Weighted Region Problem in Arrangement of Lines

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Abstract

In this paper, a geometric shortest path problem in weighted regions is discussed. An arrangement of lines $\mathcal{A}$, a source s, and a target t are given. The objective is to find a weighted shortest path, π_{st}, from s to t. Existing approximation algorithms for weighted shortest paths work within bounded regions (typically triangulated). To apply these algorithms to unbounded regions, such as arrangements of lines, there is a need to bound the regions. Here, we present a minimal region that contains π_{st}, called SP-Hull of $\mathcal{A}$. It is a closed polygonal region that only depends on the geometry of the arrangement $\mathcal{A}$ and is independent of the weights. It is minimal in the sense that for any arrangement of lines $\mathcal{A}$, it is possible to assign weights to the faces of $\mathcal{A}$ and choose s and t such that π_{st} is arbitrary close to the boundary of SP-Hull of $\mathcal{A}$. We show that SP-Hull can be constructed in $\mathcal{O}(n \log n)$ time, where n is the number of lines in the arrangement. As a direct consequence we obtain a shortest path algorithm for weighted arrangements of lines.

1 Introduction

The geometric shortest path problem ranks among the fundamental problems studied in Computational Geometry and related fields. In this problem, the input is a set of regions (often a triangulation), where each region (triangle) has a corresponding weight, and two points, source s and target t. The output is the weighted shortest path from s to t, π_{st}, which is the path with minimum cost. The cost of the path is the total sum of the length of each segment multiplied by the corresponding region's weight.

Mitchell and Papadimitriou [2] introduced this problem and proposed a $(1+\varepsilon)$-approximation algorithms. Subsequently, positioning Steiner points to discretize the triangulation became a common technique to obtain an approximation for the geometric shortest path problem in weighted regions (cf. [3, 4]). The general idea of this technique is to place a set of Steiner points in each triangle and then build a graph by connecting them. The approximation solution is achieved by finding a shortest path inside this graph, by using well-known combinatorial algorithms (e.g., Dijkstra's, BUSHWHACK[4]). Some geometric factors (such as segment lengths, angles) are taken into account in the process of Steiner point placement. Therefore, the number of Steiner points depends on these geometric factors.

To the best of our knowledge, nobody has studied the weighted shortest path problem when the input is an arrangement of lines. It is impossible to cover the whole length of the lines with Steiner points, because lines are infinite and we cannot afford an infinite number of Steiner points. Therefore, in this context, the first challenge is to bound the number of Steiner points. Consequently, we need a bound on the region that weighted shortest paths, from s to t, lie on. After establishing this bound (i.e., a closed region) the infinite lines can be clipped to bounded length segments, and the faces of the arrangement inside that region can be triangulated. Finally, by using the algorithm in [3] a $(1+\varepsilon)$-approximation can be obtained.

The formal problem statement is as follows: let s and t be two points in the plane $\mathbb{R}^2$ and let $\mathcal{A}$ be an arrangement of n lines l_i, $i = 1 \ldots n$. For simplicity, assume no two lines in $\mathcal{A}$ are parallel to each other and no three lines have a common intersection. Each face of $\mathcal{A}$ is assigned positive weight w_i. By convention, the weight of each edge of $\mathcal{A}$ is the minimum of the weights of its adjacent faces. The task is to find a closed region in $\mathbb{R}^2$ that contains a weighted shortest path from s to t, π_{st}.

A naive solution is a circle, centered at s whose radius is the Euclidean distance between s and t multiplied by $w_{max} = \max_i w_i$. It is easy to see that π_{st} will be inside this circle. This circle clips the lines to segments and the lengths of segments are bounded by the diameter of the circle. However, this bound is very sensitive to outliers and if w_{max} is large, then so is the size of the circle. In this paper, we propose an algorithm to construct a closed polygonal region, called SP-Hull (*Shortest Path Hull*), that only depends on the geometry of the arrangement and is independent of the weights. This al-

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gorithm exploits the fact that in an arrangement of lines, the lines outside of the convex hull of $\mathcal{A}$ diverge. Therefore, any shortest path, started and ended inside of the convex hull of $\mathcal{A}$, cannot go arbitrarily far from the convex hull (i.e., there is a bound). We show that there are some polygonal chains that define this bound for shortest paths, and they intersect in a restricted way. From this, we construct the SP-Hull. We will prove that any π_{st} lies inside the SP-Hull. We also justify that this is an optimally bounded region, one in which π_{st} is located in the absence of any assumptions on the weights.

The structure of this extended abstract is as follows. In Section 2, necessary preliminaries are presented. In Section 3, some relevant geometric properties are discussed. The construction algorithm for SP-Hull is described and analyzed in Section 4. Due to space limitation, some of the proofs of the lemmas are removed.

2 Preliminaries

Let $\mathcal{A}$ be an arrangement of n lines l_i, $i = 1 \dots n$, and P be the set of intersection points of l_i, $P = \{p_1, p_2, \cdots, p_{n(n-1)/2}\}$. The convex hull of P is denoted by $\mathcal{CH}(P) = \langle c_1, \cdots, c_H \rangle$. Each line l_i either intersects $\mathcal{CH}(P)$ twice, at $\mathfrak{a}_{i_1}$ and $\mathfrak{a}_{i_2}$, or contributes a segment to the boundary of the convex hull, $\partial\mathcal{CH}(P)$, from $\mathfrak{a}_{i_1} \in l_i$ to $\mathfrak{a}_{i_2} \in l_i$. For each l_i, $i = 1 \dots n$, we define two non-intersecting rays (subset of l_i) from $\mathfrak{a}_{i_1}$ and $\mathfrak{a}_{i_2}$, respectively toward infinity. Sort all the rays based on their slopes, and arrange them in a counter-clockwise order around $\mathcal{CH}(P)$. This defines an order for the rays $R = \langle r_1, r_2, \cdots, r_{2n} \rangle$ (Figure 1). This is a circular order and the relation "<" is well-defined. Note that all the rays diverge and there is no intersection between any two of them in the exterior of $\mathcal{CH}(P)$.

For simplicity, it is assumed that s and t are inside (or on the boundary of) $\mathcal{CH}(P)$. If they are not, a set of at most three lines, passing through s and t, can be added to the arrangement. This ensures that s and t are not outside of $\mathcal{CH}(P)$. However, π_{st} does not necessarily lie inside $\mathcal{CH}(P)$. For example, in Figure 1, suppose the weight of the face f_i is "very large" and the weight of the face f_{i+1} is "very cheap". Then, the shortest path from s to t goes outside of $\mathcal{CH}(P)$, as depicted in the figure.

In this paper, each ray is identified by a pair $r = \langle \mathfrak{a}, \vec{d} \rangle$, where $\mathfrak{a}$ is the starting point on the boundary of $\mathcal{CH}(P)$ and $\vec{d}$ is a vector pointing away from $\mathcal{CH}(P)$. W.l.o.g., it can be assumed for the remainder of the paper that the angle between any two consecutive rays, $r_1 = \langle \mathfrak{a}_1, \vec{d}_1 \rangle, r_2 = \langle \mathfrak{a}_2, \vec{d}_2 \rangle \in R$, is less than $\frac{\pi}{2}$. If it is not, (since this angle is less than π) one extra ray $r' = \langle \mathfrak{a}', \vec{d}_1 + \vec{d}_2 \rangle$ can be added in between, where $\mathfrak{a}'$ is a point on the boundary of $\mathcal{CH}(P)$, between $\mathfrak{a}_1$ and $\mathfrak{a}_2$. The total number of such angles greater than or equal to $\frac{\pi}{2}$ in R is at most 4. Therefore, by adding a constant number of rays to R this assumption holds.

Definition 1 (Order of the points on a ray) *For two points x and y on a ray $r_i = \langle \mathfrak{a}, \vec{d} \rangle$, $x \prec y$ if $|\vec{\mathfrak{a}x}| < |\vec{\mathfrak{a}y}|$, where $|.|$ denotes the length of a vector.*

Note that this is defined for points on a ray $r_i \subset l_j$. The point $\mathfrak{a}$ is mapped to zero, and the points on the ray r_i are mapped to $\mathbb{R}^+$, in the direction of $\vec{d}$.

Definition 2 (Chains: chain$_i^{ccw}$ and chain$_i^{cw}$) *Let c_i be a vertex of $\mathcal{CH}(P)$ corresponding to the intersection of rays r_{i-1} and r_i. The chain$_i^{ccw}$ is a polygonal chain, starting from c_i, defined as follows. Find the normal from c_i to r_{i+1}. Let it be incident at the point h_{i+1}. Find the normal from h_{i+1} to r_{i+2} and repeat until, either the normal is incident on a vertex of $\mathcal{CH}(P)$ or is incident on a point in the interior of $\mathcal{CH}(P)$. Then, chain$_i^{ccw} = \langle c_i, h_{i+1}, \cdots, h_j \rangle$, where $h_j \in \mathcal{CH}(P)$ (see Figure 1). The chain$_i^{cw}$ is defined analogously.*

The inner angle between two consecutive segments of chain$_i^{ccw}$ is the angle on the left-hand side, when the direction is from c_i towards h_j. Analogously, for chain$_i^{cw}$, it is the angle on the right-hand side.

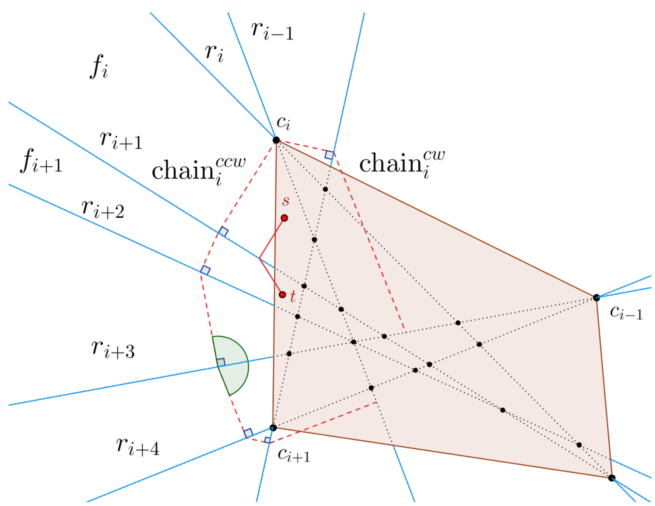


Figure 1: For each line in the arrangement there are two rays (in blue). Also, each vertex of $\mathcal{CH}(P)$, denoted by c_i, has two chains, chain$_i^{ccw}$ and chain$_i^{cw}$ (the red dashed lines in the figure). One of the inner angles of chain$_i^{ccw}$ is shown in the figure (incident at r_{i+3}). Furthermore, suppose the weight of f_i is "very large" and the weight of f_{i+1} is "very cheap". Then, π_{st} goes outside of $\mathcal{CH}(P)$.

3 Geometric Properties

In this section, some of the geometric properties on the order of the rays in the set R are discussed. Based on these properties, some lemmas about the chains, which are the primitive elements for constructing SP-Hull are proven.

Property 1 *Let $r_h < r_i < r_j$ be three rays in R and the angles between r_h and r_i, and, r_i and r_j, are both less than $\frac{\pi}{2}$. Let x (y) be a point on r_h (r_j).*

(a) The normal from x to r_j, lies on the left side of the normal from x to r_i, directed from x toward r_i. Analogously, the normal from y to r_h, lies on the right side of the normal from y to r_i, directed from y toward r_i (Figure 2 a).

(b) The normals from x and y to r_i lie on the opposite sides of the straight line that connects x to y, $\overline{xy}$ (or both coincide with $\overline{xy}$) (Figure 2 b).

Proof. The property follows from the fact that rays in R diverge and do not intersect in the exterior of $\mathcal{CH}(P)$. □

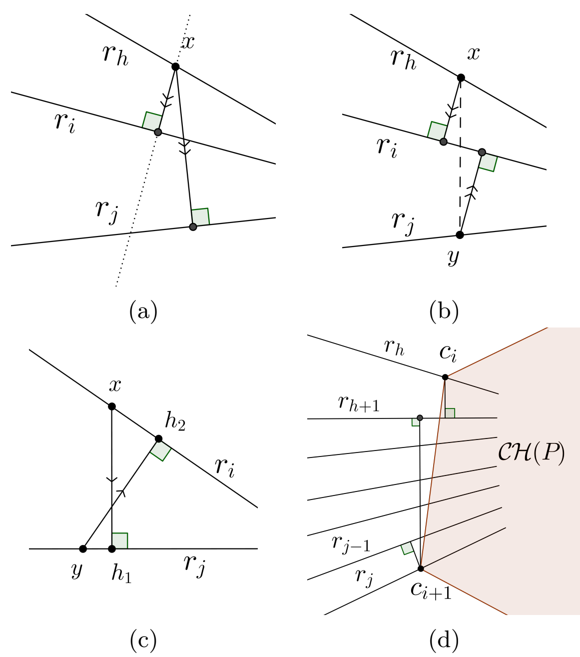


Figure 2: a) Property 1a, the normal from x to r_j lies on the left side of the normal from x to r_i. b) Property 1b, the normals from x and y to r_i lie on the opposite sides of $\overline{xy}$. c) Property 2b, if $\overline{xh_1}$ intersects with $\overline{yh_2}$, then $h_2 \prec x$ and $h_1 \prec y$. d) Lemma 1, one of the normals, either from c_i to r_{i+1} or from c_{i+1} to r_{i+k}, lies outside of $\mathcal{CH}(P)$.

Lemma 1 *Let $c_i \in r_h$ and $c_{i+1} \in r_j$ be two consecutive vertices of $\mathcal{CH}(P)$. (i) If $r_h < r_i < r_j$, then one of the normals from c_i or c_{i+1} to r_i lies outside of $\mathcal{CH}(P)$ (or on its boundary). (ii) One of the normals, either from c_i to r_{h+1} or from c_{i+1} to r_{j-1}, lies outside of $\mathcal{CH}(P)$ (or on it) (see Figure 2 d).*

Proof. (i) There is an edge e of $\mathcal{CH}(P)$ which is connecting c_i and c_{i+1}. By Property 1b, normals from c_i and c_{i+1} lie on the different sides of e or they coincide. Thus, either one of the normals lies outside or both are on the boundary of $\mathcal{CH}(P)$. (ii) If the normal from c_i to r_{h+1} lies outside the lemma is proved. Otherwise, by first part of this lemma, the normal from c_{i+1} to r_{h+1} lies outside (or on) the $\mathcal{CH}(P)$. Therefore, by Property 1a, the normal from c_{i+1} to r_{j-1} lies outside (or on it). □

Property 2 *Let $r_i < r_j$ be two rays in R so that the angle between them is less than $\frac{\pi}{2}$.*

(a) Let $x \prec y$ be two points on r_i. If the normal from x (y) to r_j is at h_1 (h_2), then $h_1 \prec h_2$.

(b) Let x and y be two points on r_i and r_j, respectively. If the normal from x (y) to r_j (r_i) is at h_1 (h_2), and $\overline{xh_1}$ intersects with $\overline{yh_2}$, then $h_2 \prec x$ and $h_1 \prec y$ (Figure 2 c).

Proof. The proof of (a) follows directly from the fact that rays in R diverge. To prove (b) assume that the axes are rotated until r_i is horizontal. Therefore, $\overline{yh_2}$ is vertical. Since r_i and r_j diverge, if x is chosen s.t. $x \prec h_2$ then $h_1 \prec y$. It implies that there will be no intersection. Therefore, to obtain an intersection between $\overline{xh_1}$ and $\overline{yh_2}$, x should be chosen s.t. $h_2 \prec x$. By this selection for x the only possible choice to pick y is $h_1 \prec y$. □

Lemma 2 *(i) All inner angles of a chain, are less than π. (ii) Furthermore, let $chain_i^{ccw} = \langle c_i, h_{i+1}, \cdots, h_{s-1}, h_s, h_{s+1}, \cdots\rangle$ and $chain_j^{cw} = \langle c_j, h'_{j-1}, \cdots, h'_{s+1}, h'_s, h'_{s-1}, \cdots\rangle$ intersect between r_s and r_{s+1} (see Figure 3a). Then, the common tangent l_t of $chain_i^{ccw}$ and $chain_j^{cw}$ passes through $h_s \in chain_i^{ccw}$ and $h'_{s+1} \in chain_j^{cw}$.*

Proof. (i) It follows directly from the fact that the rays diverge and chains are defined by the normals to the rays. (ii) We provide a proof by contradiction for one the cases, when l_t passes through h_{s-1} and h'_{s+1}. Other cases for other pairs of vertices are analogous. Since chain_i^{ccw} and chain_j^{cw} are intersecting, both lie on the same side of l_t. Therefore, the normal from h_{s-1} to r_s and from h'_{s+1} to r_s both lie on the same side of l_t. This contradicts Property 1b. □

Definition 3 (Complete revolution) *Suppose $R = \langle r_1, \cdots, r_{2n}\rangle$ is the counter-clockwise order of the rays and $chain_i^{ccw}$ ($chain_i^{cw}$) is initiated at $c_i \in \mathcal{CH}(P)$ where $c_i \in r_j$. A $chain_i^{ccw}$ ($chain_i^{cw}$), initiated at a point $x \in r_j$, is said to achieve a complete revolution, if it successively traverse all the rays in (reverse) order and returns back to r_j at a point x' such that x' is equal to x or $x \prec x'$.*

Lemma 3 *No chain starting at a vertex $c_i \in \mathcal{CH}(P)$ achieves a complete revolution.*

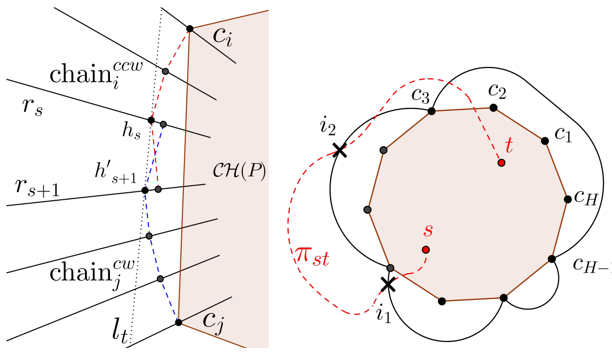


Figure 3: a) Two chains, chain$_i^{ccw}$ (the red dashed chain) and chain$_j^{cw}$ (the blue dashed chain), and their common tangent, l_t. b) An example of topological structure of the SP-Hull is shown in black solid lines. The red dashed line is the assumed weighted shortest path between s and t.

The proof of this lemma is based on the following observation. Always there exists a circle c_{max}, passing through c_i with the center inside $\mathcal{CH}(P)$, such that encloses $\mathcal{CH}(P)$. We can prove that the initiated chain at this vertex lies inside c_{max}. It implies that this chain does not achieve a complete revolution.

Lemma 4 *Let chain$_i^{ccw}$= $\langle c_i, h_{i+1}, \cdots, h_{s-1}, h_s, h_{s+1}, \cdots\rangle$ and chain$_j^{cw}$= $\langle c_j, h'_{j-1}, \cdots, h'_{s+1}, h'_s, h'_{s-1}, \cdots\rangle$ intersect between r_s and r_{s+1} (Figure 3a). Then, chain$_{ij}$= $\langle c_i, h_{i+1}, \cdots, h_{s-1}, h_s, h'_{s+1}, h'_{s+2}, \cdots, h'_{j-1}, c_j\rangle$ is a polygonal chain, connecting c_i to c_j and the inner angles of chain$_{ij}$ are less than π.*

Proof. By Lemma 2, chain$_{ij}$ from c_i to h_s and from h'_{s+1} to c_j is convex. Therefore, it is enough to show that $\angle h_{s-1}h_sh'_{s+1}$ and $\angle h_sh'_{s+1}h'_{s+2}$ are less than π. In Lemma 2 we showed that the common tangent of chain$_i^{ccw}$ and chain$_j^{cw}$, l_t, passes through h_s and h'_{s+1}. Since l_t is a straight line and both chains lie on the same side of l_t, $\angle h_{s-1}h_sh'_{s+1}$ and $\angle h_sh'_{s+1}h'_{s+2}$ are less than π. □

Let $\mathbb{CW}$ = {chain$_i^{cw}$| i = 1..H} and $\mathbb{CCW}$ = {chain$_i^{ccw}$| i = 1..H}.

Lemma 5 *Every $r_i \in R$ intersects with at least one of chain$_j^{ccw} \in \mathbb{CCW}$ or chain$_{j+1}^{cw} \in \mathbb{CW}$.*

Proof. Every $r_i \in R$ is between two consecutive vertices of $\mathcal{CH}(P)$, c_j and c_{j+1}. By Lemma 1, one of the normals from c_j and c_{j+1} to r_i is not inside $\mathcal{CH}(P)$. W.l.o.g. assume that the normal from c_j to r_i is not inside. By Property 1a, chain$_j^{ccw}$ lies on the left side (or on) the normal from c_j to r_i. Therefore, chain$_j^{ccw} \in \mathbb{CCW}$ intersects r_j. □

Lemma 6 *Any two chains in $\mathbb{CW}$ (or $\mathbb{CCW}$) are either disjoint or share an end-point at a vertex of $\mathcal{CH}(P)$.*

Proof. This proof uses contradiction. Suppose two chains, chain$_i^{cw}$ and chain$_j^{cw}$, intersect between two rays, r_s and r_{s+1}, not at a vertex of $\mathcal{CH}(P)$. Suppose chain$_i^{cw}$ intersects r_s at x and r_{s+1} at h. Also, chain$_j^{cw}$ intersects r_s at y and r_{s+1} at h'. W.l.o.g. assume $x < y$. If they intersect, it implies $h' < h$. This contradicts Property 2a. □

Definition 4 (Maximal chain) *Suppose chain$_i^{ccw}$ starts at r_j and ends at r_{j+k}, that is, chain$_i^{ccw}$ covers rays from r_j to r_{j+k-1}. We represent chain$_i^{ccw}$ by a range $[j, \cdots, j+k-1]$. It is a subrange[1] of a circular range of integers $[1, \cdots, 2n]$. We say chain$_i^{ccw}$ is maximal if there is no chain$_x^{ccw} \in \mathbb{CCW}$ or chain$_x^{cw} \in \mathbb{CW}$ such that its representative range fully covers the range $[j, \cdots, j+k-1]$. Analogously, the maximal chain$_i^{cw}$ is defined.*

Let $\mathbb{CCW}_{max}$ = {chain$_i^{ccw}$ | i = 1..H, s.t. chain$_i^{ccw}$ is maximal} and $\mathbb{CW}_{max}$ = {chain$_i^{cw}$ | i = 1..H, s.t. chain$_i^{cw}$ is maximal}. By Lemma 6, $\mathbb{CCW}_{max}$ ($\mathbb{CW}_{max}$) is a set of chains such that their representative ranges are disjoint.

Lemma 7 *Suppose chain$_i^{ccw} \in \mathbb{CCW}_{max}$ and it covers the starting point of chain$_x^{cw} \in \mathbb{CW}_{max}$. Then they do not intersect.*

Proof. By definition, chain$_i^{ccw}$ starts at the boundary of $\mathcal{CH}(P)$ and ends inside. Therefore chain$_i^{ccw}$ forms a closed region with the boundary of $\mathcal{CH}(P)$. By the assumption of the lemma, chain$_x^{cw}$ starts from inside the corresponding region of chain$_i^{ccw}$. If these two chains intersect, the intersection contradicts Property 2b. □

Corollary 1 *Let chain$_i^{ccw}$, chain$_j^{ccw}$ $\in \mathbb{CCW}_{max}$ (chain$_i^{cw}$, chain$_j^{cw}$ $\in \mathbb{CW}_{max}$) be two disjoint chains. There is no chain$_x^{cw} \in \mathbb{CW}_{max}$ (chain$_x^{ccw} \in \mathbb{CCW}_{max}$) that intersects both of them.*

Proof. If chain$_x^{cw}$ intersects chain$_i^{ccw}$ and chain$_j^{ccw}$ without intersecting $\mathcal{CH}(P)$, then by Lemma 7 it must intersects one of them at least twice. W.l.o.g. assume that chain$_x^{cw}$ intersects chain$_i^{ccw}$ twice, once to enter the closed region formed by chain$_i^{ccw}$ and once to leave it. The second intersection contradicts Property 2b. □

Lemma 8 *Each chain$_i^{ccw} \in \mathbb{CCW}_{max}$ intersects exactly one chain$_j^{cw} \in \mathbb{CW}_{max}$, or it ends at a vertex $c_x \in \mathcal{CH}(P)$.*

[1]For simplicity, we are omitting "modulo" as this is a circular range.

Proof. By definition, $\text{chain}_i^{ccw} \in \mathbb{CCW}_{max}$ starts at a vertex of $\mathcal{CH}(P)$. We prove that if it does not end at another vertex of $\mathcal{CH}(P)$, then it intersects exactly one $\text{chain}_j^{cw} \in \mathbb{CW}_{max}$.

Suppose r_s is the ray that chain_i^{ccw} ends on. Thus, the intersection of chain_i^{ccw} and r_s is in the interior of $\mathcal{CH}(P)$. By Lemma 5, there exists another chain, chain_x, that intersects r_s outside (or on) $\mathcal{CH}(P)$. This chain_x cannot be a member of $\mathbb{CCW}_{max}$ because it either intersects with chain_i^{ccw} (which contradicts Lemma 6) or fully covers chain_i^{ccw} (which contradicts maximality). Therefore, it is a member of $\mathbb{CW}$ and intersects chain_i^{ccw}. If it is maximal, we have proved that there exist at least one chain in $\mathbb{CW}_{max}$ that intersects. If it is not maximal, then there exists a maximal chain, chain_y, that fully covers chain_x. By the same reasoning, chain_y cannot be a member of $\mathbb{CCW}_{max}$. Therefore, chain_y is a member of $\mathbb{CW}_{max}$ and intersects chain_i^{ccw} (if it does not intersect, it should fully cover chain_i^{ccw} which contradicts maximality of chain_i^{ccw}).

Now suppose there are two chains chain_x^{cw} and $\text{chain}_y^{cw} \in \mathbb{CW}_{max}$, that intersect chain_i^{ccw}. By Corollary 1, chain_x^{cw} and chain_y^{cw} should either intersect each other (which contradicts Lemma 6) or one should fully cover the other one (which contradicts maximality). Therefore, there exists exactly one $\text{chain}_j^{cw} \in \mathbb{CW}_{max}$ that intersects chain_i^{ccw}. □

4 The construction algorithm

In this section, we present an algorithm to construct the SP-Hull (Algorithm 1). The input is an arrangement of lines $\mathcal{A}$, a source s, and a target t. The assumption is that s and t are inside $\mathcal{CH}(P)$. The output is a simple closed polygonal region SP-Hull that encloses $\mathcal{CH}(P)$. The idea to construct SP-Hull is to cover all vertices of $\mathcal{CH}(P)$ by some polygonal chains, chain_{ij}, which lie outside of $\mathcal{CH}(P)$ (see Figure 3b). We will prove that any weighted shortest path from s to t lies inside SP-Hull. Furthermore, we will argue its minimality.

Theorem 9 *Let $\mathcal{A}$ be an arrangement of lines. Any weighted shortest path between two points inside $\mathcal{CH}(P)$, lies inside SP-Hull of $\mathcal{A}$, constructed by Algorithm 1.*

Proof. This proof has two main steps. First, we prove that SP-Hull, generated by Algorithm 1, is a simple polygon that encloses $\mathcal{CH}(P)$. In the second step we prove, by contradiction, that any weighted shortest path between s and t, π_{st}, does not go outside of SP-Hull, where $s, t \in \mathcal{CH}(P)$.

Based on the construction in Algorithm 1, SP-Hull is a sequence of chains, chain_{ij}, which do not overlap and cover all of the rays (Lemma 5). Figure 3b shows an example of topological structure of SP-Hull around $\mathcal{CH}(P)$. By Lemma 4, each chain_{ij} is a simple chain in

Algorithm 1 SP-Hull

Input: Source (s), target (t), an arrangement of n lines ($\mathcal{A}$)

Output: A simple closed polygon, SP-Hull

Compute $convex\_hull(\mathcal{A})$, $CH = \langle c_1, \cdots, c_H \rangle$;
Mark all $c_i \in CH$ as not covered;
Find $\mathbb{CCW}_{max}$ and $\mathbb{CW}_{max}$ sets and sort them based on chains' subscripts;
while all c_i's are not covered **do**
    chain_i^{ccw}= First element of $\mathbb{CCW}_{max}$;
    if chain_i^{ccw} intersects with chain_k^{cw} **then**
        chain_{ik}=Merge (chain_i^{ccw} , chain_k^{cw});
        Mark all c_j ($j = i..k$) as covered;
    else/*chain_i^{ccw} ends at $c_x \in CH$*/
        $\text{chain}_{ix} = \text{chain}_i^{ccw}$;
        Mark all c_j ($j = i..x$) as covered;
return the list of chain_{ij}, sorted by their first index (i.e., i), as the SP-Hull;

which its inner angles are less than π. It starts and ends at the vertices of $\mathcal{CH}(P)$. Therefore, the SP-Hull is a closed simple polygon. Also, each chain_{ij} by definition is outside of $\mathcal{CH}(P)$. Therefore, SP-Hull encloses $\mathcal{CH}(P)$.

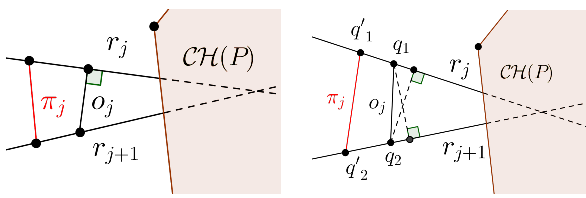
Before continuing the proof, let us introduce some notation. If π_x is a polygonal chain and a and b are two points on π_x, then $\pi_x[a,b]$ denotes the subpath of π_x from a to b.

In the second step of the proof, we show that no point of π_{st} lies in the exterior of SP-Hull. We prove this by contradiction. Since s and t are inside $\mathcal{CH}(P)$, π_{st} intersects SP-Hull at least twice. Let i_1 and i_2 be the first two consecutive intersections of π_{st} and SP-Hull (see Figure 3b). Our claim is SP-Hull$[i_1, i_2]$ is shorter than $\pi_{st}[i_1, i_2]$ which is a contradiction to the fact that π_{st} is a shortest path.

Suppose there are k regions between i_1 and i_2 which are separated by $k-1$ rays. W.l.o.g., let the rays in order be $\langle r_1, \cdots, r_{k-1} \rangle$. The number of segments in SP-Hull$[i_1, i_2]$ is at most k. Furthermore, the number of segments in $\pi_{st}[i_1, i_2]$ is at least k, as it must traverse through k diverging regions. We will show that each segment of SP-Hull$[i_1, i_2]$, o_j, is shorter than the corresponding segment of $\pi_{st}[i_1, i_2]$ in that region, π_j. Then, the total length of SP-Hull$[i_1, i_2]$ is smaller than the total length of $\pi_{st}[i_1, i_2]$ and we will arrive at a contradiction.

From the fact that there is no intersection between SP-Hull$[i_1, i_2]$ and $\pi_{st}[i_1, i_2]$ from i_1 to i_2, o_j and π_j do not intersect. There are two cases: the segment o_j is one of the normals in a chain that is contributing to SP-Hull, or it is a segment introduced by merging of two chains. The first case is shown in Figure 4a. In this case, even if π_j is perpendicular, o_j is shorter because the rays are diverging.

For case 2, assume that the endpoints of o_j are q_1 and q_2, and the endpoints of π_j are q'_1 and q'_2 (see Figure 4b). Since r_j and r_{j+1} diverge, translating π_j toward $\mathcal{CH}(P)$ makes it shorter. Therefore, the shortest possible length for π_j while avoiding an intersection between o_j and π_j, is when one of the endpoints of π_j is as close as possible to one of the endpoints of o_j. Assume q'_1 is equal to q_1. Then, o_j is shorter than π_j because of the following observation. The distance function from a point x to a ray r is a convex function (i.e., there is one line segment that connects x to a point $x_{opt} \in r$ such that it has the minimum length). Assume a line segment from x to $x_0 \in r$. By translating x_0 on r toward x_{opt} it length becomes shorter. □



(a) (b)

Figure 4: a) Proof of Theorem 9, case 1. b) Proof of Theorem 9, case 2.

Theorem 10 *For an arrangement of n lines, SP-Hull can be computed in $\mathcal{O}(n \log n)$ time.*

Proof. Computing the convex hull of P takes $\mathcal{O}(n \log n)$ time and its size is $\mathcal{O}(n)$ [1].

The key here is that it is possible to find $\mathbb{CCW}_{max}$ ($\mathbb{CW}_{max}$) in linear time without computing all chain_i^{ccw} (chain_i^{cw}), $i = 1 \cdots H$. Lemma 6 implies that if $c_j \in CH$ is covered by a chain_i^{ccw} then we can skip computing chain_j^{ccw} and chain_j^{cw}, because they are not maximal. Also, members of $\mathbb{CCW}_{max}$ do not overlap. Therefore, the computation of $\mathbb{CCW}_{max}$ requires at most two traversals of the rays.

In the While-loop, $\mathbb{CCW}_{max}$ ($\mathbb{CW}_{max}$) is a set of non-overlapping ranges that are sorted. Based on Lemma 8, each member of $\mathbb{CCW}_{max}$, either has exactly one intersection with a member of $\mathbb{CW}_{max}$, or both endpoints of that chain are vertices of CH. Therefore, finding the intersecting chains takes constant time, by comparing only the endpoints of the first and the last chains in the sets. When an intersection is detected, then remove both chains from the sets, merge them and repeat. Since the total number of operations for merging all intersected chains is equal to the number of rays, the While-loop takes linear time. □

Minimality of SP-Hull

In Theorem 9, we have shown that π_{st} lies inside SP-Hull. Now we address its minimality. We show that for any arrangement of lines, $\mathcal{A}$, it is possible to assign weights to the faces of $\mathcal{A}$ and choose $s, t \in \mathcal{CH}(P)$ such that π_{st} is arbitrarily close to the boundary of SP-Hull.

The procedure is as follows. Assign the weight "infinity" to the bounded faces of $\mathcal{A}$. By this assignment, we make sure that π_{st} does not traverses these faces. Choose one of the chains in SP-Hull, say chain_{ij}. This chain is either chain_i^{ccw}, or chain_j^{cw}, or the result of merging them. Here, we prove the minimality for the merging case. The other cases are analogous.

Let chain_{ij} be the result of merging chain_i^{ccw} and chain_j^{cw}. W.l.o.g., assume that chain_i^{ccw} starts at $c_i \in \mathcal{CH}(P)$ and intersects $\mathcal{CH}(P)$ at point $x \in \partial\mathcal{CH}(P)$. Place s on c_i and t on x. Assume chain_i^{ccw} traverses k unbounded faces in order, $\langle f_1, \cdots, f_k \rangle$. The weight for the other unbounded faces that are not visited by this chain, is set to infinity. To make π_{st} close enough to chain_i^{ccw}, the corresponding weights for f_i, $i = 1 \ldots k$, are set in such a way that $w_1 \gg w_2 \gg \cdots \gg w_k$. It suffices to set the weights of f_i, $i = 1 \ldots k$, as z^i. If z goes to zero, then $w_i \gg w_{i+1}$ and π_{st} is arbitrarily close to chain_i^{ccw}. An analogous argument can be used to become as close as possible to chain_j^{cw}.

5 Further Work

Analogous question arises for existence of such bounded region for an arrangement of line segments in an appropriately defined weighted region problem. Suppose P is the set of endpoints of line segments and their intersections. It is not difficult to show that if s and t are inside the $\mathcal{CH}(P)$, then π_{st} will not go further than the boundary of $\mathcal{CH}(P)$. Also, an interesting extension of this problem is the question of existence of such bound for a given arrangement of curves (e.g., algebraic curves).

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References

[1] M. Atallah. *Computing the convex hull of line intersections.* J. Algorithms 7, 285-288, 1986.

[2] J.S.B. Mitchell and C.H. Papadimitriou. *The weighted region problem: finding shortest paths through a weighted planar subdivision.* J. ACM 38, 1:18-73, 1991.

[3] L. Aleksandrov, A. Maheshwari, and J.-R. Sack. *Determining approximate shortest paths on weighted polyhedral surfaces.* J. ACM 52, 1:25-53, 2005.

[4] Z. Sun, J.H. Reif. *On finding approximate optimal paths in weighted regions.* J. Algorithms 58, 132, 2006.