lemma 5.18).}$$

Also, one might ponder (writing $1 \triangleq 0 \to 0$) whether

$$\begin{aligned}
1 \to 1 \to 0 \to 0 &\leq_{\beta\eta} 1 \to 0 \to 0 && \text{(no);}\\
1 \to 1 \to 1 \to 0 \to 0 &\leq_{\beta\eta} 1 \to 1 \to 0 \to 0 && \text{(yes!);}\\
[0,0] \to 0 \to 0 &\leq_{\beta\eta} 1 \to 1 \to 0 \to 0 && \text{(no).}
\end{aligned}$$

The general problem whether $A \leq_{\beta\eta} B$ (for given types A and B) is solved by the Hierarchy Theorem (printed on page 4, due to Richard Statman [Sta80a, Sta80b]), which describes (among other things) the equivalence classes of $\leq_{\beta\eta}$ in terms of (relatively) simple syntactic properties.

We give a new proof, which is self-contained, syntactic and constructive. We assume only basic knowledge of the simply typed lambda calculus (long normal form, rank, ...), and recall the most important notions before using them. Roughly speaking, the proof is one long syntactic analysis of inhabitants of simple types and reductions between them; we make no use of term models and the like. The proof is constructive in the sense that we do not use the law of the excluded middle, and so one may easily ignore this feature of the proof (except perhaps when reading Theorem 1.8). For applications of the Hierarchy Theorem (and another proof), see Section 3.4 of [BDS13].

1.1. **Hierarchy Theorem.** To formulate the theorem we first recall some notions and notation from the simply typed lambda calculus, see Section 1.1 and Section 3.4 of [BDS13].

Definition 1.1.

(i) Let A be a type. The **components** of A are the unique types $A_1, \ldots, A_n$ such that
$$A \;=\; A_1 \to \cdots \to A_n \to 0.$$

(ii) Each type A has a **rank** denoted by $\mathrm{rk}\, A$; it is defined recursively by
$$\mathrm{rk}\, 0 \triangleq 0; \qquad \mathrm{rk}(A \to B) \triangleq \max\{\, \mathrm{rk}\, A + 1,\ \mathrm{rk}\, B \,\}.$$

(iii) A type $A \equiv A_1 \to \cdots \to A_n \to 0$ is **fat** when $n \geq 2$.

(iv) A type $A \equiv A_1 \to \cdots \to A_n \to 0$ is **large** if either A has a fat component A_i, or one of A's components $A_i \equiv A_{i1} \to \cdots \to A_{im} \to 0$ has a large component A_{ij}.

(v) A type which is not large, is called **small**.

(vi) Let A, B be types and k, n natural numbers. The following notation is used.
$$n+1 \triangleq n \to 0; \qquad \left[\begin{array}{l} A^0 \to B \triangleq B \\ A^{k+1} \to B \triangleq A \to A^k \to B. \end{array}\right.$$

Definition 1.2 (Reducibility relations)**.**
Let $A \equiv A_1 \to \cdots \to A_n \to 0$ and $B \equiv B_1 \to \cdots \to B_m \to 0$ be types.

(i) A **$\beta\eta$-reduces to** B, notation $A \leq_{\beta\eta} B$, if for some $R \in \Lambda^\varepsilon(A \to B)$
$$RM_1 =_{\beta\eta} RM_2 \implies M_1 =_{\beta\eta} M_2 \qquad (M_1, M_2 \in \Lambda^\varepsilon(A)\,).$$
This R is then called a **reducing term** from A to B.

(ii) A **head reduces to** B, notation $A \leq_h B$, if $A \leq_{\beta\eta} B$ with a reducing term of the form $R \equiv \lambda m^A b_1^{B_1} \cdots b_m^{B_m}.\ m\varrho_{a_1} \cdots \varrho_{a_n}$ where $\varrho_{a_i}{:}A_i$ are open terms with free variables from $b_1^{B_1}, \ldots, b_m^{B_m}$. We call a term of this form a *Böhm term*.

(iii) A **reduces multi-head to** B, notation $A \leq_{h^+} B$, provided there exist Böhm terms $R^{(1)}, \ldots, R^{(\ell)}$ which are *jointly injective*, that is, for $M_1, M_2 \in \Lambda^\varepsilon(A)$,
$$\forall i\,[\ \ R^{(i)}M_1 =_{\beta\eta} R^{(i)}M_2\ \ ] \quad \Longrightarrow \quad M_1 =_{\beta\eta} M_2.$$

Theorem 1.3 (Statman Hierarchy)**.** *The relations $\leq_{h^+}$, $\leq_{\beta\eta}$ and $\leq_h$ are increasingly fine.*[1] *Their equivalence classes are listed below vertically in ascending order. The types H_α in the last column (called **canonical types**) are representatives for the equivalence classes of $\leq_h$.*

$\mathbb{T}_5$	$\mathbb{B}_{\omega+3}$	$\mathbb{H}_{\omega+4}$	$H_{\omega+4} \triangleq (0 \to 0 \to 0) \to 0 \to 0$
$\mathbb{T}_4$	$\mathbb{B}_{\omega+2}$	$\mathbb{H}_{\omega+3}$	$H_{\omega+3} \triangleq 3 \to 0 \to 0$
$\mathbb{T}_3$	$\mathbb{B}_{\omega+1}$	$\mathbb{H}_{\omega+2}$	$H_{\omega+2} \triangleq 1 \to 1 \to 0 \to 0$
$\mathbb{T}_2$	$\mathbb{B}_\omega$	$\mathbb{H}_{\omega+1}$	$H_{\omega+1} \triangleq 2 \to 0$
		$\mathbb{H}_\omega$	$H_\omega \triangleq 1 \to 0 \to 0$
$\mathbb{T}_1$	$\mathbb{B}_k$	$\mathbb{H}_k$	$H_k \triangleq 0^k \to 0$
	$\mathbb{B}_2$	$\mathbb{H}_2$	$H_2 \triangleq 0^2 \to 0$
$\mathbb{T}_0$	$\mathbb{B}_1$	$\mathbb{H}_1$	$H_1 \triangleq 0 \to 0$
$\mathbb{T}_{-1}$	$\mathbb{B}_0$	$\mathbb{H}_0$	$H_0 \triangleq 0$
$\leq_{h^+}$	$\leq_{\beta\eta}$	$\leq_h$	

Moreover, the equivalence classes $\mathbb{H}_\alpha$ of $\leq_h$ have the following syntactic description, and the relations $\leq_{h^+}$, $\leq_{\beta\eta}$ and $\leq_h$ are (hence) decidable.

$$\begin{aligned}
\mathbb{H}_{\omega+4} &= \{\, A\colon A \textit{ is inhabited and large} \,\};\\
\mathbb{H}_{\omega+3} &= \{\, A\colon A \textit{ is inhabited, small and } \mathrm{rk}(A) > 3 \,\};\\
\mathbb{H}_{\omega+2} &= \{\, A;\ A \textit{ is inhabited, small, } \mathrm{rk}(A) \in \{2,3\}\\
&\qquad\qquad \textit{and } A \textit{ has at least two components of rank} \geq 1 \,\};\\
\mathbb{H}_{\omega+1} &= \{\, A\colon\ A \textit{ is inhabited, small, } \mathrm{rk}(A) = 3\\
&\qquad\qquad \textit{and } A \textit{ has exactly one component of rank} \geq 1 \,\};\\
\mathbb{H}_{\omega} &= \{\, A\colon\ A \textit{ is inhabited, small, } \mathrm{rk}(A) = 2\\
&\qquad\qquad \textit{and } A \textit{ has exactly one component of rank } 1 \,\};\\
\mathbb{H}_{k} &= \{\, A\colon\ A \textit{ is inhabited, small, } \mathrm{rk}(A) = 1\\
&\qquad\qquad \textit{and } A \textit{ has exactly } k \textit{ components of rank } 0 \,\};\\
\mathbb{H}_{0} &= \{\, A\colon A \textit{ is not inhabited} \,\}.
\end{aligned}$$

[1] *Viz.* $A \leq_{h^+} B \implies A \leq_{\beta\eta} B$ and $A \leq_{\beta\eta} B \implies A \leq_h B$ for all types A and B.

We give an overview of the proof of Theorem 1.3 in Subsection 1.4. To be able to do this, we first expose the precise relation between the syntactic structure of a type and the shape of its (long normal form) inhabitants in Subsection 1.2, and we examine the inhabitants of the canonical types H_0, H_1, ... in Subsection 1.3.

While technical details are unavoidable in a paper like this, we make them more palatable by introducing some syntactic sugar in Section 2. With it we can already prove the inequalities between the canonical types (such as $[0,0] \not\leq_h [0]$) in Section 3. We proceed by developing a general theory about reductions in Section 4 to establish the order type of $\leq_h$ in Section 5, and the order types of $\leq_{\beta\eta}$ and $\leq_{h^+}$ in Section 6.

Since we are in the fortunate position to have strong normalization, every term has a *long normal form* (lnf), which is the $\beta\eta^{-1}$-normal form. As default we will only consider terms in lnf. The few exceptions will not pose a problem to the reader.

1.2. **Syntactic structure and inhabitants.** Recall that for any type A, there are unique types $A_1, \ldots, A_n$ such that $A = A_1 \to \ldots \to A_n \to 0$. Hence it is natural to write

Definition 1.4. $[A_1, \ldots, A_n] \triangleq A_1 \to \cdots \to A_n \to 0$.

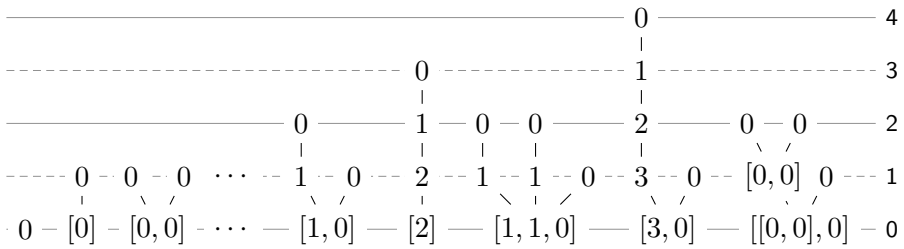
Observation 1.5. Any (lnf-)inhabitant M of a type A is of the form

$$\lambda a_1 \cdots a_n.\ b^B\, N_1 \cdots N_m.$$

Writing $A = [A_1, \ldots, A_n]$ and $B = [B_1, \ldots, B_m]$ we have that

(i) the types of the variables $a_1, \ldots, a_n$ are respectively $A_1, \ldots, A_n$ and
(ii) the types of the terms $N_1, \ldots, N_m$ are respectively $B_1, \ldots, B_m$.

Observation 1.6. We can write every type A using only the operation [ ] in a unique way. For example, $0 = [\,]$ and $n+1 = [n]$. In this way we can consider types to be finite trees. For instance, the canonical types are represented by the following trees.



From this we see that given a type A, the nodes on odd height of the tree A are the types of the variables which might occur in closed terms of type A, while the possible types of the subterms are those on even height. (E.g., in a closed term of type $[3,0]$ — such as $\lambda\Phi^3 c^0.\ \Phi\lambda f_1^1.\ f_1\Phi\lambda f_2^1.\ f_1 f_2\, c$ — the introduced variables are of type 3 and 1, while the subterms are of type 2 and 0.)

Observation 1.7. The syntactic properties mentioned in Theorem 1.3 are more easily defined and understood when considering a type to be a tree.

(i) The *rank* of a type A is its height as a tree. If the rank of A is restricted from above to, say 2, the variables occurring in a closed term M of type A are of rank 0 or 1 so that all the variables in M are introduced at the head, contrary to types like $[3,0]$.

(ii) A type A is *fat* if it has more than one component. Fat types are important, because a variable of fat type can be used to construct a pairing (see Lemma 4.42).

Moreover, A is *large* if A as tree has a fat type on odd height. One can show that if A is inhabited, then A is large if and only if there is closed inhabitant M of A which contains a (bound) variable of fat type.

In particular, if a type is *small* (= not large), then its inhabitants are "strings" of a variable followed by the mandatory abstractions

$$\lambda a_1^{A_1} \dots a_n^{A_n}.\ \underline{G\, \lambda b_1^{B_1} \dots b_m^{B_m}.}\ H\, \lambda c_1^{C_1} \dots c_k^{C_k}.\quad \cdots\quad .$$

1.3. **Inhabitants of the canonical types H_α.** By the preceding observations we can determine the (lnf-)inhabitants of a given type. As an example (and also since we will need them), we list by an iconic shorthand (explained by $\rightsquigarrow$) the inhabitants of the canonical types below. The verification is left to the reader.

To make terms more readable, we leave out parentheses. There is only one way to place parentheses to get a term obeying the typing rules. E.g., we read

$$\lambda f^1 g^1 c^0.\ fggfc \qquad \text{as} \qquad \lambda f^1 g^1 c^0.\ f(g(g(fc)));$$
$$\lambda b^{[0,0]} c^0.\ bcbcc \qquad \text{as} \qquad \lambda b^{[0,0]} c^0.\ (bc)((bc)c).$$

We abbreviate $[A, A, A, C, C]$ to $[A^3, C^2]$, etcetera.

1.3.1. $H_k = [0^k]$. The inhabitants (of $[0^k]$) are the projections on k elements:

$$U_i^k \ \rightsquigarrow\ \lambda x_1^0 \cdots x_k^0.\ x_i \qquad \text{for } 0 < i \leq k$$

1.3.2. $H_\omega = [1, 0]$. The inhabitants are the Church-numerals,

$$c_n \ \rightsquigarrow\ \lambda f^1 c^0.\ f^{(n)} c,$$

where $f^{(0)}c = c$ and $f^{(n+1)}c = f(f^{(n)}c)$. As a warm-up for what is coming, note that the inhabitants of $[1, 0]$ are produced by the following two-level grammar.

$$\lambda f^1 c^0.\ \mathsf{N} \quad \text{where} \quad \mathsf{N} ::= (f\mathsf{N}) \mid c$$

1.3.3. $H_{\omega+1} = [2]$. An inhabitant can be identified by a pair of natural numbers

$$\langle i, j \rangle \ \rightsquigarrow\ \lambda F^2.\ F\lambda x_1^0.\ \cdots F\, \lambda x_i^0.\ x_j \qquad \text{where } j \leq i.$$

These terms are produced by the following grammar.

$$\lambda F^2.\ \mathsf{P}_0 \quad \text{where} \quad \mathsf{P}_n ::= (F\lambda x_{n+1}^0.\ \mathsf{P}_{n+1}) \mid x_1 \mid \cdots \mid x_n$$

1.3.4. $H_{\omega+2} = [1, 1, 0]$. The inhabitants are essentially 'words over a two element alphabet',

$$\lambda f^1 g^1 c^0.\ \mathsf{W} \quad \text{where} \quad \mathsf{W} ::= (f\mathsf{W}) \mid (g\mathsf{W}) \mid c.$$

Hence we use words over $\{f, g\}$ as shorthands. For instance,

$$ffggf \ \rightsquigarrow\ \lambda f^1 g^1 c^0.\ ffggfc.$$

1.3.5. $H_{\omega+3} = [3, 0]$. The inhabitants are produced by the following grammar.

$$\lambda\Phi^3 c^0.\ \mathsf{M}_1 \quad \text{where} \quad \begin{aligned} \mathsf{M}_n &::= (\Phi\, \lambda f_n^1.\ \mathsf{W}_n) \mid c \\ \mathsf{W}_n &::= (f_1 \mathsf{W}_n) \mid \cdots \mid (f_n \mathsf{W}_n) \mid \mathsf{M}_{n+1} \end{aligned}$$

By replacing "$\Phi(\lambda f_i^1.\ \cdots)$" with "/", hiding "$\lambda\Phi^3 c^0.$ " and hiding the "c" at the end, we obtain a shorthand for the inhabitants of $[3, 0]$. For instance,

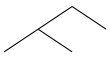
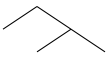
$$/1//23 \ \rightsquigarrow\ \lambda\Phi^3 c^0.\ \Phi\lambda f_1^1.\ f_1\Phi\lambda f_2^1.\ \Phi\lambda f_3.\ f_2 f_3 c.$$

So we identify an inhabitant of $[3, 0]$ with a list of words $w_1, \ldots, w_n$ with $w_i \in \{1, \ldots, i\}^*$.

1.3.6. $H_{\omega+4} = [[0, 0], 0]$. The inhabitants are (like) binary trees:

$$\lambda b^{[0,0]} c^0.\ \mathsf{T} \quad \text{where} \quad \mathsf{T} ::= (b\mathsf{T}\mathsf{T}) \mid c.$$

We will denote them as such. For instance,

 $\rightsquigarrow\ \lambda b^{[0,0]} c^0.\ bbccc$ and  $\rightsquigarrow\ \lambda b^{[0,0]} c^0.\ bcbcc.$

1.4. **Structure of the proof.** In this Subsection, we present the proof of the Hierarchy Theorem. We delegate most of the work to the remainder of this article by using statements proved later on. What is left is the compact skeleton of the proof.

Proof of Theorem 1.3. We need to prove the following.

(I) The relations $\leq_h$, $\leq_{\beta\eta}$ and $\leq_{h^+}$ are as displayed on page 4.
(II) The relations $\leq_h$, $\leq_{\beta\eta}$ and $\leq_{h^+}$ are decidable.

Concerning (I). We first consider the relation $\leq_h$. Let the sets $\mathbb{H}_\alpha$ be defined as on page 4. One easily verifies that the $\mathbb{H}_\alpha$ form a partition of $\mathbb{T}^0$, and that $H_\alpha \in \mathbb{H}_\alpha$ for all α.

To show that $\leq_h$ is of the form as on page 4 it suffices to show that for all $A, B \in \mathbb{T}^0$ and $\alpha, \beta \in \omega + 5$ with $A \in \mathbb{H}_\alpha$ and $B \in \mathbb{H}_\beta$, we have that

$$A \leq_h B \quad \Longleftrightarrow \quad H_\alpha \leq_h H_\beta \quad \Longleftrightarrow \quad \alpha \leq \beta \tag{1.1}$$

For this we use the following four facts proved later on.

(i) $A \in \mathbb{H}_\alpha \implies H_\alpha \leq_h A$ (see **Subsection 5.1**).
(ii) $A \in \mathbb{H}_\alpha \implies A \leq_h H_\alpha$ (see **Subsection 5.2**).
(iii) $\alpha \leq \beta \implies H_\alpha \leq_h H_\beta$ (see **Subsection 5.3**).
(iv) $\alpha \not\leq \beta \implies H_\alpha \not\leq_h H_\beta$ (see **Section 3**).

Before we prove Statement (1.1) let us spend some words on fact (iv). In Section 3 we do not directly prove that $\alpha \not\leq \beta \implies H_\alpha \not\leq_h H_\beta$. Instead we show the inequalities listed below in Statement (1.2) (writing $A \not\leq_{h,\beta\eta} B$ for $A \not\leq_{\beta\eta} B$ & $A \not\leq_h B$, etcetera). Together with fact (iii), this is sufficient to establish fact (iv). Indeed suppose that $\alpha \not\leq \beta$ and $H_\alpha \leq_h H_\beta$ for some α and β in order to obtain a contradiction. Then $\beta < \alpha$, so $\beta + 1 \leq \alpha$. Thus

$$H_{\beta+1} \ \leq_h \ H_\alpha \ \leq_h \ H_\beta$$

by fact (iii). This contradicts the inequality $H_{\beta+1} \not\leq_h H_\beta$ from Statement (1.2).

It is interesting to note that we will prove the inequalities from Statement (1.2) of the form $H_\beta \not\leq_{h,\beta\eta,h^+} H_\alpha$ (except one) by showing that there are distinct terms $N_1, N_2 \in \Lambda^\varepsilon(H_\beta)$ such that $R\,N_1 =_{\beta\eta} R\,N_2$ for all $R \in \Lambda^\varepsilon(A \to B)$ (see Lemma 3.2). These terms N_1, N_2 are listed on the right in Statement (1.2) using the notation from Subsection 1.3.

$$
\begin{array}{rlll}
[[0,0],0] & \not\leq_{h,\beta\eta,h^+} & [3,0] & N_1 = \quad N_2 = \\
[3,0] & \not\leq_{h,\beta\eta,h^+} & [1,1,0] & N_1 = /1/21 \quad N_2 = /1/22 \\
[1,1,0] & \not\leq_{h,\beta\eta,h^+} & [2] & N_1 = fgfg \quad N_2 = fggf \\
[2] & \not\leq_{h} & [1,0] & \\
[1,0] & \not\leq_{h,\beta\eta,h^+} & [0^{k+2}] & N_1 = c_1 \quad N_2 = c_2 \\
[0^{k+2}] & \not\leq_{h,\beta\eta} & [0^{k+1}] & \\
[0^{k+1}] & \not\leq_{h,\beta\eta,h^+} & 0 &
\end{array}
\tag{1.2}
$$

Let us prove Statement (1.1). The first equivalence follows from facts (i) and (ii), the second equivalence follows directly from facts (iii) and (iv).

We now turn to the order type of the reducibility relation $\leq_{\beta\eta}$. To show that $\leq_{\beta\eta}$ is of the form as depicted on page 4, we need to prove that

$$A \leq_{\beta\eta} B \qquad \Longleftrightarrow \qquad \alpha \leq \beta \quad \text{or} \quad \alpha, \beta \in \{\omega, \omega+1\} \tag{1.3}$$

for all $\alpha, \beta \in \omega + 5$ and all $A \in \mathbb{H}_\alpha$ and $B \in \mathbb{H}_\beta$. Note that

$$A \leq_h B \quad \Longrightarrow \quad A \leq_{\beta\eta} B \qquad \text{for all types } A \text{ and } B. \tag{1.4}$$

Hence $A \sim_{\beta\eta} H_\alpha$ for all $A \in \mathbb{H}_\alpha$, since $A \sim_h H_\alpha$ for $A \in \mathbb{H}_\alpha$ by facts (i) & (ii). So to prove Statement (1.3), it suffices to show that

$$H_\alpha \leq_{\beta\eta} H_\beta \qquad \Longleftrightarrow \qquad \alpha \leq \beta \quad \text{or} \quad \alpha, \beta \in \{\omega, \omega+1\}. \tag{1.5}$$

The implication "$\Longleftarrow$" follows from Statements (1.1), Statement (1.4) and

$$H_{\omega+1} \ \leq_{\beta\eta} \ H_\omega \qquad \text{(see } \textbf{Subsection 6.2}\text{)}.$$

Concerning "$\Longrightarrow$". Let $\alpha, \beta \in \omega + 5$ be given. Suppose that $H_\alpha \leq_{\beta\eta} H_\beta$. Then since $\leq$ and $=$ on $\omega + 5$ are decidable, it suffices to show that the negation of the right-hand side of Statement (1.5) leads to a contradiction. Suppose that $\beta < \alpha$ and not $\alpha, \beta \in \{\omega, \omega + 1\}$. Then $\beta \leq \gamma < \gamma + 1 \leq \alpha$ for some $\gamma \in \omega + 5$ with $\gamma \neq \omega$. (Pick $\gamma = \beta$ if $\beta \neq \omega$, or pick $\gamma = \omega + 1$ otherwise.) Then $H_{\gamma+1} \not\leq_{\beta\eta} H_\gamma$ by Statement (1.2), but we also have that $H_{\gamma+1} \leq_{\beta\eta} H_\alpha \leq_{\beta\eta} H_\beta \leq_{\beta\eta} H_\gamma$, a contradiction. We have proven Statement (1.3).

We continue with the order type of $\leq_{h^+}$. We need to prove that

$$
\begin{array}{rlll}
A \leq_{h^+} B & \Longleftrightarrow & \alpha \leq \beta & \text{or} \quad \alpha, \beta \in \{\omega, \omega+1\} \\
& & & \text{or} \quad \alpha, \beta \in \{2, \ldots, \omega\}
\end{array}
\tag{1.6}
$$

for all $\alpha, \beta \in \omega + 5$ and all $A \in \mathbb{H}_\alpha$ and $B \in \mathbb{H}_\beta$. Again, we have

$$A \leq_h B \quad \Longrightarrow \quad A \leq_{h^+} B \qquad \text{for all types } A \text{ and } B, \tag{1.7}$$

and $A \sim_{h^+} H_\alpha$ for all $A \in \mathbb{H}_\alpha$. So it suffices to show that

$$
\begin{array}{rlll}
H_\alpha \leq_{h^+} H_\beta & \Longleftrightarrow & \alpha \leq \beta & \text{or} \quad \alpha, \beta \in \{\omega, \omega+1\} \\
& & & \text{or} \quad \alpha, \beta \in \{2, \ldots, \omega\}.
\end{array}
\tag{1.8}
$$

The implication "$\Longleftarrow$" follows from Statement (1.1), and Statement (1.7) and

$$H_{\omega+1} \leq_{h^+} H_\omega \qquad \text{and} \qquad H_{k+1} \leq_{h^+} H_k \quad (k \geq 2) \qquad \text{(see } \textbf{Subsection 6.1}\text{)}.$$

The implication "$\Longrightarrow$" can be proven using the inequalities of Stat. (1.2) in a similar fashion as the implication "$\Longrightarrow$" of Stat. (1.5) was proven above. We leave this to the reader.

Concerning (II). To show that the reducibility relations $\leq_h$, $\leq_{\beta\eta}$ and $\leq_{h^+}$ are decidable, we prove that for every type A an $\alpha \in \omega + 5$ can be computed with $A \in \mathbb{H}_\alpha$. (This is sufficient because if $A \in \mathbb{H}_\alpha$ and $B \in \mathbb{H}_\beta$ then $A \leq_h B$ can be decided using Statement (1.1); $A \leq_{\beta\eta} B$ using Statement (1.3); $A \leq_{h^+} B$ using Statement (1.6).) Of course, algorithms to determine the rank of a type, the number of components and whether the type is large or small are defined easily enough; the difficulty here is how to decide whether a given type is inhabited.

By Proposition 2.4.4 of [BDS13] (which is proven using the law of the excluded middle) a type $A \equiv [A_1, \ldots, A_n]$ is inhabited iff A_i is uninhabited for some i. From this fact a recursive algorithm to determine whether a type A is inhabited is easily concocted. Since we want constructive proof of the Hierarchy Theorem, we have formulated and proven a constructive variant of Proposition 2.4.4 of [BDS13], see Theorem 1.8 below. Note that the algorithm to determine inhabitation in the constructive case is the same as in the classical case; only the proof that the algorithm is correct is different.

This concludes the proof of the Hierarchy Theorem. □

Theorem 1.8. *Let $A = [A_1, \ldots, A_n]$ be a type. Then either A is inhabited or not, and*

$$A \text{ is uninhabited} \quad \Longleftrightarrow \quad \text{all } A_i \text{ are inhabited.} \tag{1.9}$$

Proof. Concerning "$\Longleftarrow$". Suppose towards a contradiction that all A_i are inhabited, and A is inhabited too. Pick $M \in \Lambda^\varepsilon(A)$ and $N_i \in \Lambda^\varepsilon(A_i)$ for each i. Then $N\, M_1 \cdots M_n$ is a closed inhabitant of 0, which is impossible.

We prove "$\Longrightarrow$" and "either A is inhabited or not" by induction on the buildup of types as 'tuples' using the operation [ ]. Let $A = [A_1, \ldots, A_n]$ with $A_i = [A_{i1}, \ldots, A_{im_i}]$ be given. For all $i \in \{1, \ldots, n\}$, suppose the following.

(i) Either A_i is inhabited or A_i is uninhabited.
(ii) If A_i is uninhabited then A_{ij} is inhabited for all j.

We need to prove that all A_i are inhabited provided that A is uninhabited, and that either A is inhabited or A is uninhabited.

Assume that A is uninhabited in order to show that all A_i are inhabited. By (i), either all A_i are inhabited or some A_i is uninhabited. In the former case we are done; so let us prove the latter case leads to a contradiction. Assume A_i is uninhabited for some i. By (ii) A_{ij} is inhabited for all j. Pick $N_j \in \Lambda^\varepsilon(A_{ij})$ for all j. Then $\lambda a_1^{A_1} \cdots a_n^{A_n}.\ a_i N_1 \cdots N_{m_i}$ is a closed inhabitant of A, contradicting that A is uninhabited. Therefore, all A_i are inhabited.

Consequently, A is inhabited iff not all A_i are inhabited. Since (by (i)) either all A_i are inhabited or not, it follows A is either inhabited or not. □

2. Reductions and contexts

In this section we introduce some syntactic sugar that will save ink later on.

Definition 2.1.

(i) A **context** is a sequence of distinct typed variables, $c_1^{C_1}, \ldots, c_k^{C_k}$.
The letters Γ, Δ, Θ, and Ξ denote contexts. The empty context is denoted by ε; concatenation of contexts is written as Γ, Δ.

(ii) For a context $\Gamma = c_1^{C_1}, \cdots, c_k^{C_k}$ write

$$\begin{aligned} \lambda\Gamma.\ N &\triangleq \lambda c_1^{C_1} \ldots c_k^{C_k}.\ N \\ \{\Gamma\} &\triangleq \{c_1^{C_1}, \ldots, c_k^{C_k}\} \\ \Lambda^{\Gamma}(A) &\triangleq \{\, M \in \Lambda(A) \mid \mathrm{FV}(M) \subseteq \{\Gamma\}\,\} \\ [\Gamma] &\triangleq [C_1, \ldots, C_k]. \end{aligned}$$

(iii) Let $\Gamma = c_1^{C_1}, \ldots, c_k^{C_k}$ be a context. We say $\vec{P}$ **fits** in Γ if $\vec{P} = P_1, \ldots, P_k$ is a tuple of (open) terms, and $P_i : C_i$ for every i. In that case we write

$$M[\Gamma{:=}\vec{P}] \triangleq M[c_1{:=}P_1] \ldots [c_k{:=}P_k].$$

Remark 2.2. Recall that we have assumed that all terms are in long normal form. In particular, if $M \in \Lambda^{\Gamma}([\Delta])$, then M is of the form $M \equiv \lambda\Delta.N$ where $\mathrm{FV}(N) \subseteq \{\Gamma, \Delta\}$.

Using contexts one can formulate statements such as $N \in \Lambda^{\Gamma}(0) \implies \lambda\Gamma.\ N \in \Lambda^{\varepsilon}([\Gamma])$, and $(\lambda\Gamma.\ N)\vec{P} =_{\beta} N[\Gamma{:=}\vec{P}]$ for any term N and $\vec{P}$ which fits in Γ. Also contexts lighten the study of reductions as will be shown in the following.

We study the relation $A_1 \leq_h A_2$ for types A_1, A_2 (see Definition 1.2(ii)). Note that $A_1 \leq_h A_2$ if and only if there is a *Böhm transformation* $\Phi\colon \Lambda^{\varepsilon}(A_1) \to \Lambda^{\varepsilon}(A_2)$, which is injective (on lnf-terms). That is, Φ should be of the form $\Phi(M) =_{\beta\eta} RM$ where R is some Böhm term (see Definition 1.2(ii)). More explicitly, writing $A_1 \equiv [\Delta_1]$ and $A_2 \equiv [\Delta_2]$, the map Φ should be of the form $\Phi(M) = \lambda\Delta_2.\ M\vec{P}$ where $\vec{P}$ fits in Δ_1 and $\mathrm{FV}(\vec{P}) \subseteq \{\Delta_2\}$.

Let Φ be such a Böhm transformation, then it transforms

$$\lambda\Delta_1.\ N \quad \text{to} \quad \lambda\Delta_2.\ N[\Delta_1{:=}\vec{P}].$$

To see if Φ is injective, we only need to focus on the transformation mapping

$$N \quad \text{to} \quad N[\Delta_1{:=}\vec{P}].$$

A map $\Lambda^{\Delta_1}(0) \to \Lambda^{\Delta_2}(0)$ of this form is also called a Böhm transformation.

In order to construct these Böhm transformations it pays off to consider the more general Böhm transformations from $\Lambda^{\Gamma_1}([\Delta_1]) \to \Lambda^{\Gamma_2}([\Delta_2])$ which map

$$\lambda\Delta_1.\ N \quad \text{to} \quad \lambda\Delta_2.\ N\,[\Delta_1{:=}\vec{P}]\,[\Gamma_1{:=}\vec{Q}],$$

where $\vec{P}$, $\vec{Q}$ fit in Δ_1, Γ_1, respectively, having free variables from Γ_2, Δ_2. Note that the core of these transformations is the *substitution* of $\vec{P}, \vec{Q}$ for Δ_1, Γ_1.

These considerations lead to the next set of definitions.

Definition 2.3.

(i) A pair ${}^{\Gamma}A$ is called a **context–type** and has as intended meaning the set $\Lambda^{\Gamma}(A)$ of terms $M : A$ with $\mathrm{FV}(M) \subseteq \{\Gamma\}$ (see Definition 2.1(ii)).

(ii) Define for such ${}^{\Gamma}A$ the type

$$\Gamma \to A \triangleq C_1 \to \ldots \to C_k \to A = [\Gamma, \Delta],$$

if $\Gamma \equiv c_1^{C_1}, \ldots, c_k^{C_k}$ and $A \equiv [\Delta]$.

(iii) We say ${}^{\Gamma_1}A_1$ **reduces** to ${}^{\Gamma_2}A_2$ and write ${}^{\Gamma_1}A_1 \leq {}^{\Gamma_2}A_2$ provided that

$$\Gamma_1 \to A_1 \leq_h \Gamma_2 \to A_2.$$

(iv) We write ${}^{\Gamma_1}A_1 \sim {}^{\Gamma_2}A_2$ provided that ${}^{\Gamma_1}A_1 \leq {}^{\Gamma_2}A_2$ and ${}^{\Gamma_1}A_1 \geq {}^{\Gamma_2}A_2$.

(v) Let Γ, Δ be contexts. A **substitution** from Γ to Δ is a map ϱ from $\{\Gamma\}$ to terms such that $\varrho_c \triangleq \varrho(c) \in \Lambda^{\Delta}(C)$ for all $c^C \in \{\Gamma\}$.

(vi) Let ${}^{\Gamma_1}[\Delta_1]$ and ${}^{\Gamma_2}[\Delta_2]$ be given. A **substitution** ϱ from ${}^{\Gamma_1}[\Delta_1]$ to ${}^{\Gamma_2}[\Delta_2]$ is a substitution from Γ_1, Δ_1 to Γ_2, Δ_2.

(vii) Let ϱ be as in (v). For every context $\Theta \equiv d_1^{D_1}, \ldots, d_k^{D_\ell}$ with $\{\Theta\} \subseteq \{\Gamma\}$, define $\vec{\varrho}_\Theta \triangleq \varrho_{d_1}, \ldots, \varrho_{d_\ell}$. (Then $\vec{\varrho}_\Theta$ fits in Θ, see Definition 2.1(iii).)

(viii) Let ϱ be as in (vi). Writing $A_1 = [\Delta_1]$ & $A_2 = [\Delta_2]$, define $\hat{\varrho}\colon \Lambda^{\Gamma_1}(A_1) \to \Lambda^{\Gamma_2}(A_2)$ by

$$\begin{aligned} \hat{\varrho}(\,\lambda\Delta_1.\, N\,) &\triangleq \lambda\Delta_2.\, N\,[\Gamma_1{:=}\vec{\varrho}_{\Gamma_1}]\,[\Delta_1{:=}\vec{\varrho}_{\Delta_1}] \\ &=_\beta \lambda\Delta_2.\, (\lambda\Delta_1.\, N)[\Gamma_1{:=}\vec{\varrho}_{\Gamma_1}]\,\vec{\varrho}_{\Delta_1}. \end{aligned}$$

Such a map $\hat{\varrho}$ is called a **(Böhm-)transformation** from ${}^{\Gamma_1}A_1$ to ${}^{\Gamma_2}A_2$.

Proposition 2.4. *Let ${}^{\Gamma_1}A_1$ and ${}^{\Gamma_2}A_2$ be context–types. Then*

$${}^{\Gamma_1}A_1 \leq {}^{\Gamma_2}A_2 \quad \Longleftrightarrow \quad \left[\begin{array}{l} \textit{There is a substitution } \varrho \textit{ from } {}^{\Gamma_1}A_1 \textit{ to } {}^{\Gamma_2}A_2 \textit{ such that} \\ \textit{the transformation } \hat{\varrho}\colon \Lambda^{\Gamma_1}(A_1) \to \Lambda^{\Gamma_2}(A_2) \textit{ is injective.} \end{array}\right.$$

Proof. Just unfold the definitions. □

Hence, if ${}^{\Gamma_1}A_1$ reduces to ${}^{\Gamma_2}A_2$, then there is an injective Böhm transformation $\Phi = \hat{\varrho}$ from $\Lambda^{\Gamma_1}(A_1)$ to $\Lambda^{\Gamma_2}(A_2)$. We will focus on ϱ instead of R as the following convention shows. (The benefit of this becomes clear later, see Remark 4.4.)

Convention 2.5. A **reduction** from ${}^{\Gamma_1}A_1$ to ${}^{\Gamma_2}A_2$ is a *substitution* ϱ from ${}^{\Gamma_1}A_1$ to ${}^{\Gamma_2}A_2$ such that the Böhm transformation $\hat{\varrho}$ is injective.

Since $A \leq_h B \iff {}^{\varepsilon}A \leq {}^{\varepsilon}B$ for all types A and B, it is natural to regard the types part of the context–types by identifying A with ${}^{\varepsilon}A$. As such, any notion defined for context–types can be applied to types as well.

For notational brevity, we also identify Γ and ${}^{\Gamma}0$ for any context Γ. In this way we also regard the contexts as part of the context–types. As such, any notion defined for context–types is applicable to contexts. In particular, we obtain a notion of reduction between contexts; $\Gamma \leq \Delta \iff {}^{\Gamma}0 \leq {}^{\Delta}0$.

Note that with these identifications we have $\Gamma, \Delta \sim {}^{\Gamma}[\Delta] \sim [\Gamma, \Delta]$ for all Γ, Δ.

3. Inequalities between canonical types

In this section we will prove the inequalities listed in Statement (1.2) on page 8. This is one of the bits left out of the proof of the Hierarchy Theorem in Subsection 1.4.

We start with two of the simpler inequalities.

3.1. **Ad $H_{k+1} \not\leq_{\beta\eta} H_k$.** As H_k has exactly k inhabitants, there is no injection from $\Lambda^{\varepsilon}(H_{k+1})$ to $\Lambda^{\varepsilon}(H_k)$, and hence no $\beta\eta$-reduction from H_{k+1} to H_k (see Definition 1.2(i)).

3.2. **Ad $H_{\omega+1} \not\leq_h H_\omega$.** We need to prove that $[2] \not\leq_h [1,0]$. Or in other words, we must prove that $[2] \not\leq f^1, c^0$ (see Definition 2.3(iii)). Let ϱ be a substitution from $[2]$ to f^1, c^0; we will prove that the Böhm transformation $\hat{\varrho}$ is not injective. (Hence $[2] \not\leq f^1, c^0$ by Proposition 2.4.) To this end we simply calculate $\hat{\varrho}(M)$ for $M \in \Lambda^\varepsilon([2])$.

Recall that an inhabitant of $[2]$ is of the following form (see 1.3.3).

$$\langle i,j\rangle \;\triangleq\; \lambda F^2.\, F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, x_j$$

So we have

$$\hat{\varrho}(\,\langle i,j\rangle\,) \;\equiv\; \langle i,j\rangle\, \varrho_F \;=_\beta\; \varrho_F\lambda x_1^0.\, \cdots \varrho_F\lambda x_i^0.\, x_j. \tag{3.1}$$

Further, note that since ϱ_F is an element of $\Lambda^{f^1,c^0}(2)$ it must be of the following form.

$$\varrho_F \;\equiv\; \lambda g^1.\, f^{(k_0)}gf^{(k_1)}\cdots gf^{(k_n)}c.$$

We first exclude a pathological case. If $n = 0$, then $\hat{\varrho}(\,\langle i,j\rangle\,) =_\beta f^{(k_0)}c$. So $\hat{\varrho}$ is constant and hence not injective. So let us assume that $n > 0$.

To reduce Equation (3.1), note the following.

(i) Let $G \triangleq \lambda y^0.\, f^{(m)}x$; we calculate $\varrho_F\, G$. To start, $GM =_\beta f^{(m)}x$ for all terms M. So if $M \triangleq f^{(k_1)}Gf^{(k_2)}\cdots Gf^{(k_n)}c$, then

$$\varrho_F G \;\equiv\; \varrho_F\,\lambda y^0.\, f^{(m)}x \;=_\beta\; f^{(k_0)}GM \;=_\beta\; f^{(k_0)}f^{(m)}x$$

(ii) Let $G \triangleq \lambda y^0.\, f^{(m)}y$. We have

$$\begin{aligned}\varrho_F\lambda y^0.\, f^{(m)}y \;&=_\beta\; f^{(k_0)}Gf^{(k_1)}\cdots Gf^{(k_n)}c\\ &=_\beta\; f^{(k_0+m+k_1+\cdots+m+k_n)}c \;=_\beta\; f^{(m\cdot n+\sum_i k_i)}c.\end{aligned}$$

So if we apply (i) and (ii) to Equation (3.1), in this order, (i), (ii), (i), we obtain

$$\begin{aligned}\varrho_F(\,\langle i,j\rangle\,) \;&=_\beta\; \varrho_F\lambda x_1^0.\, \cdots \varrho_F\lambda x_j^0.\; f^{((i-j)k_0)}\; x_j\\ &=_\beta\; \varrho_F\lambda x_1^0.\, \cdots \varrho_F\lambda x_{i-1}^0.\; f^{(n(i-j)k_0+\sum_i k_i)}\; c\\ &=_\beta\; f^{((i-1)k_0+n(i-j)k_0+\sum_i k_i)}\; c.\end{aligned}$$

Consequently, $\langle 3,1\rangle$ and $\langle n+3, n+2\rangle$ are both sent to $f^{(2(n+1)k_0+\sum_i k_i)}c$ by $\hat{\varrho}$.

3.3. **Indiscernibility.** The remaining inequalities are of the form $H_\alpha \not\leq_{h,\beta\eta,h^+} H_\beta$. In this subsection we develop some general theory to prove them. In fact, we prove a stronger statement: there are terms $M_1 \neq M_2$ in $\mathbb{H}_\alpha$ (listed in Statement (1.2) on page 8) such that

$$RM_1 =_{\beta\eta} RM_2 \qquad \text{for all} \qquad R\colon H_\alpha \to H_\beta. \tag{3.2}$$

That is, M_1 and M_2 are indiscernible for any term $R\colon H_\alpha \to H_\beta$. (This is called *observational equivalence* in the literature[BDS13].) As Proposition 3.6 shows, instead of proving that M_1 and M_2 are indiscernible for every $R\colon H_\alpha \to H_\beta$, it suffices to prove that for certain variants H'_β of H_β, the terms M_1 and M_2 are indiscernible for any Böhm transformation from H_α to H'_β. (This is called *existential equivalence*.) This general method of proving that $H_\alpha \not\leq_{h,\beta\eta,h^+} H_\beta$ has been extracted from the proof in [Dek88] of $[3,0] \not\leq_h [1,1,0]$.

Definition 3.1. Let ${}^\Gamma A$ and Δ be given. For $M_1, M_2 \in \Lambda^\Gamma(A)$ define

$$M_1 \approx_\Delta^{\mathrm{Ob}} M_2 \quad \Longleftrightarrow \quad \forall R\,[\, R(\lambda\Gamma.\, M_1) =_{\beta\eta} R(\lambda\Gamma.\, M_2)\,],$$
$$M_1 \approx_\Delta^{\mathrm{Ex}} M_2 \quad \Longleftrightarrow \quad \forall \varrho\,[\, \hat{\varrho} M_1 = \hat{\varrho} M_2\,],$$

where R ranges over the closed terms $R\colon (\Gamma \to A) \to [\Delta]$ and ϱ ranges over the substitutions from ${}^\Gamma A$ to Δ. (So $\hat{\varrho}$ is a Böhm transformation.)

Lemma 3.2. *Let A and $B \equiv [\Delta]$ be types and $M_1, M_2 \in \Lambda^\varepsilon(A)$ with $M_1 \neq M_2$.*

(i) $M_1 \approx_\Delta^{\mathrm{Ob}} M_2$ implies $A \not\leq_{\beta\eta} B$.

(ii) $M_1 \approx_\Delta^{\mathrm{Ex}} M_2$ implies $A \not\leq_h B$ and $A \not\leq_{h^+} B$.

Proof. (i). Suppose $M_1 \approx_\Delta^{\mathrm{Ob}} M_2$ and $A \leq_{\beta\eta} B$ towards a contradiction. Since $A \leq_{\beta\eta} B$, there a reducing term $R \in \Lambda^\varepsilon(A \to B)$ (see Definition 1.2(i)). Since $M_1 \approx_\Delta^{\mathrm{Ob}} M_2$, we get $RM_1 =_{\beta\eta} RM_2$ (see Definition 3.1). But this implies that $M_1 = M_2$ (as R is a reducing term), contradicting $M_1 \neq M_2$. Hence $A \not\leq_{\beta\eta} B$.

(ii). Assume that $M_1 \approx_\Delta^{\mathrm{Ex}} M_2$. We will that prove $A \not\leq_{h^+} B$, and hence a fortiori $A \not\leq_h B$ (see Definition 1.2). Suppose that $A \leq_{h^+} B$ towards a contradiction. Pick a family of substitutions $\varrho^1, \ldots, \varrho^n$ from A to $[\Delta]$ such that

$$\forall i\,[\, \hat{\varrho}^i(M) = \hat{\varrho}^i(N)\,] \implies M = N \qquad (M, N \in \Lambda^\varepsilon(A)). \tag{3.3}$$

Let i and $M \in \Lambda^\varepsilon(A)$ be given. Then we know that $\hat{\varrho}^i(M) = \lambda\Delta.\, M\vec{\varrho}_\Gamma^i$ and $M\vec{\varrho}_\Gamma^i \in \Lambda^\Delta(0)$ where $A = [\Gamma]$. Hence $M \mapsto M\vec{\varrho}_\Gamma^i$ is a Böhm transformation from A to Δ. Thus we get $M_1\vec{\varrho}_\Gamma^i = M_2\vec{\varrho}_\Gamma^i$ by $M_1 \approx_\Delta^{\mathrm{Ex}} M_2$ (see Definition 3.1). Then $\hat{\varrho}^i(M_1) = \hat{\varrho}^i(M_2)$. So Statement (3.3) implies that $M_1 = M_2$, contradicting $M_1 \neq M_2$. Hence $A \not\leq_{h^+} B$. □

To formulate Proposition 3.6, we need one more notion.

Observation 3.3. An inhabitant M of a context Γ, i.e. $M \in \Lambda^\Gamma(0)$, is of the form

$$M \equiv a^A\,(\lambda\Gamma_1.\, M_1)\cdots(\lambda\Gamma_k.\, M_k),$$

where $a^A \in \{\Gamma\}$, $A = [[\Gamma_1], \ldots, [\Gamma_k]]$ and M_i is an inhabitant of Γ, Γ_i.

Definition 3.4. Let Γ be a context and $a^A \in \{\Gamma\}$ with $A \equiv [[\Gamma_1], \ldots, [\Gamma_k]]$. Then for all i the context Γ, Γ_i is said to be a *direct derivative* of Γ. A context Γ' is **a derivative** of Γ if there is a chain of direct derivatives from Γ to Δ.[2]

Examples 3.5.
(1) The only derivative of x^0, f^1 is x^0, f^1 itself. In fact, a context Γ has only one derivative (c.q. itself) iff $\mathrm{rk}[\Gamma] \leq 2$.
(2) Any derivative of F^2 is of the form $F^2, x_1^0, \ldots, x_n^0$ for some n, and any derivative of Ω^3 is of the form $\Omega^3, f_1^1, \ldots, f_n^1$ for some n.
(3) The context Φ^4, F^2, x^0, G^2 is a derivative of Φ^4. Any derivative of Φ^4 is of the form Φ^4, Δ', where Δ' is a context with $\{\Delta'\} = \{F_1^2, \ldots, F_m^2, x_1^0, \ldots, x_n^0\}$ for some n, m with $n \neq 0 \implies m \neq 0$.

Proposition 3.6. *Given a type A, terms $M_1, M_2 \in \Lambda^\varepsilon(A)$ and a context Δ,*

$$\forall \Delta'\,[\, M_1 \approx_{\Delta'}^{\mathrm{Ex}} M_2\,] \quad \Longrightarrow \quad M_1 \approx_\Delta^{\mathrm{Ob}} M_2.$$

Here Δ' ranges over contexts such that m^A, Δ' is a derivative of m^A, Δ.

[2]Equivalently, the relation on contexts of being a derivative is the *transitive–reflexive closure* of the relation on contexts of being a direct derivative.

Proof. Assume $M_1 \approx^{\mathrm{Ex}}_{\Delta'} M_2$ for every Δ'. We will prove that

$$N[m{:=}M_1] = N[m{:=}M_2] \qquad \text{for each } \Delta' \text{ and } N \in \Lambda^{m^A,\Delta'}(0). \tag{3.4}$$

This is sufficient. Indeed, suppose that $R\colon A \to [\Delta]$ with $R \equiv \lambda m^A.\, N$. Then we have that $RM_1 =_\beta N[m{:=}M_1] = N[m{:=}M_2] =_\beta RM_2$. Hence $M_1 \approx^{\mathrm{Ob}}_{\Delta} M_2$, as required.

To prove Statement (3.4), we use induction (over the long normal form of N). Let $N \in \Lambda^{m^A,\Delta'}(0)$ be given for some Δ' such that m^A, Δ' is a derivative of m^A, Δ. We have

$$N \;\equiv\; c\,(\lambda\Gamma_1.\, N_1)\cdots(\lambda\Gamma_k.\, N_k),$$

where $c \in \{m^A, \Delta'\}$ and $N_j \in \Lambda^{m^A,\Delta',\Gamma_j}(0)$ for all j (see Observation 3.3). To use induction over N, we need to prove that every N_j falls in the scope of Statement (3.4), i.e. that $N_j \in \Lambda^{m^A,\Delta''}(0)$ for some Δ'' such that m^A, Δ'' is a derivative of m^A, Δ. That is, we need to have that m^A, Δ', Γ_j is a derivative of m^A, Δ. This is indeed the case because m^A, Δ', Γ_j is a direct derivative of m^A, Δ', and m^A, Δ' itself is a derivative of m^A, Δ (see Definition 3.4).

We need to prove that $N[m{:=}M_1] = N[m{:=}M_2]$, and by induction we may assume that $N_i[m{:=}M_1] = N_i[m{:=}M_2]$ for all i. Note that either $c \in \{\Delta'\}$ or $c = m^A$.

In the former case, $m \neq c$, so

$$N[m{:=}M_j] \;=\; c\,(\lambda\Gamma_1.\, N_1[m{:=}M_j])\cdots(\lambda\Gamma_k.\, N_k[m{:=}M_j]),$$

hence $N[m{:=}M_1] = N[m{:=}M_2]$, as required.

In the latter case, we have $c = m^A$ (and thus $A = [[\Gamma_1], \ldots, [\Gamma_k]]$), so

$$N[m{:=}M_j] \;=\; M_j(\lambda\Gamma_1.\, N_1[m{:=}M_j])\cdots(\lambda\Gamma_k.\, N_k[m{:=}M_j]) = \hat{\varrho}M_j,$$

where ϱ is the substitution from $A = [a_1^{[\Gamma_1]}, \ldots, a_k^{[\Gamma_k]}]$ to Δ' given by

$$\varrho_{a_i} \;\triangleq\; \lambda\Gamma_i.\, N_i[m{:=}M_j],$$

but since $M_1 \approx^{\mathrm{Ex}}_{\Delta'} M_2$, we have $\hat{\varrho}M_1 = \hat{\varrho}M_2$ and $N[m{:=}M_1] = N[m{:=}M_2]$. □

Corollary 3.7. *Let A and $B = [\Delta]$ be types. Let $M_1, M_2 \in \Lambda^{\varepsilon}(A)$ with*

$$M_1 \neq M_2 \qquad \textit{and} \qquad \forall\Delta'\ [\ M_1 \approx^{\mathrm{Ex}}_{\Delta'} M_2\ ],$$

where Δ' ranges over contexts such that m^A, Δ' is a derivative of m^A, Δ. Then

$$A \not\leq_{\beta\eta} B; \qquad A \not\leq_{h} B; \qquad A \not\leq_{h^+} B.$$

Proof. Combine Proposition 3.6 and Lemma 3.2. □

3.4. **Ad $H_\omega \not\leq_{h^+} H_{k+1}$ and $H_\omega \not\leq_{\beta\eta} H_{k+1}$.** We need to prove that $[1,0] \not\leq_{h^+} [0^{k+1}]$ and $[1,0] \not\leq_{\beta\eta} [0^{k+1}]$. We use Corollary 3.7 with $M_i \triangleq c_i$ (see 1.3.2) and $\Delta \triangleq x_1^0, \ldots, x_{k+1}^0$. Let $m^{[0,1]}, \Delta'$ be a derivative of $m^{[0,1]}, \Delta$. We need to prove that $c_1 \approx^{\mathrm{Ex}}_{\Delta'} c_2$. Note that $\Delta' \equiv x_1^0, \ldots, x_m^0$ for some $m \geq k+1$. Let ϱ be a substitution from $[f^1, c^0]$ to Δ'. In order to show that $M_1 \approx^{\mathrm{Ex}}_{\Delta'} M_2$, we need to prove that $\hat{\varrho}c_1 = \hat{\varrho}c_2$.

The term $\varrho_f \in \Lambda^{\Delta'}(1)$ is either $\lambda y^0.\, y$ or $\lambda y^0.\, x_i$ for some i.

(i) In the former case, $\hat{\varrho}c_i = \varrho_f^{(i)}\varrho_c = \varrho_c$, so $\hat{\varrho}c_1 = \hat{\varrho}c_2$.
(ii) In the latter, $\varrho_f M =_\beta x_i$ for each M, so in particular $\hat{\varrho}c_1 = x_i = \hat{\varrho}c_2$.

3.5. **Ad $H_{\omega+2} \not\leq_{h^+} H_{\omega+1}$ and $H_{\omega+2} \not\leq_{\beta\eta} H_{\omega+1}$.** Again we use Corollary 3.7, but now with $M_1 = fgfg$ and $M_2 = fggf$ (see 1.3.4).

Let ϱ be a substitution from $[f^1, g^1, c^0]$ to a context Δ', such that $m^{[1,1,0]}, \Delta'$ is a derivative of $m^{[1,1,0]}, F^2$; we need to show that $\hat{\varrho}M_1 = \hat{\varrho}M_2$. Note that $\Delta' = F^2, d_1^0, \ldots, d_\ell^0$ for some ℓ. Let us first study $\varrho_f \in \Lambda^{\Delta'}(1)$; it is of the form

$$\varrho_f \equiv \lambda z^0.\, F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, e,$$

for some i, where either $e = z$, $e = d_k$ or $e = x_k$ for some k. So for any term M,

$$\varrho_f M \;=_\beta \begin{cases} F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, M & \text{if } e = z \\ F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, e & \text{otherwise.} \end{cases}$$

In the latter case, $\hat{\varrho}M_1 \equiv \hat{\varrho}(fgfg) =_{\beta\eta} \varrho_f\hat{\varrho}(gfg) =_\beta F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, e$ and similarly $\hat{\varrho}M_2 =_{\beta\eta} F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, e$, so $\hat{\varrho}M_1 = \hat{\varrho}M_2$. So let us instead assume that

$$\rho_f \;=\; \lambda z^0.\, F\lambda x_1^0.\, \cdots F\lambda x_i^0.\, z.$$

By similar reasoning for g, we are left with the case that, for some j,

$$\varrho_g = \lambda z^0.\, F\lambda x_1^0.\, \cdots F\lambda x_j^0.\, z.$$

Abusing notation, one could set $h \triangleq$ "$F\lambda z^0.\,$" and write $\varrho_f = h^i z$. Then

$$\hat{\varrho}M_1 \equiv \hat{\varrho}(fgfg) \;=_\beta\; h^i h^j h^i h^j \varrho_c \;=\; h^{2(i+j)}\varrho_c \;=_\beta\; \hat{\varrho}(fggf) \equiv \hat{\varrho}M_2.$$

3.6. **Ad $H_{\omega+3} \not\leq_{h^+} H_{\omega+2}$ and $H_{\omega+3} \not\leq_{\beta\eta} H_{\omega+2}$.** We use Corollary 3.7 with (see 1.3.5)

$$M_i \triangleq /1/2i.$$

Let $m^{[3,0]}, \Delta'$ be a derivative of $m^{[3,0]}, f^1, g^1, d^0$, and ϱ a substitution from $[\Phi^3, c^0]$ to Δ'. One easily verifies that $\{\Delta'\} = \{f, g, d, G_1^2, \ldots, G_\nu^2, d_1^0, \ldots, d_\mu^0\}$ for some ν, μ.

We need to prove the following equality.

$$\hat{\varrho}M_1 = \hat{\varrho}M_2 \tag{3.5}$$

To this end, we first calculate $\varrho_\Phi M$ for $M \in \Lambda^\Xi(2)$ where $\Xi \triangleq h_1^1, \ldots, h_m^1, \Delta'$. The result is recorded in Lemma 3.8. We start with two remarks.

First, note that $\varrho_\Phi \in \Lambda^{\Delta'}(3)$ is of the form

$$\varrho_\Phi \;\equiv\; \lambda F^2.\, w_0 F\lambda z_1^0.\, w_1 \cdots F\lambda z_n^0.\, w_n e, \tag{3.6}$$

where w_i are words on the alphabet

$$\mathcal{A} = \{f, g, G_1\lambda y^0.\,, \ldots, G_\nu\lambda y^0.\,\}$$

and e is a variable of type 0, so either $e = d$, $e = z_i$ for some $i \in \{1, \ldots, n\}$, $e = d_i$ for some $i \in \{1, \ldots, m\}$, or $e = y$ for some y introduced by a G_j in one of the w_k.

Secondly, we know that any $M \in \Lambda^\Xi(2)$ is of the form

$$M \equiv \lambda h^1.\, H_1 h H_2 h \cdots H_\ell h R$$

where $H_i \in \Lambda^\Xi(1)$ and $R \in \Lambda^\Xi(0)$.

Lemma 3.8. *Let $M \equiv \lambda h^1.\, H_1 h H_2 h \cdots H_\ell h R$ from $\Lambda^\Xi(2)$ be given.*

(i) If h does not occur in M (i.e. $M \equiv \lambda h^1.\, R$), then $\varrho_\Phi M =_\beta w_0 R$.

(ii) If h occurs in M then

$$\varrho_\Phi M =_\beta \begin{cases} W^H(TP_i^H) & \text{if } e = z_i, \\ W^H e & \text{otherwise,} \end{cases} \tag{3.7}$$

where $H \triangleq H_1$ *and* $T \triangleq \lambda h^1.\ H_2 h \cdots H_\ell h R$ *and*

$$W^H \triangleq w_0 H w_1 \cdots H w_n \quad \text{and} \quad P_i^H \triangleq \lambda z_i^0.\ w_i H w_{i+1} \cdots H w_n z_i.$$

Proof. (i). Assume h does not occur in M. Then $MK =_\beta R$ for every $K : 1$. We apply this to Equation (3.6). Writing $K \triangleq \lambda z_1^0.\ w_1 \cdots M \lambda z_n^0.\ w_n e$, we have

$$\varrho_\Phi M =_\beta w_0 M(\lambda z_1^0.\ w_1 \cdots M \lambda z_n^0.\ w_n e) \equiv w_0 M K =_\beta w_0 R.$$

(ii). Assume h occurs in M. Note that by definition of H and T,

$$M =_\beta \lambda h^1.\ H h (T h). \tag{3.8}$$

In particular, for any term $K : 1$ in which z_j does not occur, we have

$$M(\lambda z_j^0.\ K) =_\beta HK. \tag{3.9}$$

Either $e = z_i$ for some i or not. If $e \neq z_i$, then

$$\begin{aligned} \varrho_\Phi M &=_\beta w_0 M \lambda z_1^0.\ w_1 \cdots M \lambda z_n^0.\ w_n e && \text{by Eq. (3.6)} \\ &=_\beta w_0 H w_1 \cdots H w_n e && \text{by Eq. (3.9)} \\ &= W^H e && \text{by def. of } W^H. \end{aligned}$$

If $e = z_i$, then

$$\begin{aligned} \varrho_\Phi M &=_\beta w_0 M \lambda z_1^0.\ w_1 \cdots M \lambda z_n^0.\ w_n z_i && \text{by Eq. (3.6)} \\ &=_\beta w_0 H w_1 \cdots M \lambda z_i^0.\ w_i \cdots H w_n z_i && \text{by Eq. (3.9)} \\ &= w_0 H w_1 \cdots M P_i^H && \text{by def. of } P_i^H \\ &=_\beta w_0 H w_1 \cdots H P_i^H (T P_i^H) && \text{by Eq. (3.8)} \\ &=_\beta w_0 H w_1 \cdots H w_i \cdots H w_n (T P_i^H) && \text{by def. of } P_i^H \\ &= W^H (T P_i^H) && \text{by def. of } W^H. \end{aligned}$$

We have proven Statement (3.7) and so we are done. □

We will use the special case of Lemma 3.8 where $H = \lambda x^0.\ x$.

Corollary 3.9. *Define* $W \triangleq w_0 w_1 \cdots w_n$ *and* $P_i \triangleq \lambda z_i^0.\ w_i w_{i+1} \cdots w_n z_i$. *Then for any term* $M \in \Lambda^\Xi(2)$ *of the form* $M =_\beta \lambda h^1.\ h(Th)$ *with* $T \in \Lambda^\Xi(2)$ *we have*

$$\varrho_\Phi M =_\beta \begin{cases} W(TP_i) & \text{if } e = z_i, \\ We & \text{otherwise.} \end{cases}$$

Proof. Follows immediately from Lemma 3.8. □

We are now ready to prove Equation (3.5).

Corollary 3.10. $\hat{\varrho}M_1 = \hat{\varrho}M_2$.

Proof. For brevity, let $K_j \triangleq \lambda x_1 x_2.\ x_j$. We have

$$M_j = \lambda\Phi^3 c^0.\ \Phi\lambda f_1^1.\ f_1\Phi\lambda f_2^1.\ f_2(K_j f_1 f_2)c. \tag{3.10}$$

We distinguish two cases: either $e = z_i$ or not.

Assume $e = z_i$ for some i. We apply Corollary 3.9 twice, first to $M^{\mathrm{I}} \triangleq \lambda f_2^1.\ f_2(K_j f_1 f_2)\varrho_c$ and then to $M^{\mathrm{II}} \triangleq \lambda f_1^1.\ f_1 W(K_j f_1 P_i)\varrho_c$. Indeed,

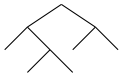
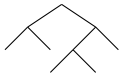
$$\begin{aligned}
\hat{\varrho}M_j &=_\beta \varrho_\Phi\lambda f_1^1.\ f_1\varrho_\Phi\lambda f_2^1.\ f_2\,(K_j f_1 f_2)\varrho_c && \text{by Equation (3.10)}\\
&= \varrho_\Phi\lambda f_1^1.\ f_1\varrho_\Phi M^{\mathrm{I}} && \text{by def. of } M^{\mathrm{I}}\\
&=_\beta \varrho_\Phi\lambda f_1^1.\ f_1 W(K_j f_1 P_i)\varrho_c && \text{by Corollary 3.9}\\
&=_\beta \varrho_\Phi M^{\mathrm{II}} && \text{by def. of } M^{\mathrm{II}}\\
&=_\beta WW(K_j P_i P_i)\varrho_c && \text{by Corollary 3.9}\\
&=_\beta WWP_i\varrho_c && \text{by def. of } K_j.
\end{aligned}$$

Assume $e \neq z_i$. By Corollary 3.9 applied to $M \triangleq \lambda f_1^1.\ f_1\varrho_\Phi\lambda f_2^1.\ f_2 f_i\varrho_c$,

$$\hat{\varrho}M_j =_\beta \varrho_\Phi(\,\lambda f_1^1.\ f_1\varrho_\Phi\lambda f_2^1.\ f_2 f_i\varrho_c\,) \equiv \varrho_\Phi M =_\beta We.$$

So in both cases the value of $\hat{\varrho}M_j$ does not depend on j. □

3.7. **Ad $H_{\omega+4} \not\leq_{h^+} H_{\omega+3}$ and $H_{\omega+4} \not\leq_{\beta\eta} H_{\omega+3}$.** We use Corollary 3.7 with (see 1.3.6)

$M_1 =$  $M_2 =$  .

That is, $M_1 \triangleq \lambda b^{[0,0]}c^0.\ bbcbccbcc$ and $M_2 \triangleq \lambda b^{[0,0]}c^0.\ bbccbbccc$. Let $m^{[[0,0],0]}, \Delta'$ be a derivative of $m^{[[0,0],0]}, \Phi^3, c^0$, and let ϱ be a substitution from $[b^{[0,0]}, c^0]$ to Δ'. Note that

$$\{\Delta'\} = \{\Phi^3, c^0, f_1^1, \ldots, f_\ell^1, d_1^0, \ldots, d_\nu^0\}.$$

We need to prove that

$$\hat{\varrho}M_1 = \hat{\varrho}M_2.$$

Consider $\varrho_b \in \Lambda^{\Delta'}([0,0])$. It is of the form

$$\varrho_b \equiv \lambda x^0 y^0.\ w_0\,\Phi\lambda g_1^1.\ w_1 \cdots \Phi\lambda g_\mu^1.\ w_\mu e,$$

where $e \in \{x, y, c, d_1, \ldots, d_\nu\}$ and w_i is a word over $\{f_1, \ldots, f_\ell, g_1, \ldots, g_i\}$.

We see that either $e \in \{x, c, d_1, \ldots, d_\nu\}$ or $e = y$.

In the former case, we have $e \neq y$. Then y is not used in ϱ_b, so we have $\varrho_b MN =_\beta \varrho_b MN'$ for all terms $M, N, N' : 0$. In particular,

$$\begin{aligned}
\hat{\varrho}M_1 &\equiv \varrho_b(\varrho_b\varrho_c(\varrho_b\varrho_c\varrho_c))(\varrho_b\varrho_c\varrho_c)\\
&=_\beta \varrho_b(\varrho_b\varrho_c N)N' && \text{for any } N,\ N'\\
&=_\beta \varrho_b(\varrho_b\varrho_c\varrho_c)(\varrho_b(\varrho_b\varrho_c\varrho_c)\varrho_c)\\
&\equiv \hat{\varrho}M_2.
\end{aligned}$$

Similarly, if $e = y$, then $e \neq x$, so $\varrho_b MN =_\beta \varrho_b M'N$ for all M, M', N, and

$$\hat{\varrho}M_1 =_\beta \varrho_b M(\varrho_b M' \varrho_c) =_\beta \hat{\varrho}M_2 \quad \text{for all } M,\ M'.$$

4. Calculus of reductions

Before we proceed, we establish some general calculation rules for reducibility. Although one can find some trivial rules for $\leq_h$ such as $[A_1, A_2] \leq_h [A_2, A_1]$, the notion of head reduction is otherwise uncooperative. Therefore, we work with *strong reductions* (Subsection 4.1) and *atomic reductions* (Subsection 4.2) instead, yielding the more tractable relations $\leq^s$ and $\leq^a$, respectively. We prove later on that for types A and B we have

$$A \leq^a B \quad \Longrightarrow \quad A \leq^s B \quad \Longrightarrow \quad A \leq_h B.$$

So to show that $A \leq_h B$ it suffices to prove that either $A \leq^s B$ or $A \leq^a B$.

One of the calculation rules provided in this section concerns types A with $[1, 1] \leq^a A$. It states that for such A and any contexts Γ_1, Γ_2, we have

$$[\Gamma_1] \leq^s A \quad \text{and} \quad [\Gamma_2] \leq^s A \quad \Longrightarrow \quad [\Gamma_1, \Gamma_2] \leq^s A.$$

We will call these types *atomic types* and study them in Subsection 4.3.

4.1. **Strong reductions.** For the sake of familiarity we begin with strong reductions between *types*. Let A_1 and A_2 be types. Recall that a reducing term from A_1 to A_2 is a closed term R of type $A_1 \to A_2$ which is injective on closed terms (see Definition 1.2), that is, the map $\Phi\colon \Lambda^\varepsilon(A_1) \to \Lambda^\varepsilon(A_2)$ given by $\Phi(M) =_\beta RM$ is injective.

If R is also injective on open terms, then R is called *strong*:

Definition 4.1. (i) Let A_1 and A_2 be types. A **strong reducing term** from A_1 to A_2 is a closed term $R\colon A_1 \to A_2$ that is injective on open terms, that is, for every context Ξ the term R is injective on open terms with free variables from $\{\Xi\}$, that is, the term R is injective considered as a map $\Lambda^\Xi(A_1) \to \Lambda^\Xi(A_2)$.

(ii) If there is a strong reducing term $R\colon A_1 \to A_2$ that is a Böhm term (see Definition 1.2(ii)), we say that A_1 **strongly head reduces to** A_2, notation $A_1 \leq_h^s A_2$.

(iii) For context–types (see Definition 2.3) ${}^{\Gamma_1}A_1$ **strongly reduces** to ${}^{\Gamma_2}A_2$ if

$$\Gamma_1 \to A_1 \ \leq_h^s \ \Gamma_2 \to A_2,$$

and we write ${}^{\Gamma_1}A_1 \leq^s {}^{\Gamma_2}A_2$. If in addition ${}^{\Gamma_2}A_2 \leq^s {}^{\Gamma_1}A_1$, we write ${}^{\Gamma_2}A_2 \sim^s {}^{\Gamma_1}A_1$.

(iv) Let ϱ be a substitution from ${}^{\Gamma_1}A_1$ to ${}^{\Gamma_2}A_2$, and let Ξ be a fresh context. With ϱ^Ξ we denote the natural extension of ϱ to a substitution from ${}^{\Xi,\Gamma_1}A$ to ${}^{\Xi,\Gamma_2}B$ given by $\varrho^\Xi_c = c$ for all $c^C \in \{\Xi\}$. Then $\hat{\varrho}^\Xi\colon \Lambda^{\Xi,\Gamma_1}(A_1) \to \Lambda^{\Xi,\Gamma_2}(A_2)$.

Proposition 4.2. *Let ${}^{\Gamma_1}A_1$ and ${}^{\Gamma_2}A_2$ be context–types. Then*

$${}^{\Gamma_1}A_1 \leq^s {}^{\Gamma_2}A_2 \quad \Longleftrightarrow \quad \left[\begin{array}{l} \textit{There is a substitution } \varrho \textit{ from } {}^{\Gamma_1}A_1 \textit{ to } {}^{\Gamma_2}A_2 \\ \textit{such that } \hat{\varrho}^\Xi\colon \Lambda^{\Xi,\Gamma_1}(A_1) \to \Lambda^{\Xi,\Gamma_2}(A_2) \textit{ is in-} \\ \textit{jective for every context } \Xi. \end{array}\right.$$

Proof. Just unfold the definitions. □

Definition 4.3. Let ${}^{\Gamma_1}A_1$, ${}^{\Gamma_2}A_2$ be context–types. A **strong reduction from** ${}^{\Gamma_1}A_1$ **to** ${}^{\Gamma_2}A_2$ is a substitution ϱ from ${}^{\Gamma_1}A_1$ to ${}^{\Gamma_2}A_2$ such that all $\hat{\varrho}^{\Xi}$ are injective. We write $\varrho\colon {}^{\Gamma_1}A_1 \leq^s {}^{\Gamma_2}A_2$.

Remark 4.4. It would not make sense to define a strong reduction to be the Böhm transformation $\Phi \equiv \hat{\varrho}$ because one can not always reconstruct ϱ—and hence the $\hat{\varrho}^{\Xi}$s—from $\hat{\varrho}$, which acts only on closed terms.

The merit of strong reductions (over regular ones) is that it is easy to built complex strong reductions from simpler ones. Moreover, almost all reductions encountered in this text are strong.

Remarks 4.5. (i) Not every reduction is also a strong reduction: the substitution ϱ from the context f^1 to the empty context ε given by $\varrho_f = \lambda x^0.\ x$ is a reduction, because $\Lambda^{f^1}(0)$ is empty, and thus $\hat{\varrho}\colon \Lambda^{f^1}(0) \to \Lambda^{\varepsilon}(0)$ is injective (see Convention 2.5); but ϱ is not a strong reduction since $\hat{\varrho}^{\Xi}$ with $\Xi \triangleq d^0$ maps both fd and ffd to $\lambda x^0.\ x$ and is hence not injective (see Definition 4.3).

(ii) Note that $\Gamma \leq^s \Delta$ implies $\Xi, \Gamma \leq \Xi, \Delta$ for all contexts Ξ. It is not evident whether the reverse implication holds as well. If $\Gamma \leq^s \Delta$ then there is *one* substitution ϱ which yields a family of similar reductions $\varrho^{\Xi}\colon \Xi, \Gamma \leq \Xi, \Delta$; on the other hand, if $\Xi, \Gamma \leq \Xi, \Delta$ for all Ξ, we only know there is a family of (potentially quite dissimilar reductions) $\varrho_{\Xi}\colon \Xi, \Gamma \leq \Xi, \Delta$. As it turns out, the reverse implication does hold; we will not prove this in this article.

(iii) As $\Gamma \leq^s \Delta \implies [\Gamma] \leq^s [\Delta]$ (by Definition 4.1(iii)) one could conjecture that we also have that $A \leq^s B \implies [A] \leq^s [B]$. These however are quite different statements. By the Hierarchy Theorem, the conjecture is false. Indeed: later on we will see that $[1,0] \leq^s [2]$. If one had $[[1,0]] \leq^s [[2]]$, then also $[0,[1,0]] \leq_h [0,[2]]$, quod non as $[0,[1,0]] \in \mathbb{H}_{\omega+4}$, while $[0,[2]] \in \mathbb{H}_{\omega+3}$.

(iv) Nevertheless we do have $A \leq^s B \implies [[A]] \leq^s [[B]]$ (see Lemma 4.21).

(v) Similarly, we have $\Gamma \sim^s [\Gamma]$ for every context Γ (by Definition 4.1(iii)), but *never* $A \sim^s [A]$ for a type A. Indeed, if $A \sim^s [A]$, then $[A]$ is inhabited iff A is inhabited, while by Theorem 1.8, $[A]$ is inhabited iff A is uninhabited.

Lemma 4.6. $(\varrho^{\Xi_1})^{\Xi_2} = \varrho^{\Xi_2,\Xi_1}$ *for every substitution* ϱ *and contexts* Ξ_1, Ξ_2.

Proof. By Definition 2.3(vi) we may assume that ϱ is a substitution between contexts, say from Γ to Δ. Recall that ϱ^{Ξ_1} is a substitution from Ξ_1, Γ to Ξ_1, Δ with $\varrho_c^{\Xi_1} = \varrho_c$ for all $c \in \{\Gamma\}$ and $\varrho_d^{\Xi_1} = d$ for all $d \in \{\Xi_1\}$ (see Definition 4.1(iv)). So both $(\varrho^{\Xi_1})^{\Xi_2}$ and ϱ^{Ξ_2,Ξ_1} are a substitution σ from Ξ_2, Ξ_1, Γ to Ξ_2, Ξ_1, Δ such that $\sigma_c = \varrho_c$ for all $c \in \{\Gamma\}$ and $\sigma_d = d$ for all $d \in \{\Xi_1, \Xi_2\}$. Hence $(\varrho^{\Xi_1})^{\Xi_2}$ and ϱ^{Ξ_2,Ξ_1} are the same. □

Lemma 4.7. *Given contexts* Θ, Γ *and* Δ, *we have*

$$\Gamma \leq^s \Delta \quad \implies \quad \Theta, \Gamma \leq^s \Theta, \Delta.$$

Proof. Assume that $\Gamma \leq^s \Delta$, that is, that there is some strong reduction ϱ from Γ to Δ (see Proposition 4.2 and Definition 4.3). To show that $\Theta, \Gamma \leq^s \Theta, \Delta$, we prove that ϱ^{Θ} is a strong reduction from Θ, Γ to Θ, Δ. Writing $\varrho^{\Theta\,\Xi} \triangleq (\varrho^{\Theta})^{\Xi} = \varrho^{\Xi,\Theta}$ (see Lemma 4.6), we need to prove that the map $\hat{\varrho}^{\Theta\,\Xi}\colon \Lambda^{\Xi,\Theta,\Gamma}(0) \to \Lambda^{\Xi,\Theta,\Delta}(0)$ is injective for every context Ξ (see Proposition 4.2). Since ϱ is a strong reduction, we know that $\hat{\varrho}^{\Xi'}\colon \Lambda^{\Xi',\Gamma}(0) \to \Lambda^{\Xi',\Delta}(0)$ is injective for every Ξ'. Now, pick $\Xi' = \Xi, \Theta$. □

Before we get to the more serious reductions, we study the workings of a Böhm transformation $\hat{\varrho}$ (see Definition 2.3(viii)) more closely in Proposition 4.10.

Definition 4.8. Let ϱ be a substitution from ${}^{\Gamma_1}[\Delta_1]$ to ${}^{\Gamma_2}[\Delta_2]$ (see Definition 2.3(vi)). For every type $A \equiv [\Xi]$, let ϱ^A denote the natural extension of ϱ to a substitution from ${}^{\Gamma_1}[\Xi, \Delta_1]$ to ${}^{\Gamma_2}[\Xi, \Delta_2]$ given by $\hat{\varrho}^A(c) = c$ for all $c \in \{\Xi\}$.

Remarks 4.9. Let Ξ be a context and let ϱ a substitution from ${}^{\Gamma_1}[\Delta_1]$ to ${}^{\Gamma_2}[\Delta_2]$.

(i) $\varrho^{[\Xi]}$ (Definition 4.8) and ϱ^{Ξ} (Definition 4.1(iv)) are essentially the same substitution since we have $\varrho_c^{[\Xi]} = \varrho_c^{\Xi}$ for all $c \in \{\Xi, \Gamma_1, \Delta_1\}$ (see Definition 2.3(vi)).

(ii) Using Definition 2.3(viii) we see that for all contexts Ξ, Θ and $M \in \Lambda^{\Xi,\Gamma_1}([\Theta, \Delta_1])$,

$$\varrho^{\Xi\,[\Theta]} M \;=_{\beta\eta}\; M\,[\Gamma_1, \Delta_1 := \vec{\varrho}_{\Gamma_1,\Delta_1}].$$

(iii) We have $\varrho^0 = \varrho = \varrho^{\varepsilon}$.

Proposition 4.10. *A substitution ϱ from Γ to Δ satisfies the 'recursion':*

$$\left\lceil \begin{array}{l} \textit{Given contexts } \Xi,\ \Theta \textit{ and } a^A \in \{\Xi, \Gamma\} \textit{ with } A \equiv [A_1, \ldots, A_n], \textit{ we have} \\ \qquad \hat{\varrho}^{\Xi}(\, a\, M_1 \ldots M_n \,) =_{\beta\eta} \varrho_a^{\Xi}\ (\, \hat{\varrho}^{\Xi\, A_1} M_1 \,) \cdots (\, \hat{\varrho}^{\Xi\, A_n} M_n \,), \\ \qquad \hat{\varrho}^{\Xi\,[\Theta]}(\, \lambda\Theta.\, M \,) = \quad \lambda\Theta.\, (\, \hat{\varrho}^{\Theta,\Xi} M \,), \\ \textit{for all } M_i \in \Lambda^{\Xi,\Gamma}(A_i) \textit{ and } M \in \Lambda^{\Theta,\Xi,\Gamma}(0). \end{array} \right.$$

Proof. It is only a matter of expanding definitions. Indeed,

$$\begin{array}{lll} & \hat{\varrho}^{\Xi}(\, a\, M_1 \cdots M_n \,) & \\ =_{\beta\eta} & (\, a\, M_1 \cdots M_n \,)[\Gamma := \vec{\varrho}_{\Gamma}\,] & \text{by Rem. 4.9(ii),(iii)} \\ =_{\beta\eta} & \varrho_a^{\Xi}\ M_1[\Gamma := \vec{\varrho}_{\Gamma}\,]\ \cdots\ M_n[\Gamma := \vec{\varrho}_{\Gamma}\,] & \\ =_{\beta\eta} & \varrho_a^{\Xi}\ (\, \hat{\varrho}^{\Xi\, A_1} M_1 \,) \cdots (\, \hat{\varrho}^{\Xi\, A_n} M_n \,) & \text{by Rem. 4.9(ii),} \end{array}$$

where $a^A \in \{\Xi, \Gamma\}$ with $A \equiv [A_1, \ldots, A_n]$ and $M_i \in \Lambda^{\Xi,\Gamma}(A_i)$. Similarly,

$$\begin{array}{lll} \hat{\varrho}^{\Xi\,[\Theta]}(\, \lambda\Theta.\, M \,) =_{\beta\eta} (\lambda\Theta.\, M)[\Gamma := \vec{\varrho}_{\Gamma}] & & \text{by Rem. 4.9(ii)} \\ \qquad = \quad \lambda\Theta.\, (\, M[\Gamma := \vec{\varrho}_{\Gamma}]\,) & & \\ \qquad =_{\beta\eta} \lambda\Theta.\, (\, \hat{\varrho}^{\Theta,\Xi} M \,) & & \text{by Rem. 4.9(ii),(iii)} \end{array}$$

for every term $M \in \Lambda^{\Theta,\Xi,\Gamma}(0)$. □

We now give an important condition for a substitution to be a strong reduction.

Theorem 4.11. *Let Γ and Δ be contexts. Let ϱ be a substitution from Γ to Δ. If ϱ has the following property, then ϱ is a strong reduction.*

$$\left\lceil \begin{array}{l} \textit{Given } a^A, b^B \in \{\Xi, \Gamma\} \textit{ with } A \equiv [A_1, \ldots, A_n] \textit{ and } B \equiv [B_1, \ldots, B_m]. \textit{ Then} \\ \qquad \varrho_a^{\Xi}\, M_1 \cdots M_n \;=_{\beta\eta}\; \varrho_b^{\Xi}\, N_1 \cdots N_m \quad \Longrightarrow \quad a = b \;\textit{ and }\; M_i = N_i \qquad (4.1) \\ \textit{for all } M_i \in \Lambda^{\Xi,\Delta}(A_i) \textit{ and } N_i \in \Lambda^{\Xi,\Delta}(B_i) \textit{ and every context } \Xi. \end{array} \right.$$

Proof. To prove that ϱ is a strong reduction, we need to show that for each context Ξ, the Böhm transformation $\hat{\varrho}^{\Xi} \colon \Lambda^{\Xi,\Gamma}(0) \to \Lambda^{\Xi,\Delta}(0)$ is injective (see Definition 4.3). So, consider for each context–type ${}^{\Xi}C$ (see Definition 2.3) and $M \in \Lambda^{\Xi,\Gamma}(C)$ the property $P(M)$:

$$\hat{\varrho}^{\Xi\, C}(M) = \hat{\varrho}^{\Xi\, C}(N) \quad \Longrightarrow \quad M = N \qquad \text{for all } N \in \Lambda^{\Xi,\Gamma}(C).$$

It suffices to prove that $P(M)$ for all M, because then (taking $C = 0$),

$$\hat{\varrho}^{\Xi}(M) = \hat{\varrho}^{\Xi}(N) \qquad \Longrightarrow \quad M = N \qquad \text{for all } M, N \in \Lambda^{\Xi,\Gamma}(0)$$

for each context Ξ, so each Böhm transformation $\hat{\varrho}^{\Xi}$ is injective.

To prove that $P(M)$ for all M, we use induction on M. There are two cases.

$$\text{(I)}\ \ M \equiv a^A\, M_1 \cdots M_n; \qquad \text{(II)}\ \ M \equiv \lambda\Theta.\, M'.$$

(I) We have $M \equiv a^A\, M_1 \cdots M_n$ where $a^A \in \{\Xi, \Gamma\}$ and where writing $A \equiv [A_1, \ldots, A_n]$ we have $M_i \in \Lambda^{\Xi,\Gamma}(A_i)$. Assume $P(M_i)$ in order to show that $P(M)$.

Let $N \in \Lambda^{\Xi,\Gamma}(0)$ with $\hat{\varrho}^{\Xi} M = \hat{\varrho}^{\Xi} N$ be given. We need to prove that $M = N$. We have $N \equiv b^B\, N_1 \cdots N_m$ for some $b^B \in \{\Xi, \Gamma\}$ with $B \equiv [B_1, \ldots, B_m]$ and $N_i \in \Lambda^{\Xi,\Gamma}(B_i)$. Then by Proposition 4.10, $\hat{\varrho}^{\Xi} M = \hat{\varrho}^{\Xi} N$ implies

$$\varrho_a^{\Xi}\ (\ \hat{\varrho}^{\Xi\, A_1} M_1\ ) \cdots (\ \hat{\varrho}^{\Xi\, A_n} M_n\ )\ =_{\beta\eta}\ \varrho_b^{\Xi}\ (\ \hat{\varrho}^{\Xi\, B_1} N_1\ ) \cdots (\ \hat{\varrho}^{\Xi\, B_m} N_m\ ).$$

Now, $\hat{\varrho}^{\Xi\, A_i} M_i \in \Lambda^{\Xi,\Delta}(A_i)$ and $\hat{\varrho}^{\Xi\, B_i} N_i \in \Lambda^{\Xi,\Delta}(B_i)$, so by Statement (4.1), $a = b$ and $\hat{\varrho}^{\Xi\, A_i} M_i = \hat{\varrho}^{\Xi\, B_i} N_i$. Then $M_i = N_i$ by $P(M_i)$, so $M = N$. Hence $P(M)$.

(II) We have $M \equiv \lambda\Theta.\, M'$ where Θ is some context and $M' \in \Lambda^{\Theta,\Xi,\Gamma}(0)$. Assume that $P(M')$ in order to show that $P(M)$.

Let $N \in \Lambda^{\Xi,\Gamma}([\Theta])$ with $\hat{\varrho}^{\Xi\,[\Theta]} M = \hat{\varrho}^{\Xi\,[\Theta]} N$ be given. We need to prove that $M = N$. Write $N \equiv \lambda\Theta.\, N'$ where $N' \in \Lambda^{\Theta,\Xi,\Gamma}(0)$. Then by Proposition 4.10 we have

$$\lambda\Theta.\ (\ \hat{\varrho}^{\Theta,\Xi} M'\ )\ =\ \lambda\Theta.\ (\ \hat{\varrho}^{\Theta,\Xi} N'\ ).$$

Then $\hat{\varrho}^{\Theta,\Xi} M' = \hat{\varrho}^{\Theta,\Xi} N'$, and thus $M' = N'$ by $P(M')$. Hence $M = N$ and so $P(M)$.

So we see that $P(M)$ for all M. Hence ϱ is a strong reduction. □

4.2. **Atomic reductions.**

Definition 4.12. (i) Let Γ and Δ be contexts. A substitution from Γ to Δ is called an **atomic reduction** if it satisfies condition (4.1) of Theorem 4.11.

(ii) A substitution ϱ from ${}^{\Gamma_1}[\Delta_1]$ to ${}^{\Gamma_2}[\Delta_2]$ is called an **atomic reduction** if ϱ, considered as substitution from Γ_1, Δ_1 to Γ_2, Δ_2 (see Definition 2.3(vi)), is an atomic reduction. In that case we write $\varrho\colon {}^{\Gamma_1}[\Delta_1] \leq^a {}^{\Gamma_2}[\Delta_2]$.

(iii) We say that ${}^{\Gamma_1}[\Delta_1]$ **atomically reduces to** ${}^{\Gamma_2}[\Delta_2]$ if there is an atomic reduction from ${}^{\Gamma_1}[\Delta_1]$ to ${}^{\Gamma_2}[\Delta_2]$. In that case we write ${}^{\Gamma_1}[\Delta_1] \leq^a {}^{\Gamma_2}[\Delta_2]$.

Remark 4.13. Given variables a and b, we have (cf. Statement (4.1))

$$a\, M_1 \cdots M_n\ =_{\beta\eta}\ b\, N_1 \cdots N_m \qquad \Longrightarrow \qquad a = b\ \text{ and } M_i = N_i$$

for all terms M_i and N_i. In this respect the terms ϱ_a^{Ξ}s of an *atomic reduction* ϱ behave similar to *atomic terms* (=variables). Hence the name.

Remark 4.14. Given context–types ${}^{\Gamma_1}\Delta_1$ and ${}^{\Gamma_2}\Delta_2$ we have

$${}^{\Gamma_1}[\Delta_1] \leq^a {}^{\Gamma_2}[\Delta_2] \qquad \Longleftrightarrow \qquad \Gamma_1, \Delta_1 \leq^a \Gamma_2, \Delta_2$$

by Definition 4.12(ii). Cf. Definition 4.1(iii).

Proposition 4.15. *For context–types* ${}^{\Gamma_1}[\Delta_1]$ *and* ${}^{\Gamma_2}[\Delta_2]$ *we have*

$${}^{\Gamma_1}[\Delta_1] \leq^a {}^{\Gamma_2}[\Delta_2] \qquad \Longrightarrow \qquad {}^{\Gamma_1}[\Delta_1] \leq^s {}^{\Gamma_2}[\Delta_2].$$

Proof. Assume that ${}^{\Gamma_1}[\Delta_1] \leq^a {}^{\Gamma_2}[\Delta_2]$. That is, there is some atomic reduction ϱ from ${}^{\Gamma_1}[\Delta_1]$ to ${}^{\Gamma_2}[\Delta_2]$. By Theorem 4.11, ϱ is also a strong reduction. Hence ${}^{\Gamma_1}[\Delta_1] \leq^s {}^{\Gamma_2}[\Delta_2]$ by Proposition 4.2 and Definition 4.3. □

Below we have collected the calculation rules for $\leq^a$ which we use later on. The reader can chose to skip them at first and proceed to Remark 4.31.

Lemma 4.16. *Let Γ, Δ and Θ be contexts.*

(i) If $\{\Gamma\} \subseteq \{\Delta\}$ then $\Gamma \leq^a \Delta$.
(ii) If $\Gamma \leq^a \Delta$ then $\Theta, \Gamma \leq^a \Theta, \Delta$.

Proof. (i). Assume that $\{\Gamma\} \subseteq \{\Delta\}$. To prove $\Gamma \leq^a \Delta$, we need to find an atomic reduction from Γ to Δ (see Definition 4.12(iii)). Let ϱ be the substitution from Γ to Δ given by $\varrho_c = c$ for all $c^C \in \{\Gamma\}$ (see Def. 2.3(v)). To prove that ϱ is an atomic reduction, we need to show that given a context Ξ and $a^A, b^B \in \{\Xi, \Gamma\}$ with $A \equiv [A_1, \ldots, A_n]$ and $B \equiv [B_1, \ldots, B_m]$,

$$\varrho_a^\Xi\, M_1 \cdots M_n =_{\beta\eta} \varrho_b^\Xi\, N_1 \cdots N_m \quad \Longrightarrow \quad a = b \text{ and } M_i = N_i$$

for all $M_i \in \Lambda^{\Xi,\Delta}(A_i)$ and $N_i \in \Lambda^{\Xi,\Delta}(B_i)$. Since all ϱ_a^Ξ are distinct variables, this follows immediately from Remark 4.13.
(ii). Assume that $\Gamma \leq^a \Delta$, that is, that there is some atomic reduction ϱ from Γ to Δ (see Definition 4.12(iii)). To prove that $\Theta, \Gamma \leq^a \Theta, \Delta$, we show that ϱ^Θ is an atomic reduction from Θ, Γ to Θ, Δ. For this we must prove that

$$(\varrho^\Theta)_a^{\Xi'}\, M_1 \cdots M_n =_{\beta\eta} (\varrho^\Theta)_b^{\Xi'}\, N_1 \cdots N_m \quad \Longrightarrow \quad a = b \text{ and } M_i = N_i \tag{4.2}$$

for every context Ξ' and appropriate a, b, M_i and N_i (see Definition 4.12(ii)). Since we have that $(\varrho^\Theta)^{\Xi'} = \varrho^{\Xi',\Theta}$ (see Lemma 4.6), Statement (4.2) follows immediately from the fact that ϱ is an atomic reduction. (Indeed, pick $\Xi = \Xi', \Theta$). □

Corollary 4.17. *Given types $C_1, \ldots, C_k$ and a permutation φ of $\{1, \ldots, k\}$, we have*

$$[C_1, \ldots, C_k] \leq^a [C_{\varphi(1)}, \ldots, C_{\varphi(k)}].$$

Proof. Write $[C_1, \ldots, C_k] = [\Gamma]$ and $[C_{\varphi(1)}, \ldots, C_{\varphi(k)}] = [\varphi{\cdot}\Gamma]$. We must prove that $\Gamma \leq^a \varphi{\cdot}\Gamma$ (see Remark 4.14). This follows immediately from Lemma 4.16(i) since $\{\Gamma\} = \{\varphi \cdot \Gamma\}$. □

Lemma 4.18. *A substitution ϱ from Γ to Δ is an atomic reduction provided that*

(i) If $a^A, b^B \in \{\Gamma\}$ with $A \equiv [A_1, \ldots, A_n]$ and $B \equiv [B_1, \ldots, B_m]$, then

$$\varrho_a\, M_1 \cdots M_n =_{\beta\eta} \varrho_b\, N_1 \cdots N_m \quad \Longrightarrow \quad a = b \text{ and } M_i = N_i$$

for all $M_i \in \Lambda^{\Xi,\Delta}(A_i)$ and $N_i \in \Lambda^{\Xi,\Delta}(B_i)$ and every context Ξ.

(ii) If $a^A \in \{\Gamma\}$, $d^D \in \{\Xi\}$ with $A \equiv [A_1, \ldots, A_n]$, $D \equiv [D_1, \ldots, D_\ell]$, then

$$\varrho_a\, M_1 \cdots M_n \neq_{\beta\eta} d\, N_1 \cdots N_\ell$$

for all $M_i \in \Lambda^{\Xi,\Delta}(A_i)$ and $N_i \in \Lambda^{\Xi,\Delta}(D_i)$.

Proof. Let $a, b \in \{\Gamma, \Xi\}$ with $A \equiv [A_1, \ldots, A_n]$ and $B \equiv [B_1, \ldots, B_m]$ and

$$\varrho_a^\Xi\, M_1 \cdots M_n =_{\beta\eta} \varrho_b^\Xi\, N_1 \cdots N_m$$

for some $M_i \in \Lambda^{\Xi,\Delta}(A_i)$ and $N_i \in \Lambda^{\Xi,\Delta}(B_i)$ and some context Ξ. We need to prove $a = b$ and $M_i = N_i$ (see Definition 4.12(i)). We distinguish four cases.

(i) If $a, b \in \{\Gamma\}$ then $\varrho_a^\Xi = \varrho_a$, so $a = b$ and $M_i = N_i$ by Assumption (i).
(ii) If $a, b \in \{\Xi\}$ then $\varrho_a^\Xi = a$, so $a = b$ and $M_i = N_i$ by Remark 4.13.

(iii) The situation that $a \in \{\Gamma\}$ and $b \in \{\Xi\}$ does not occur, since by Assumption (ii) we have $\varrho_a^\Xi\, M_1 \cdots M_n \neq_{\beta\eta} b\, N_1 \cdots N_m \equiv \varrho_b^\Xi\, N_1 \cdots N_m$.
(iv) Similarly, the situation that $a \in \{\Xi\}$ and $b \in \{\Gamma\}$ does not happen. □

Lemma 4.19. *Given types $C_1, \ldots, C_k$ and a permutation φ of $\{1, \ldots, k\}$, we have*

$$[[C_1, \ldots, C_k]] \leq^a [[C_{\varphi(1)}, \ldots, C_{\varphi(k)}]].$$

Proof. Writing $[C_1, \ldots, C_k] = [\Gamma]$ and $[C_{\varphi(1)}, \ldots, C_{\varphi(k)}]$, we must prove that $[[\Gamma]] \leq^a [[\varphi \cdot \Gamma]]$. By Remark 4.14 we need to find an atomic reduction from $F^{[\Gamma]}$ to $G^{[\varphi\cdot\Gamma]}$. We show that the substitution ϱ from F to G given by

$$\varrho_F \triangleq \lambda\Gamma.\, G(\varphi \cdot \Gamma) \tag{4.3}$$

is an atomic reduction. For this we use Lemma 4.18. Let Ξ be a context.

(i) Suppose that $\varrho_F M_1 \cdots M_k =_{\beta\eta} \varrho_F N_1 \cdots N_k$ for some $M_i, N_i \in \Lambda^{\Xi,G}(C_i)$. We need to prove that $M_i = N_i$. Indeed, $\varrho_F M_1 \cdots M_k =_{\beta\eta} \varrho_F N_1 \cdots N_k$ yields
$$GM_{\varphi(1)} \cdots M_{\varphi(k)} = GN_{\varphi(1)} \cdots N_{\varphi(k)}.$$
Hence $M_{\varphi(i)} = N_{\varphi(i)}$ and thus $M_i = N_i$.
(ii) Let $d^D \in \{\Xi\}$ with $D \equiv [D_1, \ldots, D_\ell]$ and $\varrho_F M_1 \cdots M_k =_{\beta\eta} \varrho_d^\Xi N_1 \cdots N_\ell$ for some $M_i \in \Lambda^{\Xi,G}(C_i)$ and $N_i \in \Lambda^{\Xi,G}(D_i)$. We need to reach a contradiction. Indeed, we get $GM_{\varphi(1)} \cdots M_{\varphi(k)} = bN_1 \cdots N_k$, so $G = b$, quod non. □

Remark 4.20. Given a type $C \equiv [C_1, \ldots, C_k]$, the order $C_1, \ldots, C_k$ of the components is largely immaterial. Witnesses of this principle include Corollary 4.17, Lemma 4.19 and Definition 2.3(v). We will often use this principle implicitly. For instance, we will use Lemma 4.7 to argue that $\Gamma \leq^s \Delta \implies \Gamma, \Theta \leq^s \Delta, \Theta$. (Of course, this is licit by Corollary 4.17.)

Lemma 4.21. *Let A, B and $C_1, \ldots, C_k$ be types. Then*

$$A \leq^s B \quad \Longrightarrow \quad [[A, C_1, \ldots, C_k]] \leq^a [[B, C_1, \ldots, C_k]].$$

Proof. Assume that $A \leq^s B$ to find an atomic reduction from $F^{[A,C_1,\ldots,C_k]}$ and $G^{[B,C_1,\ldots,C_k]}$ (see Remark 4.14). Pick a strong reduction $\sigma\colon A \leq^s B$ with reducing term $S\colon A \to B$. Define a substitution ϱ from F to G by

$$\varrho_F \triangleq \lambda a^A \Gamma.\, G(Sa)\Gamma,$$

where $\Gamma \triangleq c_1^{C_1}, \ldots, c_k^{C_k}$. We prove that ϱ is an atomic reduction by Lemma 4.18.

(i) Let Ξ be a context and suppose that

$$\varrho_F\, M\, M_1 \cdots M_k =_{\beta\eta} \varrho_F\, N\, N_1 \cdots N_k \tag{4.4}$$

for some $M, N \in \Lambda^{\Xi,G}(A)$ and $M_i, N_i \in \Lambda^{\Xi,G}(C_i)$ in order to prove $M_i = N_i$ and $M = N$. Equation (4.4) yields $G(SM)M_1 \cdots M_k =_{\beta\eta} G(SN)N_1 \cdots N_k$, so $M_i = N_i$ and $SM =_{\beta\eta} SN$. Hence $M = N$ too, since σ is a strong reduction.
(ii) Let Ξ be a context and $d^D \in \{\Xi\}$ with $D = [D_1, \ldots, D_\ell]$. Suppose that

$$\varrho_F\, M\, M_1 \cdots M_k =_{\beta\eta} d\, N_1 \cdots N_\ell$$

for some $M \in \Lambda^{\Xi,G}(A)$, $M_i \in \Lambda^{\Xi,G}(C_i)$ and $N_i \in \Lambda^{\Xi,G}(D_i)$ to reach a contradiction. We get $G(SM)M_1 \cdots M_k =_{\beta\eta} dN_1 \cdots N_\ell$. But then $G = d$, which is absurd. □

Lemma 4.22. *$A \leq^a [\![A]\!]$ for every type A.*

Proof. Write $A = [\Gamma]$ for some context Γ. It suffices to find an atomic reduction ϱ from Γ to $F^{[\![\Gamma]\!]}$ (see Remark 4.14). Let ϱ be the substitution from Γ to $F^{[\![\Gamma]\!]}$ given by

$$\varrho_d \triangleq \lambda\Delta.\, F\, \lambda\Gamma.\, d\, \Delta \quad \text{for each } d^{[\Delta]} \text{ from } \{\Gamma\}.$$

We prove that ϱ is atomic using Lemma 4.18. Let Ξ be some context.

(i) Let $d^D, e^E \in \{\Gamma\}$ with $D \equiv [D_1, \ldots, D_k]$ and $E \equiv [E_1, \ldots, E_\ell]$. Assume that

$$\varrho_d\, M_1 \cdots M_k \;=_{\beta\eta}\; \varrho_e\, N_1 \cdots N_\ell \tag{4.5}$$

for some $M_i \in \Lambda^{\Xi,F}(D_i)$ and $N_i \in \Lambda^{\Xi,F}(E_i)$, to prove $d = e$ and $M_i = N_i$. It is easy to see that Equation (4.5) implies that

$$F\lambda\Gamma.\, d\, M_1 \cdots M_k \;=\; F\lambda\Gamma.\, e\, N_1 \cdots N_\ell.$$

Hence $d\, M_1 \cdots M_k = e\, M_1 \cdots M_\ell$ and thus $d = e$ and $M_i = N_i$.

(ii) Given $d^D \in \{\Gamma\}$, $b^B \in \{\Xi\}$ with $D \equiv [D_1, \ldots, D_k]$, $B \equiv [B_1, \ldots, B_n]$, assume that there are $M_i \in \Lambda^{\Xi,F}(D_i)$ and $N_i \in \Lambda^{\Xi,F}(B_i)$ such that

$$\varrho_d\, M_1 \cdots M_k \;=_{\beta\eta}\; b\, N_1 \cdots N_m \tag{4.6}$$

in order to obtain a contradiction. This is easy; Equation (4.6) implies

$$F\lambda\Gamma.\, d\, M_1 \cdots M_k \;=\; b\, N_1 \cdots N_m,$$

and thus $F = b$, qoud non. □

Lemma 4.23. *Given types $A_1, \ldots, A_n$ we have*

$$[\,[A_1, \ldots, A_n]\,] \;\leq^a\; [\,[A_1], \ldots, [A_n], [0^n]\,].$$

Proof. By Remark 4.14 it suffices to show that the substitution from the context $F^{[A_1,\ldots,A_n]}$ to the context $\Theta \triangleq F_1^{[A_1]}, \ldots, F_n^{[A_n]}, p^{[0^n]}$ given by

$$\varrho_F \;\triangleq\; \lambda m_1^{A_1} \cdots m_n^{A_n}.\;\; p\,(F_1 m_1) \cdots (F_n m_n)$$

is an atomic reduction. To prove this, we use Lemma 4.18.

(i) Let Ξ be a context and suppose that

$$\varrho_F\, M_1 \cdots M_n \;=_{\beta\eta}\; \varrho_F\, N_1 \cdots N_n$$

for some $M_i, N_i \in \Lambda^{\Xi,\Theta}([A_1, \ldots, A_n])$ in order to show $M_i = N_i$. We have

$$p\,(F_1 M_1) \cdots (F_n M_n) \;=_{\beta\eta}\; p\,(F_1 N_1) \cdots (F_n N_n).$$

So we get $F_i M_i =_{\beta\eta} F_i N_i$. Thus $M_i = N_i$.

(ii) As easy as before. □

Lemma 4.24. *Given a set of types $\mathcal{A} \equiv \{A_1, \ldots, A_n\}$ we have*

$$[\,[\![C_1, \ldots, C_k]\!]\,] \;\leq^a\; [\,[\![A_1]\!],\, \ldots,\, [\![A_n]\!]\,]$$

for all $C_1, \ldots, C_k \in \mathcal{A}$.

Proof. Writing $\Gamma \triangleq c_1^{C_1}, \ldots, c_k^{C_k}$ it suffices to prove that the substitution ϱ from the context $F^{[\![\Gamma]\!]}$ to $\Theta \triangleq F_{A_1}^{[\![A_1]\!]}, \ldots, F_{A_n}^{[\![A_n]\!]}$ given by the assignment

$$\varrho_F \;\triangleq\; \lambda m^{[\Gamma]}.\;\; F_{C_1}\lambda c_1^{C_1}.\;\; \cdots\;\; F_{C_k}\lambda c_k^{C_k}.\;\; m\,\Gamma$$

is an atomic reduction (see Remark 4.14). For this we use Lemma 4.18.

(i) Let Ξ be a context and suppose that

$$\varrho_F M \;=_{\beta\eta}\; \varrho_F N$$

for some $M, N \in \Lambda^{\Xi,\Theta}([\Gamma])$ in order to prove $M = N$. We get

$$F_{C_1}\lambda c_1^{C_1}.\;\; \cdots\;\; F_{C_k}\lambda c_k^{C_k}.\;\; M\,\Gamma \;=_{\beta\eta}\; F_{C_1}\lambda c_1^{C_1}.\;\; \cdots\;\; F_{C_k}\lambda c_k^{C_k}.\;\; N\,\Gamma.$$

Hence $M\Gamma =_{\beta\eta} N\Gamma$, so $M = N$.

(ii) Again trivial. □

Corollary 4.25. *For every type A and $k \geq 1$, we have $[[\![A^k]\!]] \leq^a [[\![A]\!]]$.*

Proof. Apply Lemma 4.24 with $\mathcal{A} \triangleq \{A\}$ and $C_1, \ldots, C_k \triangleq A, \ldots, A$. □

Lemma 4.26. *Let A be a type. Then $[[\![A]\!]] \leq^a [3, A]$.*

Proof. Let ϱ be the substitution from $F^{[\![A]\!]}$ to $\Theta \triangleq \Phi^3, a^A$ given by

$$\varrho_F \;\triangleq\; \lambda m^{[A]}.\; \Phi\lambda f^1.\; m(fa).$$

We prove that ϱ is an atomic reduction (and thus $[[\![A]\!]] \leq^a [A, 3]$) using Lemma 4.18.

(i) Let Ξ be a context and suppose that

$$\varrho_F M \;=_{\beta\eta}\; \varrho_F N$$

for some $M, N \in \Lambda^{\Xi,\Theta}([A])$ in order to prove $M = N$. We have

$$\Phi\lambda f^1.\; M(fa) =_{\beta\eta} \Phi\lambda f^1.\; N(fa)$$

and thus $M(fa) =_{\beta\eta} N(fa)$. Since f does not occur in M and N we have

$$M(Ra) \;=_{\beta\eta} N(Ra)$$

for every term R of type 1. If we pick $R \triangleq \lambda x^0.\; a_1$ for fresh a_1^0, we get

$$Ma_1 =_{\beta\eta} Na_1.$$

So we see that $M = N$.

(ii) Easy. □

For the proof of Lemma 4.28 we need the following fact concerning terms.

Lemma 4.27. *Let E be a type and let c^E, d^E be variables. Then we have*

$$\left.\begin{array}{l} M[x{:=}c] \;=\; N[x{:=}c] \\ M[x{:=}d] \;=\; N[x{:=}d] \end{array}\right] \quad\Longrightarrow\quad M = N \tag{4.7}$$

for all terms M, N (which might contain c and d).

Proof. Write $\Theta \triangleq x^E, c^E, d^E$. Given a context–type ${}^{\Xi}C$ and $M \in \Lambda^{\Xi,\Theta}(C)$ let $P(M)$ be the property that Statement (4.7) holds for M and any $N \in \Lambda^{\Xi,\Theta}(C)$. With induction we prove that $P(M)$ holds for every M. This is sufficient.

(i) Suppose that $M \equiv a\, M_1 \cdots M_n$ for some $a^A \in \{\Xi, \Theta\}$ with $A \equiv [A_1, \ldots, A_n]$ and $M_i \in \Lambda^{\Xi,\Theta}(A_i)$ with $P(M_i)$ and let $N \in \Lambda^{\Xi,\Theta}(0)$ be such that

$$M[x{:=}c] = N[x{:=}c] \qquad \text{and} \qquad M[x{:=}d] = N[x{:=}d]. \tag{4.8}$$

Write $N \equiv b\, N_1 \cdots N_m$ where $b^B \in \{\Xi, \Theta\}$ with $B \equiv [B_1, \ldots, B_m]$. We need to prove that $M = N$. Note that Statement (4.8) implies that

$$a[x{:=}c] \;=\; b[x{:=}c] \qquad \text{and} \qquad a[x{:=}d] \;=\; b[x{:=}d]. \tag{4.9}$$

By examining the different cases for a (viz., $a = c$, $a = d$, $a = x$ and $a \in \{\Xi\}$) and similarly for b, one easily sees that Statement (4.9) implies $a = b$.

Statement (4.8) also implies $M_i[x{:=}c] \;=\; N_i[x{:=}c]$ and $M_i[x{:=}d] \;=\; N_i[x{:=}d]$.

Consequently, $M_i = N_i$ as $P(M_i)$ by assumption. Hence $M = N$.

(ii) Suppose that $M \equiv \lambda\Delta.\, M'$ for some $M' \in \Lambda^{\Delta,\Xi,\Theta}(0)$ with $P(M')$. Assume that

$$M[x{:=}c] = N[x{:=}c] \qquad \text{and} \qquad M[x{:=}d] = N[x{:=}d] \tag{4.10}$$

for some $N \in \Lambda^{\Xi,\Theta}([\Delta])$ in order to show that $M = N$. Writing $N \equiv \lambda\Delta.\, N'$ with $N' \in \Lambda^{\Delta,\Xi,\Theta}(0)$, we see that Statement (4.10) implies that

$$\lambda\Delta.\, M'[x{:=}c] \;=\; \lambda\Delta.\, N'[x{:=}c].$$

Hence $M'[x{:=}c] = N'[x{:=}c]$. Similarly, we get $M'[x{:=}d] = N'[x{:=}d]$.

Then $P(M')$ implies $M' = N'$, so that $M = N$. □

Lemma 4.28. *For any type A we have $[[\![A]\!]] \leq^a [[0,0], A, A]$.*

Proof. We need to find an atomic reduction from $F^{[\![A]\!]}$ to $\Theta \triangleq b^{[0,0]}, c^A, d^A$ (see Remark 4.14). Let ϱ be the substitution from F to Θ given by

$$\varrho_F \;\triangleq\; \lambda m^{[A]}.\, b\,(mc)(md).$$

We prove that ϱ is an atomic reduction using Lemma 4.18.

(i) Let Ξ be a context and suppose that

$$\varrho_F M \;=_{\beta\eta}\; \varrho_F N$$

for some $M, N \in \Lambda^{\Xi,\Theta}([A])$ in order to prove $M = N$. By reduction we get

$$b\,(Mc)(Md) \;=_{\beta\eta}\; b\,(Nc)(Nd).$$

Hence $Mc =_{\beta\eta} Nc$, $Md =_{\beta\eta} Nd$. Writing $M \equiv \lambda x^A.\, M'$, $N \equiv \lambda x^A.\, N'$, we get

$$N'[x{:=}c] \;=\; M'[x{:=}c] \qquad \text{and} \qquad N'[x{:=}d] \;=\; M'[x{:=}d].$$

Hence $M' = N'$ by Lemma 4.27, and thus $M = N$.

(ii) Simple. □

Lemma 4.29. *Let Γ be a context and $F^A \in \{\Gamma\}$ with $A \equiv [[\Gamma_1], \ldots, [\Gamma_n]]$. Then*

$$\Theta \leq^a \Gamma, \Gamma_k \quad \Longrightarrow \quad \Theta \leq^a \Gamma$$

for every context Θ and $k \in \{1, \ldots, n\}$ such that

$$[\Gamma_i, \Delta, \Gamma] \text{ is inhabited} \qquad \text{for all } t^{[\Delta]} \in \{\Theta\},\ i \neq k. \tag{4.11}$$

Proof. Assume that $\Theta \leq^a \Gamma, \Gamma_k$ for some Θ and k. By Definition 4.12, there is an atomic reduction ϱ from Θ to Γ, Γ_k. In order to prove that $\Theta \leq^a \Gamma$, we need to find an atomic reduction from Θ to Γ. Pick terms $H_i^t \in \Lambda^{\Gamma_i, \Delta, \Gamma}(0)$ for every $i \neq k$ and $t^{[\Delta]} \in \{\Theta\}$; this is possible by Statement (4.11). Now, let σ be the substitution from Θ to Γ given by

$$\sigma_t \triangleq \lambda\Delta.\, F\, M_1^t \cdots M_n^t; \qquad M_i^t \triangleq \begin{cases} \lambda\Gamma_k.\, \varrho_t \Delta & \text{if } i = k \\ \lambda\Gamma_i.\, H_i^t & \text{otherwise} \end{cases}$$

for every $t^{[\Delta]} \in \{\Theta\}$. We use Lemma 4.18 to prove that σ is an atomic reduction.

(i) Let $s^S, t^T \in \{\Theta\}$ with $S \equiv [S_1, \ldots, S_k]$ and $T \equiv [T_1, \ldots, T_\ell]$. Suppose

$$\sigma_s\, U_1 \cdots U_k \ =_{\beta\eta}\ \sigma_t\, V_1 \cdots V_\ell$$

for certain $U_i \in \Lambda^{\Xi,\Gamma}(S_i)$, $V_i \in \Lambda^{\Xi,\Gamma}(T_i)$ and some Ξ, to prove $U_i = V_i$. Then

$$F\, M_1^s[\Delta{:=}\vec{U}] \cdots M_n^s[\Delta{:=}\vec{U}] \ =_{\beta\eta}\ F\, M_1^t[\Delta{:=}\vec{V}] \cdots M_n^t[\Delta{:=}\vec{V}].$$

Hence $M_i^s[\Delta{:=}\vec{U}] =_{\beta\eta} M_i^t[\Delta{:=}\vec{V}]$. For $i = k$, we get

$$\lambda\Gamma_k.\, \varrho_s\, U_1 \cdots U_k \ =_{\beta\eta}\ \lambda\Gamma_k.\, \varrho_t\, V_1 \cdots V_\ell.$$

Thus $\varrho_s\, U_1 \cdots U_k =_{\beta\eta} \varrho_t\, V_1 \cdots V_\ell$. Hence $s = t$ and $U_i = V_i$.

(ii) Trivial. □

Lemma 4.30. *Let Γ be a context and Δ a derivative of Γ (see Definition 3.4). Then*

$$\Theta \leq^a \Delta \quad \Longrightarrow \quad \Theta \leq^a \Gamma$$

for every context Θ such that

$$[\Gamma, \Xi] \text{ is inhabited} \qquad \text{for all } t^{[\Xi]} \in \{\Theta\}. \tag{4.12}$$

Proof. Let Γ and Θ with $\Theta \leq^a \Gamma$ be given and suppose Statement (4.12) holds. We prove that $\Theta \leq^a \Delta$ for every derivative Δ of Γ with induction on Δ.

Let Δ be a derivative of the context Γ and let Δ' be a direct derivative of Δ. Assume that $\Theta \leq^a \Delta$. We need to prove that $\Theta \leq^a \Delta'$.

By Definition 3.4, $\Delta' \equiv \Delta, \Delta_k$ for some $F^A \in \{\Delta\}$ with $A \equiv [[\Delta_1], \ldots, [\Delta_n]]$. So we apply Lemma 4.29 to prove $\Theta \leq^a \Delta'$. We must show that $[\Delta_i, \Xi, \Delta]$ is inhabited for every $t^{[\Xi]} \in \{\Theta\}$ and $i \neq k$.

Since Δ is a derivative of Γ, we have $\{\Gamma\} \subseteq \{\Delta\}$. Hence $\{\Gamma, \Xi\} \subseteq \{\Delta_i, \Xi, \Delta\}$. So to prove that $[\Delta_i, \Xi, \Delta]$ is inhabited, it suffices to show $[\Gamma, \Xi]$ is inhabited. This is Statement (4.12). □

Remark 4.31. Surprisingly, it is not clear whether the relation $\leq^a$ is transitive.

4.3. **Atomic types.** We are interested in types A with the property

$$[\Gamma_1] \leq^s A \quad \text{and} \quad [\Gamma_2] \leq^s A \quad \Longrightarrow \quad [\Gamma_1, \Gamma_2] \leq^s A, \tag{4.13}$$

as this property makes it easier to find reductions to A. For instance, to prove $[3,0,0] \leq^s A$, it suffices to show that both $[3] \leq^s A$ and $[0] \leq^s A$. In this subsection we give a criterion (namely $[1,1] \leq^a A$) for a type to satisfy Statement (4.13).

Definition 4.32. A context–type ${}^{\Gamma}A$ is **atomic** if $[1,1] \leq^a {}^{\Gamma}A$.

Remark 4.33. A context–type ${}^{\Gamma}[\Delta]$ is atomic iff Γ, Δ is atomic by Remark 4.14. In particular, a type $[\Delta]$ is atomic iff the context Δ is atomic.

Lemma 4.34. *A context Θ is atomic iff there are terms $X_1, X_2 \in \Lambda^{\Theta}(1)$ with:*

(i) For every context Ξ and for all $M, N \in \Lambda^{\Xi,\Theta}(0)$,

$$X_i M =_{\beta\eta} X_j N \quad \Longrightarrow \quad i = j \;\; \text{and} \;\; M = N.$$

(ii) For every context Ξ and all $d^D \in \{\Xi\}$ with $D \equiv [D_1, \ldots, D_\ell]$,

$$X_i M \;\neq_{\beta\eta}\; d\, N_1 \cdots N_\ell,$$

where $M \in \Lambda^{\Xi,\Theta}(0)$ and $N_i \in \Lambda^{\Xi,\Theta}(D_i)$.

Proof. Simply expand Definition 2.3(v) in Lemma 4.18. □

Definition 4.35. Let Θ be a context. A pair of terms $X_1, X_2 \in \Lambda^{\Theta}(1)$ which satisfies conditions (i) and (ii) of Lemma 4.34 will be called an **atomic pair**.

Before we give some examples of atomic types, we prove (as promised) that an atomic type satisfies Statement (4.13). The result is recorded in Corollary 4.38.

Lemma 4.36. *Let $\Theta = t_1^{T_1}, \ldots, t_n^{T_n}$ be an atomic context. Then $\Theta_1, \Theta_2 \leq^a \Theta$, where Θ_i are clones of Θ, defined by $\Theta_i \triangleq t_{i1}^{T_1}, \ldots t_{in}^{T_n}$.*

Proof. Since Θ is atomic, $[1,1] \leq^a \Theta$ (see Definition 4.32). So there is an atomic reduction ϱ from the context f_1^1, f_2^1 to Θ (see Remark 4.14). We need to find an atomic reduction σ from $\Theta_1, \Theta_2 \equiv t_{11}, \ldots, t_{1n}, t_{21}, \ldots, t_{2n}$ to $\Theta \equiv t_1, \ldots, t_n$. We do this by replacing t_{ij} by $\varrho_{f_i} t_j$. More formally, write $T_i \equiv [\Gamma_i]$ and define the substitution σ from Θ_1, Θ_2 to Θ by

$$\sigma_{t_{ij}} \triangleq \lambda\Gamma_j.\; \varrho_{f_i} t_j\, \Gamma_j \qquad \text{for all } j \in \{1, \ldots, n\},\; i \in \{1,2\}.$$

We use Lemma 4.18 to prove that σ is an atomic reduction. Let Ξ be a context.

(i) Let $t_{ij}, t_{k\ell} \in \{\Theta_1, \Theta_2\}$ be given. Suppose that

$$\sigma_{t_{ij}}\, \vec{M} \;=_{\beta\eta}\; \sigma_{t_{k\ell}}\, \vec{N}$$

for some tuples $\vec{M}$ and $\vec{N}$ with free variables from Ξ, Θ which fit in Γ_j and Γ_ℓ, respectively (see Definition 2.1(iii)). We need to prove that $\vec{M} = \vec{N}$, $i = k$ and $j = \ell$. If we expand the definition of σ, we get

$$\varrho_{f_i}\, t_j \vec{M} \;=_{\beta\eta}\; \varrho_{f_k}\, t_\ell \vec{N}.$$

Since ϱ is an atomic reduction, this implies $f_i = f_k$ (so $i = k$) and $t_j \vec{M} = t_\ell \vec{N}$. The latter implies $t_j = t_\ell$ (so $j = \ell$) and $\vec{M} = \vec{N}$.

(ii) Let $t_{ij} \in \{\Theta_1, \Theta_2\}$ and $b^B \in \{\Theta\}$ with $B \equiv [B_1, \dots, B_m]$. Assume

$$\sigma_{t_{ij}}\, \vec{M} =_{\beta\eta} b\, \vec{N} \tag{4.14}$$

for some tuples $\vec{M}$ and $\vec{N}$ with free variables from Ξ, Θ which fit on Γ_j and Δ, respectively. Equation (4.14) implies $b\vec{N} =_{\beta\eta} \varrho_{f_i} t_j \vec{M}$. On the other hand we have $b\vec{N} \neq_{\beta\eta} \varrho_{f_i}(t_j \vec{M})$ as ϱ is an atomic reduction. A contradiction. □

Proposition 4.37. *Let Γ, Δ and Θ be contexts and suppose Θ is atomic. Then*

$$\Gamma \leq^s \Theta \quad \textit{and} \quad \Delta \leq^s \Theta \quad \Longrightarrow \quad \Gamma, \Delta \leq^s \Theta. \tag{4.15}$$

Proof. Assume that $\Gamma \leq^s \Theta$ and $\Delta \leq^s \Theta$; we must prove that $\Gamma, \Delta \leq^s \Theta$. Let Θ_1 and Θ_2 be clones of Θ. By Lemma 4.36 and Proposition 4.15 we see that we have $\Theta_1, \Theta_2 \leq^s \Theta$. Further, $\Gamma \leq^s \Theta$ & $\Delta \leq^s \Theta$ implies $\Gamma \leq^s \Theta_1$ & $\Delta \leq^s \Theta_2$. So we see that

$$\Gamma, \Delta \leq^s \Theta_1, \Delta \leq^s \Theta_1, \Theta_2 \leq^s \Theta$$

by Lemma 4.7 and transitivity of $\leq^s$. □

Corollary 4.38. *An atomic type A satisfies Statement* (4.13).

Proof. Let $A \equiv [\Theta]$ be an atomic type. Then Θ is an atomic context (see Remark 4.33). Hence Θ satisfies Statement (4.15) by Proposition 4.37. But then the type $A = [\Theta]$ satisfies Statement (4.13) because of Definition 4.1(iii). □

Atomic types are quite common; in fact, we will spend the remainder of this section showing that a type A is atomic if it is from $\mathbb{H}_{\omega+2}$ (see Corollary 4.41), $\mathbb{H}_{\omega+3}$ (Corollary 4.48) or $\mathbb{H}_{\omega+4}$ (Corollary 4.45).

Lemma 4.39. *Let Γ, Δ be contexts with $\{\Gamma\} \subseteq \{\Delta\}$. We have*

(i) Γ is atomic $\Longrightarrow$ Δ is atomic,
(ii) $\Theta \leq^a \Gamma \implies \Theta \leq^a \Delta$ for every context Θ.

Proof. By expanding Definition 4.32 and using Remark 4.14, one easily sees that part (i) is a special case of (ii). Let us prove part (ii).

Let Θ be a context with $\Theta \leq^a \Gamma$. We need to prove $\Theta \leq^a \Gamma$. That is, we need to find an atomic reduction ϱ from Θ to Δ. We know there is a substitution ϱ from Θ to Γ which is an atomic reduction. Since $\{\Gamma\} \subseteq \{\Delta\}$, the map ϱ can be considered a substitution from Θ to Δ (see Definition 2.3(v)). We prove that ϱ is an atomic reduction from Θ to Δ.

Let $a^A, b^B \in \{\Xi, \Theta\}$ with $A \equiv [A_1, \dots, A_n]$ and $B \equiv [B_1, \dots, B_m]$ be given where Ξ is some context. We need to show that

$$\varrho_a\, M_1 \cdots M_n =_{\beta\eta} \varrho_b\, N_1 \cdots N_m \quad \Longrightarrow \quad a = b \quad \text{and} \quad M_i = N_i \tag{4.16}$$

for all $M_i \in \Lambda^{\Xi,\Delta}(A_i)$ and $N_i \in \Lambda^{\Xi,\Delta}(B_i)$.

Let us shorten "Statement (4.16) holds for $M_i \in \Lambda^{\Xi_0}(A_i)$ and $N_i \in \Lambda^{\Xi_0}(B_i)$" to "(4.16) holds for Ξ_0". We need to prove that (4.16) holds for Ξ, Δ.

Recall that $\{\Gamma\} \subseteq \{\Delta\}$. Pick a context Γ^c such that $\{\Gamma^c, \Gamma\} = \{\Delta\}$. Then $\{\Xi, \Delta\} = \{\Xi, \Gamma^c, \Gamma\}$. Thus $\Lambda^{\Xi,\Delta}(C) = \Lambda^{\Xi,\Gamma^c,\Gamma}(C)$ for all types C. Hence to prove (4.16) holds for Ξ, Δ, it suffices to show that (4.16) holds for Ξ, Γ^c, Γ.

Thus, writing $\Xi' \triangleq \Xi, \Gamma^c$, we need to prove that (4.16) holds for Ξ', Γ. Since we have $a, b \in \{\Xi, \Theta\} \subseteq \{\Xi', \Theta\}$, this follows immediately from the fact that ϱ is an atomic reduction from Θ to Γ. □

Lemma 4.40. *Let A_1 and A_2 be types. Then $[[A_1], [A_2]]$ is atomic.*

Proof. By Remark 4.33 we must to show that $\Theta \triangleq F_1^{[A_1]}, F_2^{[A_2]}$ is atomic. To this end, we apply Lemma 4.34. Writing $A_i = [\Gamma_i]$, define

$$X_i \triangleq \lambda z^0.\ F_i \lambda \Gamma_i.\ z.$$

We prove that X_1, X_2 is an atomic pair (see Definition 4.35), i.e., that the terms X_1, X_2 satisfy conditions (i) and (ii) of Lemma 4.34. Let Ξ be a context.

(i) Assume $X_i M =_{\beta\eta} X_j N$ for some $M, N \in \Lambda^{\Xi,\Theta}(0)$ in order to show that $M = N$ and $i = j$. By reduction, we get an equality between lnfs,

$$F_i \lambda \Gamma_i.\ M \ =\ F_j \lambda \Gamma_j.\ N.$$

Hence $M = N$ and $F_i = F_j$. The latter implies $i = j$ as $F_1 \neq F_2$.

(ii) Trivial. Indeed, if $X_i M =_{\beta\eta} d\vec{N}$ for appropriate M, d and $\vec{N}$, then

$$F_i \lambda \Gamma_i.\ M \ =\ d\vec{N},$$

so $F_i = d$, which is absurd. □

Corollary 4.41. *If $A \in \mathbb{H}_{\omega+2}$ then A is atomic.*

Proof. Writing $A = [\Delta]$, we need to prove that Δ is atomic (see Remark 4.33). We claim there is a context $\Gamma \equiv f_1^{[B_1]}, f_2^{[B_2]}$ such that $\{\Gamma\} \subseteq \{\Delta\}$. Then since $[[B_1], [B_2]]$ (and thus Γ) is atomic by Lemma 4.40, we know that Δ (and thus A) is atomic by Lemma 4.39(i).

To ground the claim, it suffices to find two components of A of the form $[B]$. Since $A \in \mathbb{H}_{\omega+2}$, we know that A is small and has at least two components C_1, C_2 with $\operatorname{rk} C_i \geq 1$ (see Theorem 1.3). Since $\operatorname{rk} C_i \geq 1$, the type C_i must have at least one component. Also C_i has at most one component since C_i is not fat as A is small (see Definition 1.1(v)). So we see that $C_i \equiv [B_i]$ for some type B_i. □

To prove that all types $A \in \mathbb{H}_{\omega+4}$ are atomic, we need two lemmas.

Lemma 4.42. *If A is a large type (see Definition 1.1(iv)), then $[[0, 0]] \leq^a A$.*

Proof. Write $A \equiv [\Gamma]$. It suffices to prove that $b^{[0,0]} \leq^a \Gamma$ (see Remark 4.14).

One can verify that since A is large there is derivative Δ of Γ and $p^P \in \{\Delta\}$ such that P is fat (see Definition 3.4). Further, note that $[\Gamma, 0, 0]$ is inhabited. Hence to prove $b \leq^a \Gamma$, it suffices to show that $b \leq^a \Delta$ by Lemma 4.30.

Since P is fat $P \equiv [[\Gamma_1], \cdots, [\Gamma_k]]$ with $k \geq 1$ (see Definition 1.1(iii)). Define

$$\varrho_b \ \triangleq\ \lambda x^0 y^0.\ p\,(\lambda \Gamma_1.\ x)\ (\lambda \Gamma_2.\ y) \cdots (\lambda \Gamma_k.\ y).$$

Then $\varrho_b \in \Lambda^{\Delta}([0, 0])$ yields a substitution ϱ from b to Δ. We prove that ϱ is an atomic reduction (and thus $b \leq^a \Delta$) using Lemma 4.18.

(i) Given a context Ξ and $M_i \in \Lambda^{\Xi,\Delta}(0)$ and $N_i \in \Lambda^{\Xi,\Delta}(0)$ with

$$\varrho_b\, M_1 M_2 \ =_{\beta\eta}\ \varrho_b\, N_1 N_2$$

we need to prove that $M_i = N_i$. By reduction we get

$$p\,(\lambda \Gamma_1.\ M_1)\ (\lambda \Gamma_2.\ M_2) \cdots (\lambda \Gamma_k.\ M_2) \ =\ p\,(\lambda \Gamma_1.\ N_1)\ (\lambda \Gamma_2.\ N_2) \cdots (\lambda \Gamma_k.\ N_2).$$

Hence $M_1 = N_1$ and $M_2 = N_2$.

(ii) Simple as before. □

Lemma 4.43. *Let A be a type such that $[[0,0]] \leq^a A$. Then A is atomic.*

Proof. Write $A \equiv [\Theta]$. We need to prove that Θ is atomic (see Remark 4.33). We will define a pair $X_1, X_2 \in \Lambda^{\Theta}(1)$ and show it is atomic (see Definition 4.35).

Since $[[0,0]] \leq A$, there is an atomic reduction ϱ from $b^{[0,0]}$ to Θ. Define

$$\mathsf{s} \triangleq \lambda x^0.\ \varrho_b\, xx.$$

Let Ξ be a context. Given $M, N \in \Lambda^{\Xi,\Theta}(0)$, we have

$$\mathsf{s}M =_{\beta\eta} \mathsf{s}N \quad \Longrightarrow \quad M = N. \tag{4.17}$$

Moreover, we claim that $\mathsf{s}M \neq_{\beta\eta} M$ for all $M \in \Lambda^{\Xi,\Theta}(0)$.

To prove the claim, write $\mathsf{s} \equiv \lambda x^0.\ S$ for some $S \in \Lambda^{x,\Theta}(0)$. Note that either x occurs in S or not, and if x does not occur in S then $\mathsf{s}M =_{\beta\eta} \mathsf{s}N$ for all N, M, which contradicts Statement (4.17). Hence x occurs in S.

Now, let $M \in \Lambda^{\Xi,\Theta}(0)$ be given; we prove $\mathsf{s}M \neq_{\beta\eta} M$. Recall that we consider all terms to be in long normal form. In particular, S is in lnf. Note that if we replace x in S with M, the resulting term is immediately in long normal form—no reduction is needed. Hence if $S \neq x$, we see that M is a strict subterm of $S[x{:=}M] = M$, which is absurd. So $S \equiv x$ and thus $\mathsf{s} = \lambda x^0.\ x$.

This is also absurd. Indeed, we get $\varrho_b\, dd =_\beta \mathsf{s}\, d =_{\beta\eta} d$ for any fresh variable d^0, which contradicts that ϱ is an atomic reduction.

Now that we know $\mathsf{s}M \neq_{\beta\eta} M$ for all $M \in \Lambda^{\Xi,\Theta}(0)$, cunningly define

$$X_1 \triangleq \lambda x^0.\ \mathsf{b}\, xx; \qquad\qquad X_2 \triangleq \lambda x^0.\ \mathsf{b}\, x(\mathsf{s}x).$$

Then $X_i \in \Lambda^{\Theta}(1)$. We prove X_1, X_2 satisfies (i) and (ii) of Lemma 4.34.

(i) Given $M, N \in \Lambda^{\Xi,\Theta}(0)$, assume $X_iM =_{\beta\eta} X_jN$ to show $i = j$ & $M = N$. We distinguish three cases.
  (i) $X_1M =_{\beta\eta} X_1N$. Then $\mathsf{b}\, MM =_{\beta\eta} \mathsf{b}\, NN$, so $M = N$.
  (ii) $X_2M =_{\beta\eta} X_2N$. Then $\mathsf{b}\, M(\mathsf{s}M) =_{\beta\eta} \mathsf{b}\, N(\mathsf{s}N)$, so $M = N$.
  (iii) $X_1M =_{\beta\eta} X_2N$. Then $\mathsf{b}\, MM =_{\beta\eta} \mathsf{b}\, N(\mathsf{s}N)$. So we have both $M = N$ and $M =_{\beta\eta} \mathsf{s}N$. Consequently, $N =_{\beta\eta} \mathsf{s}N$, which is absurd.

(ii) As simple as before. □

Corollary 4.44. *Each large type A is atomic.* □

Proof. Combine Lemma 4.43 and Lemma 4.42.

Corollary 4.45. *If $A \in \mathbb{H}_{\omega+4}$ then A is atomic.*

Proof. Since A is large by definition of $\mathbb{H}_{\omega+4}$, A is atomic by Corollary 4.44. □

Lemma 4.46. *A context Γ is atomic if one of its derivatives Δ is atomic.*

Proof. Follows from Lemma 4.30 as the type $[\Gamma, 0]$ is inhabited. □

Lemma 4.47. *Let A be a small type with $\operatorname{rk} A \geq 4$. Then A is atomic.*

Proof. There is a component B of A such that $\operatorname{rk} B \geq 3$ (see Definition 1.1(ii)). In other words, writing $A = [\Gamma]$, there is an $b^B \in \{\Gamma\}$ such that $\operatorname{rk} B \geq 3$. Similarly, if we write $B \equiv [\![\Theta]\!]$ for some Θ (recall that A is small), then there must be a $c^C \in \{\Theta\}$ with $\operatorname{rk} C \geq 1$. So $C \equiv [D]$ for some D.

Note that Γ, Θ is a direct derivative of Γ, so to prove A is atomic, it suffices to show that Γ, Θ is atomic (by Lemma 4.46). To this end, consider the context $\Xi \triangleq b^B, c^C$. We have $\{\Xi\} \subseteq \{\Gamma, \Theta\}$ and $[\Xi] = [[\![\Theta]\!], [D]]$, so Ξ is atomic by Lemma 4.40 and hence Γ, Θ is atomic by Lemma 4.39(i). □

Corollary 4.48. *If $A \in \mathbb{H}_{\omega+3}$ then A is atomic.*

Proof. Since A is small and $\operatorname{rk} A \geq 4$ by definition, A is atomic by Lemma 4.47. □

5. Order type of $\leq_h$

The order type of the reducibility relation $\leq_h$ (see Definition 1.2(ii)) is $\omega + 5$. At least, this is what is shown in Subsection 1.4 using statements promised to be proven later on. In this section, we deliver on these promises; they are $H_\alpha \leq_h \mathbb{H}_\alpha$ (Subsection 5.1), $\mathbb{H}_\alpha \leq_h H_\alpha$ (Subsection 5.2), and $\alpha \leq \beta \implies H_\alpha \leq_h H_\beta$ (Subsection 5.3).

We refer the reader to Theorem 1.3 for the definition of H_α and $\mathbb{H}_\alpha$.

5.1. $H_\alpha \leq_h \mathbb{H}_\alpha$.

Let us begin with a harvest. We use the theory of strong reductions and atomic types to easily prove that $H_\alpha \leq^s A$ for all $A \in \mathbb{H}_\alpha$ and $\alpha \in \omega + 5$. Loosely stated, we do this by recognizing the tree of H_α as part of the tree of A (see Subsection 1.2).

Recall that an atomic reduction is also a strong reduction, so for example $H_\alpha \leq^a A$ implies $H_\alpha \leq^s A$ (see Proposition 4.15). We use this fact without further mention.

Lemma 5.1. *$H_0 \leq^s A$ for all $A \in \mathbb{H}_0$.*

Proof. We need to prove that $0 \leq^s A$ whenever A is uninhabited. We will prove $0 \leq^s A$ for all types A. Writing $A \equiv [\Gamma]$, we need to prove $[\varepsilon] \leq^s [\Gamma]$. So it suffices to show that $\varepsilon \leq^s \Gamma$ (see Definition 4.1(iii)). This follows immediately from Lemma 4.16(i). □

Lemma 5.2. *$H_n \leq^s A$ for all $A \in \mathbb{H}_n$ where $n \geq 1$.*

Proof. Trivial, since $\mathbb{H}_n = \{H_n\}$ for each $n \in \mathbb{N}$. □

Lemma 5.3. *$H_\omega \leq^s A$ for each $A \in \mathbb{H}_\omega$.*

Proof. We need to prove that $[1, 0] \leq^s A$. Recall that since $A \in \mathbb{H}_\omega$, we have A is small, $\operatorname{rk} A = 2$ and A has exactly one component of rank 1. So precisely one of the components of A is 1; the remaining components are 0. By a permutation of the components we get $A \sim^s [1, 0^k]$ for some k (see Corollary 4.17). Hence it suffices to prove that $[1, 0] \leq^s [1, 0^k]$. This follows from Lemma 4.16(i). □

Lemma 5.4. *$H_{\omega+1} \leq^s A$ for every $A \in \mathbb{H}_{\omega+1}$.*

Proof. We need to prove that $[2] \leq^s A$. Note that A is small, $\operatorname{rk} A = 3$ and A has exactly one component of rank ≥ 1. So one of the components of A is of the form $[[\![0^\ell]\!]]$ where $\ell \geq 1$ and the remaining components are 0. Hence $A \sim^s [[[\![0^{\ell+1}]\!]], 0^k]$ by Corollary 4.17. So it suffices to prove that $[2] \leq^s [[[\![0^\ell]\!]], 0^k]$.

Since $[0] \leq^s [0^\ell]$ by Lemma 4.16(i), we have

$$[2] \equiv [[[\![0]\!]]] \leq^s [[[\![0^\ell]\!]]] \leq^s [[[\![0^\ell]\!]], 0^k]$$

by Lemma 4.21 and Lemma 4.16(i), respectively. □

Lemma 5.5. *$H_{\omega+2} \leq^s A$ for every $A \in \mathbb{H}_{\omega+2}$.*

Proof. Note that A is small and has at least two components of rank ≥ 1, so after a permutation of As components we get $A \sim^s B \triangleq [[\![\Delta_1]\!], [\![\Delta_2]\!], \Gamma]$ for some contexts Δ_1, Δ_2 and Γ. We need to show that $[1,1,0] \leq^s B$. Since B is atomic by Corollary 4.41, it suffices to prove $[0] \leq^s B$ and $[1] \leq^s B$ (see Proposition 4.37).

We have $[0] \leq^s A \sim^s B$ since A is inhabited. Concerning $[1] \leq^s B$, note that $0 \leq^s [\Delta_1]$ by Lemma 4.16(i) and so $[1] \equiv [[0]] \leq^s [[\![\Delta_1]\!]] \leq^s B$ by Lemma 4.21 and Lemma 4.16(i). □

Lemma 5.6. *$H_{\omega+3} \leq^s A$ for every $A \in \mathbb{H}_{\omega+3}$.*

Proof. We need to prove that $[3,0] \leq^s A$. By Proposition 4.37 it suffices to show that $[3] \leq^s A$ and $[0] \leq^s A$ since A is atomic by Corollary 4.48.

As A is inhabited, $[0] \leq^s A$ is trivial.

Concerning $[3] \leq^s A$. Since A is small and $\operatorname{rk} A \geq 4$, there is a component $[A_1]$ of A with $\operatorname{rk} A_1 \geq 2$. Then $[[A_1]] \leq^s A$ by Lemma 4.16(i), so it suffices to show $[[[1]]] \equiv [3] \leq^s [[A_1]]$. By Lemma 4.21 it is enough to prove that $[1] \leq^s A_1$.

By similar reasoning for A_1, we are left with the problem to prove $0 \leq^s A_2$ where $[A_2]$ is some component of A_1. Lemma 4.16(i) gives the solution. □

Lemma 5.7. *$H_{\omega+4} \leq^s A$ for every $A \in \mathbb{H}_{\omega+4}$.*

Proof. We need to prove that $[[0,0],0] \leq^s A$. Since A is atomic by Corollary 4.45 it suffices to show by Proposition 4.37 that $[0] \leq^s A$ and $[[0,0]] \leq^s A$. The former inequality is trivial since A is inhabited. The latter is Lemma 4.42. □

5.2. **$\mathbb{H}_\alpha \leq_h H_\alpha$.** In this subsection we prove that $A \leq_h H_\alpha$ for all $\alpha \in \omega + 5$ and $A \in \mathbb{H}_\alpha$. (In fact, we show that $A \leq^s H_\alpha$ for all $\alpha \neq 0$.) This is more difficult than proving $H_\alpha \leq^s A$ (which involved only 'chopping'), as it requires the 'encoding' of the inhabitants of A using the simpler inhabitants of H_α.

5.2.1. *Ad $0, \ldots, \omega$ and $\omega + 1$.*

Lemma 5.8. *$A \leq_h H_0$ for all $A \in \mathbb{H}_0$.*

Proof. We need to prove that $A \leq_h 0$. Since A is uninhabited (by definition of $\mathbb{H}_0$), all the components of A are inhabited by Theorem 1.8. Write $A = [\Gamma]$ and pick for each $b^B \in \{\Gamma\}$ an inhabitant N_b of B. Then $\varrho_b \triangleq N_b$ yields a substitution ϱ from A to 0. For a rather dull reason the map $\hat{\varrho}\colon \Lambda^\varepsilon(A) \to \Lambda^\varepsilon(0)$ is injective: $\Lambda^\varepsilon(A)$ is empty. Hence $A \leq_h 0$. □

Lemma 5.9. *$A \leq^s H_k$ for all $A \in \mathbb{H}_k$ where $k > 0$.*

Proof. Trivial, since $\mathbb{H}_k = \{H_k\}$. □

Lemma 5.10. *$A \leq^s H_\omega$ for all $A \in \mathbb{H}_\omega$.*

Proof. Again we have $A \sim^s [1, 0^k]$ for some $k > 0$ (see the proof of Lemma 5.3). So we need to prove that $[1, 0^k] \leq^s [1, 0]$. By Definition 4.1(iii), it suffices to show that

$$f^1, d_1^0, \ldots, d_k^0 \ \leq^s \ f^1, c^0.$$

The terms

$$\varrho_f \triangleq f^{(k)}, \qquad \varrho_{d_{i+1}} \triangleq f^{(i)} c.$$

constitute a substitution from $f^1, d_1^0, \ldots, d_k^0$ to f^1, c^0. It suffices to prove that ϱ is an atomic reduction (see Proposition 4.15). Let Ξ be a context. Note that for all $M, N \in \Lambda^{\Xi,f,c}(0)$,

$$\begin{aligned} \varrho_{d_i} =_{\beta\eta} \varrho_{d_j} \quad &\Longrightarrow \quad i = j, \\ \varrho_f M =_{\beta\eta} \varrho_f N \quad &\Longrightarrow \quad M = N, \\ \varrho_f M \neq_{\beta\eta} \varrho_{d_i}. & \end{aligned}$$

Hence condition (i) of Lemma 4.18 is met. Since the other condition can be easily verified, Lemma 4.18 implies that ϱ is an atomic reduction. □

Before we proceed to "ad $\omega + 1$", we need a lemma.

Lemma 5.11. *$[[\![0^k]\!]] \leq^s [[\![0]\!]] \equiv [2]$ for all $k > 0$.*

Proof. Apply Corollary 4.25 with $A \triangleq 0$. □

Lemma 5.12. *Given $A \in \mathbb{H}_{\omega+1}$, we have $A \leq^s H_{\omega+1}$.*

Proof. Let $A \in \mathbb{H}_{\omega+1}$ be given. We need to prove that $A \leq^s [2]$. One can easily verify that $A \sim^s [[\![0^k]\!], 0^l]$ for some $k \geq 1$ and $l \geq 0$. By Lemma 5.11, we have $[[\![0^k]\!], 0^l] \leq^s [2, 0^k]$. So it suffices to show that $[2, 0^k] \leq^s [2]$. To this end note that the terms

$$\begin{aligned} \varrho_F &\triangleq \lambda f^1.\ F\lambda z^0.\ F\lambda x_1^0.\ \cdots F\lambda x_l^0.\ fz; \\ \varrho_{c_i} &\triangleq F\lambda x_1^0.\ \cdots F\lambda x_i^0.\ x_1. \end{aligned}$$

give a strong reduction from $[F^2, c_1^0, \ldots, c_k^0]$ to $[2]$ (cf. Lemma 5.10). □

5.2.2. *Ad $\omega + 2$.* We need to prove that $A \leq^s [1, 1, 0]$ for all $A \in \mathbb{H}_{\omega+2}$. With atomicity, we easily reduce the problem to showing that $[2] \leq^s [1, 1, 0]$ (see Lemma 5.13). Interestingly, we have $[2] \not\leq^a [1, 1, 0]$ (see Lemma 5.14). Consequently, the proof of $[2] \leq^s [1, 1, 0]$ has quite a unique flavor (see Proposition 5.15).

Lemma 5.13. *Suppose that $[2] \leq^s [1, 1, 0]$. Then $A \leq^s [1, 1, 0]$ for all $A \in \mathbb{H}_{\omega+2}$.*

Proof. Let $A \in \mathbb{H}_{\omega+2}$ be given. Then A is small and $\operatorname{rk} A \leq 3$, so

$$A \ \sim^s \ [[\![0^{k_1}]\!], \ldots, [[\![0^{k_n}]\!], 1^l, 0^m]$$

for some k_i, n, m, ℓ by Corollary 4.17. We need to prove that $A \leq^s [1, 1, 0]$. By Lemma 5.11, we have $[[\![0^{k_i}]\!]] \leq^s [2]$, and so $A \leq^s [2^n, 1^\ell, 0^m]$ by Lemma 4.7. Hence it suffices to show that $[2^n, 1^\ell, 0^m] \leq^s [1, 1, 0]$ by transitivity. Since $[1, 1, 0]$ is atomic by Lemma 4.40, we apply Proposition 4.37. It remains to be shown that

$$[2] \leq^s [1, 1, 0]; \qquad [1] \leq^s [1, 1, 0]; \qquad [0] \leq^s [1, 1, 0].$$

The first statement is valid by assumption, the latter two by Lemma 4.16(i). □

Lemma 5.14. *We have* $[2] \not\leq^a [1,1,0]$.

Proof. Let ϱ be a substitution from F^2 to $\Theta \triangleq f^1, g^1, c^0$. In order to show that $[2] \not\leq^a [1,1,0]$, we prove that ϱ is not atomic by finding terms $M, N \in \Lambda^\Theta(1)$ with $\varrho_F M =_{\beta\eta} \varrho_F N$ while $M \neq N$ (see Definition 4.12). Write

$$\varrho_F \equiv \lambda h^1.\, w_0\, h w_1 \cdots h w_n c$$

where $w_i \in \Lambda^\Theta(1)$ and define $M \triangleq \lambda x^0.\, x$ and $N \triangleq \lambda x^0.\, w_1 \cdots w_n c$. Then

$$\begin{aligned}
\varrho_F M &=_\beta w_0\, M w_1 \cdots M w_n c \\
&=_\beta w_0\, w_1 \cdots w_n c \\
&=_\beta w_0\, N K && \text{for all } K \in \Lambda^\Theta(0) \\
&=_\beta w_0\, N(w_1 \cdots N w_n c) \;=_\beta\; \varrho_F N,
\end{aligned}$$

while $N \neq M$. □

Proposition 5.15. *We have* $[2] \leq^s [1,1,0]$.

Proof. It suffices to prove that $F^2 \leq^s f^1, g^1, c^0$ (see Definition 4.1(iii)). For this we need to find a substitution ϱ from F^2 to $\Theta \triangleq f^1, g^1, c^0$ such that the map

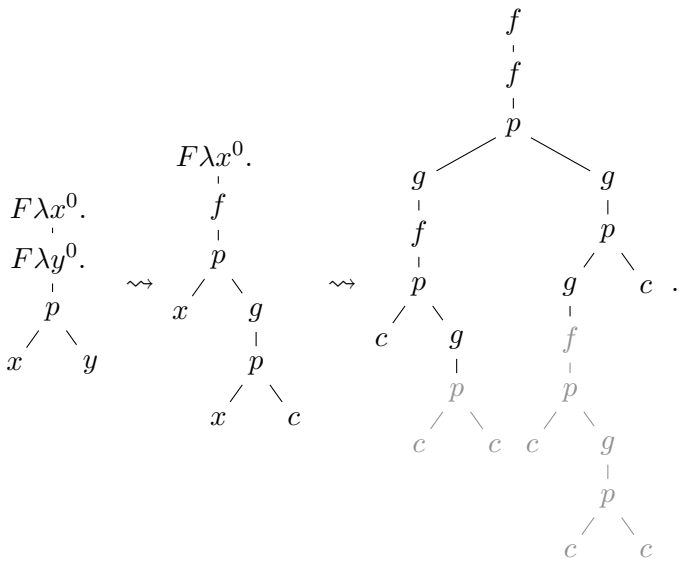
$$\varrho^\Xi \colon \Lambda^{\Xi, F^2}(0) \to \Lambda^{\Xi,\Theta}(0)$$

is injective for every context Ξ (see Proposition 4.2). The assignment

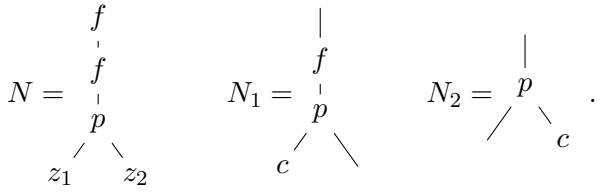
$$\varrho_F \triangleq \lambda h^1.\, f h g h c$$

gives a substitution ϱ from F^2 to Θ. We prove that ϱ^Ξ is injective for given context Ξ.

Let us examine $\hat{\varrho}^\Xi$, informally. Occurrences of $F\lambda x^0.\, M$ are recursively replaced by $fM[\,x{:=}gM[x{:=}c]\,]$. E.g., consider the inhabitant $M = F\lambda x^0.\, F\lambda y^0.\, pxy$ of $p^{[0,0]}, F^2$:



We are interested in the following "subterms" of the image:



Replacing the maximal subterms of the form gK with distinct z_i yields N. This is almost the original term: the f are to be replaced by $F\lambda$ and the z_i need to be appropriately bound to them.

If we repeat the process on the aforementioned maximal subterms gK (which we replaced with z_i to get N), but instead simply remove the maximal subterms of K of the form gK', we get the terms N_i.

An N_i is almost a subterm of N: lay N_i on top of N with c and z_i aligned. There will always be a f under the top of N_i. This is the f that has to be bound to z_i.

Thus the orignal term can be read back from the image. A rigorous proof of the correctness of this method, requires nothing but tedious bookkeeping and is therefore omitted. □

5.2.3. *Ad $\omega + 3$.* We need to show that $A \leq^s [3, 0]$ for all $A \in \mathbb{H}_{\omega+3}$. To this end, we prove that $A \leq^s [3, 0]$ for every small type A (since every $A \in \mathbb{H}_{\omega+3}$ is small).

Lemma 5.16. *Let $B_1, \ldots, B_m$ be types. We have*

$$[\,[\![B_1, \ldots, B_m]\!]\,] \;\leq^s\; [B_1, \ldots, B_m, 3^m].$$

Proof. Combine Lemma 4.26, Lemma 4.24 and Corollary 4.17. □

Lemma 5.17. *Let A be a small type. Then $A \leq^s [3, 0]$.*

Proof. One can easily verify that every component C of A is either 0 or of the form $C \equiv [B]$ where B is small. So, if we repeatedly apply Lemma 5.16 (with the help of Lemma 4.7 and Corollary 4.17) we eventually see that, for some k, ℓ and m,

$$A \;\leq^s\; [3^k, 1^\ell, 0^m].$$

We illustrate this with an example.

$$\begin{aligned}[0, [\![0, 1, 2]\!]] \;&\leq^s [0, 0, 1, 2, 3^3] \;\equiv\; [0, 0, 1, [\![0]\!], 3^3] \\ &\leq^s [0, 0, 1, 0, 3, 3^3] \;\sim^s\; [3^4, 1, 0^2].\end{aligned}$$

So it remains to be shown that $[3^k, 1^\ell, 0^m] \leq^s [3, 0]$. Since $[3, 0]$ is atomic by Lemma 4.47, it suffices (see Proposition 4.37) to prove that

$$[0] \leq^s [3, 0]; \qquad [3] \leq^s [3, 0]; \qquad [1] \leq^s [3, 0].$$

The first two statements follow immediately from Lemma 4.16(i). Concerning the last one, we have $[1] \leq^s [[\![1]\!]] \equiv [3] \leq^s [3, 0]$ by Lemma 4.22. □

5.2.4. *Ad* $\omega + 4$. We need to show that $A \leq [[0,0],0]$ for all $A \in \mathbb{H}_{\omega+4}$. We prove more.

Lemma 5.18. *Let A be a type. Then $A \leq^s [[0,0],0]$.*

Proof. Note that $[[0,0],0]$ is atomic by Lemma 4.43, since we have

$$[[0,0]] \ \leq^a \ [[0,0],0]. \tag{5.1}$$

Hence to prove that $A \leq^s [[0,0],0]$, it suffices to show that $[C] \leq^s [[0,0],0]$ for every component C of A (see Proposition 4.37).

Let $C \equiv [C_1, \ldots, C_k]$ be a component of A. We prove that $[C] \leq^s [[0,0],0]$. Using induction, we may assume that we already have $C_i \leq^s [[0,0],0]$ for all i.

Since $[[C_1, \ldots, C_k]] \leq^s [[C_1], \ldots, [C_k], [0^k]]$ by Lemma 4.23, it suffices to show that

$$[[C_1], \ldots, [C_k], [0^k]] \ \leq^s \ [[0,0],0].$$

Since $[[0,0],0]$ is atomic, this reduces to $[[C_i]] \leq^s [[0,0],0]$ and $[[0^k]] \leq^s [[0,0],0]$. For the latter inequality, note that the substitution from $p^{[0^k]}$ to $b^{[0,0]}$ given by

$$\varrho_p \triangleq \lambda x_1 \cdots x_n.\ bx_1\, bx_2\, \cdots bx_n x_n$$

is an atomic reduction and hence $[[0^k]] \leq^s [[0,0]] \leq^s [[0,0],0]$. Concerning the first inequality, write $C_i \equiv [D_1, \ldots, D_\ell]$ and note that we have

$$\begin{aligned}
[[C_i]] \equiv \ & [[\![D_1, \ldots, D_\ell]\!]] \\
\leq^s \ & [[\![D_1,]\!], [\![D_2]\!], \ldots, [\![D_\ell]\!]] && \text{by Lemma 4.24} \\
\leq^s \ & [\, D_1, D_1, [0,0],\ [\![D_2]\!], \ldots, [\![D_\ell]\!]] && \text{by Lemma 4.28} \\
& \ \vdots \\
\leq^s \ & [\, D_1, \ldots, D_\ell,\ D_1, \ldots, D_\ell,\ [0,0]^\ell\,] && \text{by Corollary 4.17.}
\end{aligned}$$

As we have $C_i \equiv [D_1, \ldots, D_\ell] \leq^s [[0,0],0]$ and $[[0,0]] \leq^s [[0,0],0]$, we get

$$[\, D_1, \ldots, D_\ell,\ D_1, \ldots, D_\ell,\ [0,0]^\ell\,] \ \leq^s \ [[0,0],0]$$

by Proposition 4.37. Hence $[[C_i]] \leq^s [[0,0],0]$. So we are done. □

5.3. $\alpha \leq \beta \implies H_\alpha \leq_h H_\beta$**.** We prove that $\alpha \leq \beta$ implies $H_\alpha \leq^s_h H_\beta$ by showing that

$$[0^k] \underset{(i)}{\leq^s_h} [0^{k+1}] \underset{(ii)}{\leq^s_h} [1,0] \underset{(iii)}{\leq^s_h} [2] \underset{(iv)}{\leq^s_h} [1,1,0] \underset{(v)}{\leq^s_h} [3,0] \underset{(vi)}{\leq^s_h} [[0,0],0].$$

(i) Follows directly from Lemma 4.16(i).
(ii) Similar to Lemma 5.10, but easier.
(iii) On the one hand, $[1,0] \leq^s_h [2,0]$ as $f^1 \leq^s F^2$ via $\varrho_f \triangleq \lambda x^0.\ F\lambda y^0.\ x$. On the other hand, $[2,0] \leq^s_h [2]$ by Lemma 5.12.
(iv) This is Proposition 5.15.
(v) Follows from Lemma 5.17 since $[1,1,0]$ is small.
(vi) A consequence of Lemma 5.18.

6. Order type of $\leq_{\beta\eta}$ and $\leq_{h^+}$

We are in the home stretch now. We have proven that the order type of the reduction relation $\leq_h$ is as depicted in the diagram on page 4. The structure of this proof was given in Subsection 1.4, and we have spent the previous sections filling in all the difficult details. In this section we provide the final and easy bits of the proof that the order types of the reduction relations $\leq_{\beta\eta}$ and $\leq_{h^+}$ (see Definition 1.2) are depicted correctly as well.

6.1. $[2] \leq_{h^+} [1,0]$ **and** $[0^{k+1}] \leq_{h^+} [0^k]$**.** Let us begin with $[0^{k+1}] \leq_{h^+} [0^k]$ for given $k \geq 2$. It suffices to prove that

$$[0^{k+1}] \ \leq_{h^+} \ [0^2],$$

because one can easily verify that $[0^2] \leq_{h^+} [0^k]$.

Lemma 6.1. *For every $k \geq 2$ we have $[0^k] \leq_{h^+} [0^2]$.*

Proof. We need to find a finite family of Böhm terms from $[0^k]$ to $[0^2]$ which is jointly injective (see Definition 1.2). Recall that (see 1.3.1)

$$\Lambda^\varepsilon([0^k]) \ = \ \{U_1^k, \ldots, U_k^k\}.$$

Given $i, j \in \{1, \ldots, k\}$ with $i \neq j$, there is a Böhm term $M_{ij} \in \Lambda^\varepsilon([0^k] \to [0])$ which separates the terms U_i^k and U_j^k in the sense that

$$M_{ij}U_i^k \ =_\beta \ U_1^2 \qquad \text{and} \qquad M_{ij}U_j^k \ =_\beta \ U_2^2.$$

Indeed, the Böhm term $M_{ij} \triangleq \lambda m^{[0^k]} x_1 x_2.\, m\, P_1 \cdots P_k$ does the job, where

$$P_\ell \triangleq \begin{cases} x_1 & \text{if } \ell = i \\ x_2 & \text{otherwise.} \end{cases}$$

Hence the family of terms $\{M_{ij} : i \neq j\}$ is jointly injective. □

Proposition 6.2. $[2] \leq_{h^+} [1,0]$.

Proof. We have shown there is no injective transformation from the type $[2]$ to $[1,0]$ (see Subsection 3.2). However, we will prove that the following Böhm transformations ϱ and σ from $[2]$ to type $[1,0]$ are jointly injective (and thus $[2] \leq_{h^+} [1,0]$).

$$\varrho_F \triangleq \lambda h^1.\, fhc \qquad\qquad \sigma_F \triangleq \lambda h^1.\, fhfhc$$

It suffices to show that i and j can be recovered from $\hat{\sigma}(\,\langle i,j\rangle\,)$ and $\hat{\varrho}(\,\langle i,j\rangle\,)$. This is indeed the case as one we have the following equalities for $\langle i,j\rangle \in \Lambda^\varepsilon([2])$.

$$\hat{\varrho}(\,\langle i,j\rangle\,) = c_i \qquad\qquad \hat{\sigma}(\,\langle i,j\rangle\,) = c_{2i-j+1}.$$

We verify the latter equality and leave the other to the reader.

$$\begin{aligned}
\hat{\sigma}(\,\langle i,j\rangle\,) &=_{\beta\eta} \sigma_F\lambda x_1^0.\ \cdots \sigma_F\lambda x_i^0.\ x_j \\
&=_\beta \ \sigma_F\lambda x_1^0.\ \cdots \sigma_F\lambda x_j^0.\ f^{(i-j)}x_j \\
&=_\beta \ \sigma_F\lambda x_1^0.\ \cdots \sigma_F\lambda x_{j-1}^0.\ ff^{(i-j)}ff^{(i-j)}c \\
&= \ \ \sigma_F\lambda x_1^0.\ \cdots \sigma_F\lambda x_{j-1}^0.\ f^{(2+2(i-j))}c \\
&=_\beta \ f^{(j-1)}f^{(2+2(i-j))}c = c_{2i-j+1}.
\end{aligned}$$

We have proven that $[2] \leq_{h^+} [1,0]$. □

6.2. $[2] \leq_{\beta\eta} [1,0]$**.** For the proof of $[2] \leq_{\beta\eta} [1,0]$ we need some preparations.

Lemma 6.3. *Addition and multiplication on the Church numerals is definable in the following sense. There are closed terms* $M_+, M_\times : [1,0] \to [1,0] \to [1,0]$ *with*

$$M_+ c_m c_n =_{\beta\eta} c_{m+n} \quad \text{and} \quad M_\times c_m c_n =_{\beta\eta} c_{m\cdot n} \qquad (n, m \in \mathbb{N}).$$

Proof. It is not hard to see that the terms

$$M_+ = \lambda a^{[0,1]} b^{[0,1]} f^1 c^0.\ af(bfc)$$
$$M_\times = \lambda a^{[0,1]} b^{[0,1]} f^1 c^0.\ a(bf)c.$$

do the job. □

Corollary 6.4. *The Church numerals contain a pairing in the following sense. There is a term* $M_p : [1,0] \to [1,0] \to [1,0]$ *such that*

$$M_p c_n c_m =_{\beta\eta} M_p c_{n'} c_{m'} \quad \Longrightarrow \quad n = n' \text{ and } m = m'.$$

Proof. The map $P\colon (n,m) \mapsto \frac{1}{2}(n+m)(n+m+1)+m$, is a well known bijection between $\mathbb{N}^2$ and $\mathbb{N}$, called the Cantor pairing. Using Lemma 6.3, we obtain a term M_p such that $M_p(c_n, c_m) =_{\beta\eta} c_{P(n,m)}$ for all n, m. □

Proposition 6.5. $[2] \leq_{\beta\eta} [1,0]$.

Proof. We need to find an $R \in \Lambda^\varepsilon([2] \to [1,0])$ such that

$$RM =_{\beta\eta} RN \quad \Longrightarrow \quad M = N \qquad (M, N \in \Lambda^\varepsilon([2]).$$

Let ϱ and σ be the substitutions from Proposition 6.2 which form a multi-head reduction from $[2]$ to $[1,0]$ and define $R \triangleq \lambda m^{[2]}.\ M_p\,(m\varrho_F)(m\sigma_F)$.

Let $M, N \in \Lambda^\varepsilon([2])$ with $RM =_{\beta\eta} RN$ be given, to prove $M = N$. Then

$$M_p\,(M\varrho_F)(M\sigma_F) =_{\beta\eta} M_p\,(N\varrho_F)(N\sigma_F).$$

So $M\varrho_F =_{\beta\eta} N\varrho_F$ & $M\sigma_F =_{\beta\eta} N\sigma_F$ by Corollary 6.4. But then $M = N$ as $\hat{\varrho}$ and $\hat{\sigma}$ are jointly injective (see Proposition 6.2). □

7. Conclusion

We have proven Statman's Hierarchy Theorem (see page 4). With it we can mechanically determine for all types A and B in $\mathbb{T}^0$ whether $A \leq_{\beta\eta} B$, whether $A \leq_h B$, and whether $A \leq_{h^+} B$ only by inspecting the syntactic form of A and B. Let us make some final remarks.

7.1. **Contributions.** The *calculus of reductions* (see Section 4) used to prove the existence of reductions is new (including the notions of strong reduction and atomic type). The method to prove the absense of reductions from Subsection 3.3 is a generalisation of the work of Dekkers in [Dek88].

The Hierarchy Theorem as presented here is slightly stronger than the one proven by Statman in that the original version only completely determined the relations $\leq_{\beta\eta}$ and $\leq_{h^+}$, but not $\leq_h$ (see Theorem 3.4.18 and Corollary 3.4.27 of [BDS13]), while our version determines the relation $\leq_h$ as well. For this we had to add one canonical type, namely [2], and prove (among other things), that $[2] \not\leq_h [1,0]$ (see Subsection 3.2).

7.2. **Outlook.** The notions and notation introduced in this paper are easily adapted to a setting with multiple base types $\alpha_1, \alpha_2, \ldots$. However, if one tries to determine the equivalence classes of $\leq_h$ in this setting one realises much more work has to be done. (Indeed, try, for instance, to prove a variant of Theorem 1.8 for multiple base types.) Perhaps the development of a software tool based on the calculation rules for strong reductions will be of use in such a project.

7.3. **Acknowledgements.** We are grateful that a reviewer spotted an error in Subsection 3.6 of a previous version of this manuscript.

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