On the Number of Arrangements of Pseudolines

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Abstract. Given a simple arrangement of n pseudolines in the Euclidean plane, associate with line i the list σ_i of the lines crossing i in the order of the crossings on line i. $\sigma_i = (\sigma_1^i, \sigma_2^i, .., \sigma_{n-1}^i)$ is a permutation of $\{1, .., n\} - \{i\}$. The vector $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an encoding for the arrangement. Define $\tau_j^i = 1$ if $\sigma_j^i > i$ and $\tau_j^i = 0$, otherwise. Let $\tau_i = (\tau_1^i, \tau_2^i, .., \tau_{n-1}^i)$, we show that the vector $(\tau_1, \tau_2, \ldots, \tau_n)$ is already an encoding.

We use this encoding to improve the upper bound on the number of arrangements of n pseudolines to $2^{0.6974 \cdot n^2}$. Moreover, we have enumerated arrangements with 10 pseudolines. As a by-product we determine their exact number and we can show that the maximal number of halving lines of 10 point in the plane is 13.

1 Introduction

Arrangements of lines and pseudolines are recognized as important and appealing objects for research in geometry and combinatorics. A general theory of arrangements is given in Grünbaum's monograph [8]. The oriented matroid point of view on arrangements is taken in Björner et al. [2]. Enumeration questions for arrangements are discussed in ([2], Subsection 6.5) and in Knuth ([9], Section 9). In most texts arrangements of pseudolines are defined with the real projective plane as ambient space. In contrast, we consider arrangements in the Euclidean plane.

Let a *pseudoline* be an x-monotone curve in the Euclidean plane. An *arrangement of pseudolines* is a family of pseudolines with the property that each pair of pseudolines has a unique point of intersection where the two pseudolines cross. An arrangement is *simple* if no three pseudolines have a common point of intersection. Throughout this manuscript the term *arrangement* if not specified further will always denote a simple arrangement of pseudolines. The *size* of an arrangement is the number of its pseudolines. Given an arrangement $\mathcal{A}$ of size n we label the pseudolines so that they cross a vertical line left of all intersections in increasing order from bottom to top.

An arrangement partitions the plane into cells of dimensions 0, 1 or 2, the *vertices, edges* and *faces* of the arrangement. The cells of an arrangement carry a natural lattice structure. Adding a **0** and a **1** element we obtain the *face lattice* of the arrangement. Two arrangements are considered to be *isomorphic* if their face lattices are isomorphic under the correspondence induced by some labeling.

Particularly nice pictures of arrangements of pseudolines are given by their *wiring diagrams* introduced in Goodman [5], see Figure 1. Let $\mathcal{W}$ be a wiring diagram of a simple arrangement of size n. For each abscissa x where no crossing takes place the vertical order (upwards) of the pseudolines at x is a permutation π_x of $\{1..n\}$. Assuming that no two crossings of $\mathcal{W}$ have the same x position we obtain $\binom{n}{2}+1$ different permutations. Denote by Σ the sequence of these permutations in left to right order. We note two properties of sequence Σ.

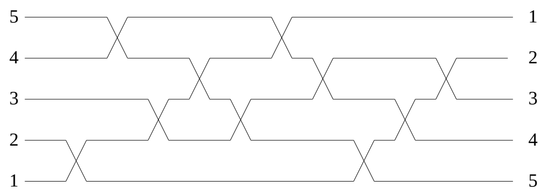

Figure 1. Wiring diagram.

(1) The first element of Σ is the identity permutation $(1,2,\ldots,n)$ and the last element of Σ is the reverse permutation $(n,\ldots,2,1)$.

(2) Two consecutive permutations in Σ differ by the reversal of an adjacent pair.

Following Goodman and Pollack [6, 7] we call a sequence Σ of $\binom{n}{2}+1$ permutations of $\{1..n\}$ satisfying the above properties a *simple allowable sequence*. In general allowable sequences it is allowed for consecutive permutations to differ by the reversal of a larger substring. A simple allowable sequence is easily transformed into a wiring diagram and, hence, an arrangement of pseudolines. Note, however, that many allowable sequences may correspond to the same arrangement. Consecutive pairs of crossings that have no pseudoline in common can be interchanged without changing the arrangement.

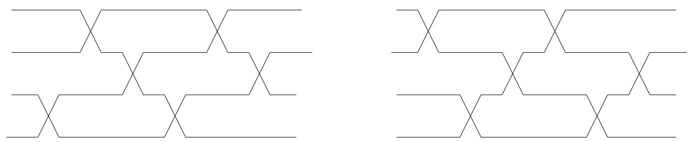

Figure 2. Wiring diagrams corresponding to one arrangement but two allowable sequences.

Simple allowable sequences are basically the same as reflection networks, see Knuth [9]. Alternatively, they can also be seen as maximal chains in the weak Bruhat order of the symmetric group. In this last context their number A_n has been determined by Stanley [10]. His remarkable formula is

$$A_n = \frac{\binom{n}{2}!}{\prod_{k=1}^{n-1}(2n-2k-1)^k}.$$

Edelman and Greene [3] prove this formula via a combinatorial bijection between different types of tableaux.

Let B_n be the number of non-isomorphic simple arrangements of size n. Besides the numbers A_n and B_n we will consider their logarithms $a_n = \log_2 A_n$ and $b_n = \log_2 B_n$. From the above remarks it follows that there are more allowable sequences than arrangements, i.e., $b_n < a_n$. From Stanley's formula an $O(n^2 \log n)$ upper bound for a_n follows. Knuth [9] proves lower and upper bounds for the number of arrangements:

$$2^{\frac{n^2}{6} - \frac{5n}{2}} \leq B_n \leq 3^{\binom{n+1}{2}}$$

This gives $b_n \leq 0.7924\ (n^2 + n)$. Knuth reports on some computations supporting a conjecture of $b_n \leq \binom{n}{2}$. From the sharpest version of the zone theorem (Bern et al. [1]) a bound of $b_n \leq 0.7194\, n^2$ is obtained. In the next section we propose a new encoding of arrangements from which we easily obtain $b_n \leq 0.7213\, n^2$. In Section 3 we work a little harder to obtain an improved bound of $b_n \leq 0.6974 n^2$.

2 An encoding for arrangements

Representing an arrangement by an allowable sequence can be seen as an encoding by an ordered sequence of vertical cuts through the arrangement. A representation by a sequence of horizontal cuts can be obtained by associating with line i the list σ_i of the lines crossing i in the order of the crossings on line i. To an arrangement $\mathcal{A}$ thus corresponds a vector $(\sigma_1, \ldots, \sigma_n)$ where σ_i is a permutation of $\{1, .., i-1, i+1, .., n\}$. As will be shown in this section it suffices to know which entries of σ_i are larger than i in order to obtain an encoding for $\mathcal{A}$.

Definition 1 *Let $\mathcal{T}_n$ be the set of n-tuples $(\tau_1, \tau_2, \ldots, \tau_n)$ with $\tau_i = (t_1^i, t_2^i, \ldots, t_{n-1}^i)$ a binary vector and $\sum_{j=1}^{n-1} t_j^i = n - i$ for all i.*

Define a mapping Φ from arrangements of size n to $\mathcal{T}_n$. Given an arrangement $\mathcal{A}$ let τ_i report the crossings of pseudoline i with the other lines from left to right. More precisely $t_j^i = 1$ if the j-th crossing on line i is a crossing with a line with index larger than i. In the wiring diagram this corresponds to a move of wire i up into the next track. Conversely $t_j^i = 0$ if line i is moving down at the j-th crossing, i.e., if the j-th crossing on line i is a crossing with a line with index smaller than i. Each of the $n-1$ lines different from i contributes exactly one crossing on line i and $n-i$ of these lines have a larger label than i. This proves that $(\tau_1, \tau_2, \ldots, \tau_n) = \Phi(\mathcal{A})$ is in $\mathcal{T}_n$. E.g., the element of $\mathcal{T}_4$ corresponding to the arrangement represented by the wiring diagram of Figure 1 is

$$T = ((1,1,1,1), (0,1,1,1), (0,1,1,0), (1,0,0,0), (0,0,0,0)).$$

Of course, not all elements of $\mathcal{T}_n$ correspond to an arrangement, e.g., for $n = 4$ we have 9 elements in $\mathcal{T}_4$ but only 8 arrangements. The element of $\mathcal{T}_4$ not in the image of Φ is $T = ((1,1,1), (1,0,1), (0,1,0), (0,0,0))$.

Theorem 1 *The mapping Φ is injective.*

Proof. Algorithmically the tool of choice for the construction of the face lattice of an arrangement of pseudolines is a topological sweep (see Edelsbrunner and Guibas [4]). Imagine a sweep of arrangement $\mathcal{A}$ as a move of a topological line continuously from left to right across the plane. All incidences between cells of the arrangement are visited by the line during this move. We discretize the line and replace it by a *cut* of edges of the arrangement. This is a list $(e_1, e_2, \ldots, e_n)$ of edges obeying the conditions:

(1) Edge e_1 is on the boundary of the bottom face, i.e., on the face containing the vertical ray to $-\infty$ and edge e_n is on the boundary of the top face, i.e., the face containing the vertical ray to $+\infty$.

(2) For each $1 \leq i \leq n-1$ there is a face F_i of the arrangement with edges e_i and e_{i+1} on its boundary.

To get from the bottom face to the top face every pseudoline has to be crossed. Since a cut consists of n edges only it follows that the order of edges of a cut represents a permutation of the lines of the arrangement. The sweep begins at the leftmost cut consisting of all left unbounded edges. The permutation corresponding to this cut is the identity permutation.

An *advance move* corresponds to shifting the topological line cross a point of the arrangement. The admissible points for advance moves are those with both left edges in the current cut (Figure 3).

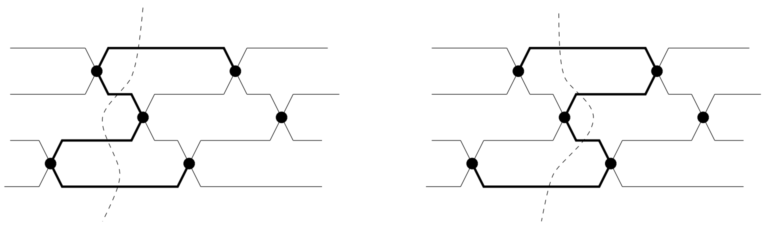

Figure 3. Advancing the cut across a vertex.

To make the algorithm deterministic our sweep always has to pick the lowest admissible point for the advance move. Formally, let i be the least index such that the right endpoints of edges e_i and e_{i+1} coincide in the current cut $(e_1, \ldots, e_n)$. The next cut is $(e_1, .., e_{i-1}, e'_i, e'_{i+1}, e_{i+2}, .., e_n)$ where e'_i is the edge right of e_{i+1} on the same pseudoline and e'_{i+1} is the edge right of e_i on the same pseudoline. In general, if two cuts differ by an advance move the corresponding permutations differ by an adjacent transposition. As long as some edges in the cut have right endpoints an advance move is possible. The algorithm terminates when the current cut has become the rightmost cut consisting of all right unbounded edges and the vertical order of the lines is reversed. The sequence of permutations of the cuts visited by the algorithm is a *cannonical* allowable sequence for the arrangement.

The next algorithm works with input $\Phi(\mathcal{A})$ and produces a sequence of permutations. The first permutation $\pi = (\pi_1, .., \pi_n)$ is the identity. We initialize an edge counter $s(i) = 1$ for each line i and let $v_i = t^{\pi_i}_{s(\pi_i)}$. The *bit-state* of the algorithm is the vector $v = (v_1, v_2, \ldots, v_n)$. It will be important to keep in mind that v depends on π and s. Initially v_i is simply the first bit of τ_i where $\Phi(\mathcal{A}) = (\tau_1, \ldots, \tau_n)$.

In each step the algorithm takes the least index i with $v_i = 1$ and $v_{i+1} = 0$. Edge counters $s(\pi_i)$ and $s(\pi_{i+1})$ are increased by one and π is changed by an adjacent transposition at position i, i.e., π becomes $(\pi_1, .., \pi_{i-1}, \pi_{i+1}, \pi_i, \pi_{i+2}, .., \pi_n)$.

The claim is that sweeping $\mathcal{A}$ and $\Phi(\mathcal{A})$ produces the same sequence of indices i for advance moves and consequently the same, i.e, the cannonical allowable sequence. We compare the two sweeps by making simultaneous advance steps in both algorithms. Let $e = (e_1, \ldots, e_n)$ be the current cut and $v = (v_1, \ldots, v_n)$ be the current bit state. The following invariant suffices to prove the claim by induction.

($\star$) The current permutation of both algorithms agree. Moreover, the least i such that the right endpoints of e_i and e_{i+1} coincide equals the least i with $v_i = 1$ and $v_{i+1} = 0$.

This is trivially verified at the beginning. Now suppose that ($\star$) is true after some fixed number of moves of both algorithms.

Both algorithms make their next advance at the same index i and the two lines involved in the crossing are determined by the permutation, hence, they are the same. It follows that the new permutations agree. Let π be the new permutation, e be the new cut and v be the new bit state. Consider any index j with $v_j = 1$ and $v_{j+1} = 0$. This means that at its next crossing line π_j is moving up while line π_{j+1} is moving down at its next crossing. Since line π_j is below line π_{j+1} and they border a common face in $\mathcal{A}$ they cross each other, i.e., edges e_j and e_{j+1} have a common right endpoint. Conversely, if edges e_j and e_{j+1} have a common right endpoint then line π_j is moving up while line π_{j+1} is moving down at the next crossing, hence, $v_j = 1$ and $v_{j+1} = 0$. This proves the invariant.

By ($\star$) the sweep algorithms for $\mathcal{A}$ and $\Phi(\mathcal{A})$ produce the same allowable sequence. The sequence characterizes arrangement $\mathcal{A}$. This proves the injectivity of mapping Φ. $\square$

We have seen that Φ is an injective mapping from arrangements of size n to elements of $\mathcal{T}_n$. Counting elements of $\mathcal{T}_n$ is a trivial task, $|\mathcal{T}_n| = \binom{n-1}{0}\binom{n-1}{1}\binom{n-1}{2}\dots\binom{n-1}{n-1}$.

Fact 1. $b_n < \sum_{k=1}^{n-1} k \log e = 0.7213\,(n^2 - n)$.

Proof. Let $f(n) = \binom{n-1}{0}\dots\binom{n-1}{n-1}$, hence $f(n) = \frac{(n-1)^{n-1}}{(n-1)!} f(n-1)$. The formula of Stirling gives $\log f(n) = (n-1)\log e + \log f(n-1)$. The claim follows by induction. $\triangle$

Compared to the best known bound $b_n \leq 0.7194\, n^2$ this was surprisingly easy to obtain.

For a better understanding of the encoding Φ it would be interesting to have some tools to discriminate between members from $\mathcal{T}_n$ that are in the image of Φ and those that are not. At this time we have little more than the second algorithm from the above proof. We can take arbitrary elements $T \in \mathcal{T}_n$ as input to this algorithm. The two possible outcomes are.

(1) The algorithm gets stuck before $\binom{n}{2}$ moves have been made, i.e., in the current vector V there is no index i with $v_i = 1$ and $v_{i+1} = 0$.

(2) T indeed corresponds to an arrangement.

Other cases can be ruled out as follows. Suppose that T can be swept and consider the sequence of permutations generated. Since line i moved up $n - i$ times and down $i - 1$ line i ends up on wire $n - i + 1$. This proves that we end up with the reverse permutation. Hence, the sequence is allowable and corresponds to an arrangement.

3 A better bound for b_n

Recall the element $T = ((1,1,1),(1,0,1),(0,1,0),(0,0,0))$ of $\mathcal{T}_4$ not in the image of Φ. Trying to sweep T we get stuck after three moves. At the second move we already note that something goes wrong since the lines involved in the crossing of the first move cross back. Call an *immediate back-cross* a situation where two lines cross twice in a row. Geometrically this corresponds to two edges with the same left and right endpoints. When sweeping $T \in \mathcal{T}_n$ we recognize an immediate back-cross when the pair $(v_i, v_{i+1}) = (1,0)$ of the move is replaced by $(v'_i, v'_{i+1}) = (1,0)$, i.e., the vectors v and v' before and after the move are identical.

Note that the sweep corresponding to $T \in \mathcal{T}_n$ is completely determined by the initial vector v and a sequence of *replace pairs* $w_1, w_2, \ldots, w_{\binom{n}{2}}$. If the jth move of the sweep interchanges π_i and π_{i+1} we replace $(v_i, v_{i+1}) = (1,0)$ by the pair $w_j = (w_j^1, w_j^2)$. A sequence of replace pairs leads to an immediate back-cross exactly if one of the pairs w_j is $(1,0)$. The number of back-cross free elements of $\mathcal{T}_n$ and, hence, the number of arrangements can thus be estimated from above by the number of initial vectors v and the number of $(1,0)$ free sequences of replace pairs. For v there are $\leq 2^n$ choices and for each pair w_j there remain 3 choices, therefore:

Fact 2. $B_n \leq 2^n 3^{\binom{n}{2}}$, i.e, $b_n \leq 0.7924\, n^2 + O(n)$.

The proof of Fact 1 made use only of the number of 0 and 1 in each τ_j. The proof of Fact 2 is based on forbidding immediate back-crossings. With the replace matrix we next define a representation that helps to take care of both aspects. Estimating the number of replace matrices will enable us to slightly improve the upper bound for b_n in Theorem 2.

Definition 2 *A* replace matrix *is a binary $n \times n$ matrix M with properties*

(1) $\displaystyle\sum_{j=1}^{n} m_{ij} = n - i$ *for* $i = 1, .., n$,

(2) $m_{ij} \geq m_{ji}$ *for all* $i < j$.

Lemma 1 *There is an injective mapping Ψ from arrangements of size n to $n \times n$ replace matrices.*

Proof. Consider $\Phi(\mathcal{A})$ and let $m_{ii} = t_1^i$, that is, we record the initial v of the sweep of $\Phi(\mathcal{A})$ along the diagonal of M. If in the kth move of the sweep of $\Phi(\mathcal{A})$ lines i and j cross we define $m_{ij} = 1$ if the next crossing (after the crossing with line j) of line i goes up and $m_{ij} = 0$ if the next crossing of line i goes down, respectivly $m_{ij} = t^i_{s(i)+1}$. If $i < j$ then at their crossing line i is going up and line j is going down. Since the lines don't back-cross we have $(m_{ij}, m_{ji}) \neq (0,1)$ or equivalently $m_{ij} \geq m_{ji}$. After the complete sweep of $\Phi(\mathcal{A})$ we remain with a single undefined entry in each row of M. Let this entry be 0. Suppose $i < j$ and m_{ij} was the last undefined entry of its row. It follows that after crossing j from below line i was not involved in further crossings. If line j had a further crossing then it had to move down there since the position above j was occupied by i, hence, $m_{ji} = 0$. Otherwise line j had no further crossings and again $m_{ji} = 0$.

Property (1) of replace matrices is easily seen to hold for M as defined above. The entries in row i of M are the entries of τ_i in $\Phi(\mathcal{A})$ and an additional 0 in some permutation. Hence, $M = \Psi(\mathcal{A})$ is a well defined replace matrix. To show that this mapping is injective we sweep $M = \Psi(\mathcal{A})$ and reconstruct $\Phi(\mathcal{A})$. The details very similar to the arguments in the proof of Theorem 1 are left to the reader. $\square$

We illustrate this encoding of arrangements by replace matrices by giving the replace matrix corresponding to the arrangement of Figure 1. In that case

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To obtain an estimate for the number of replace matrices we use probabilistic arguments. Consider the probability space Ω of all binary $n \times n$ matrices with $\sum_{j=1}^{n} m_{ij} = n - i$ for $i = 1, .., n$ and let M be an uniformly distributed random variable in Ω. Let p_i be the probability that a fixed entry in row i of M is 0, i.e., $p_i = \frac{i}{n}$, and $q_i = 1 - p_i$ be the probability that this entry is 1, i.e., $q_i = \frac{n-i}{n}$.

For $i < j$ let E_{ij} be the event $m_{ij} \geq m_{ji}$. Since $m_{ij} \not\geq m_{ji}$ is equivalent to $(m_{ij}, m_{ji}) = (0, 1)$ the probability of event E_{ij} is $\mathrm{Prob}[E_{ij}] = (1 - p_i q_j)$. For the number R_n of replace matrices we have $R_n = |\Omega| \mathrm{Prob}[\bigwedge_{i<j} E_{ij}]$.

Carelessly assuming independence of the events E_{ij} we obtain as estimate for R_n the product $\prod_{k=0}^{n-1} \binom{n}{k} \prod_{i<j} (1 - \frac{i(n-j)}{n^2})$. The logarithm of this function behaves like $0.66n^2$. Of course due to the fixed row sums of matrices in Ω the E_{ij} are not independent. There are positively and negatively correlated pairs $E_{ij}, E_{ij'}$, therefore is not obvious in which direction the error made by ignoring dependencies goes. In the remaining part of this section we derive a valid estimate for R_n.

Lemma 2 *If I is a subset of $\{(i,j) : 1 \leq i < j \leq n-1\}$ such that* $\mathrm{Prob}[E_\alpha | \bigwedge_{\beta \in J} E_\beta] \leq \mathrm{Prob}[E_\alpha]$ *for all $\alpha \in I$ and $J \subseteq I - \alpha$ then $R_n \leq |\Omega| \prod_{\alpha \in I} \mathrm{Prob}[E_\alpha]$.*

Proof. For every enumeration $\alpha_1, .., \alpha_{|I|}$ of I we have $\mathrm{Prob}[\bigwedge_{i<j} E_{ij}] \leq \mathrm{Prob}[\bigwedge_{\alpha \in I} E_\alpha] = \prod_{i=1}^{|I|} \mathrm{Prob}[E_{\alpha_i} | \bigwedge_{j<i} E_{\alpha_j}]$. The assumption on I implies $\mathrm{Prob}[E_{\alpha_i} | \bigwedge_{j<i} E_{\alpha_j}] \leq \mathrm{Prob}[E_{\alpha_i}]$ for all i. □

Lemma 3 *The set $I = \{(i,j) : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor < j \leq n\}$ obeys the condition of Lemma 2.*

Proof. Let $\Omega(i,j)$ be the set of matrices that can be obtained from matrices of Ω by removing rows i and j. Think of $\Omega(i,j)$ as the set of $(n-2) \times n$ matrices with rows indexed $1, .., i-1, i+1, .., j-1, j+1, .., n$ and $\sum_{l=1}^{n} m_{kl} = n - k$ for index k. Given $M' \in \Omega(i,j)$ let $\#(M')$ be the number of matrices M in Ω that reduce to M' by removing rows i and j, equivalently $\#(M')$ counts the number of pairs (r_i, r_j) of rows that extend M' to a matrix in Ω. Generalizing this notation let $\#(M' : E)$ be the number of pairs of rows that extend M' to a matrix M in Ω so that E holds for M. Let $\alpha = (i,j) \in I$ and $J \subseteq I - \alpha$. The following inequalities are equivalent.

$$\begin{aligned} \mathrm{Prob}[E_\alpha] &\geq \mathrm{Prob}[E_\alpha | \bigwedge_{\beta \in J} E_\beta] \\ \mathrm{Prob}[\neg E_\alpha] &\leq \mathrm{Prob}[\neg E_\alpha | \bigwedge_{\beta \in J} E_\beta] \\ \mathrm{Prob}[\neg E_\alpha] \cdot \mathrm{Prob}[\bigwedge_{\beta \in J} E_\beta] &\leq \mathrm{Prob}[\neg E_\alpha \wedge \bigwedge_{\beta \in J} E_\beta] \end{aligned}$$

$$\sum_{M'\in\Omega(i,j)} \#(M'\colon \neg E_\alpha) \sum_{M'\in\Omega(i,j)} \#(M'\colon \bigwedge_{\beta\in J} E_\beta) \;\leq\; \sum_{M'\in\Omega(i,j)} \#(M') \sum_{M'\in\Omega(i,j)} \#(M'\colon \neg E_\alpha \wedge \bigwedge_{\beta\in J} E_\beta)$$

$$\sum_{M' N'\in\Omega(i,j)} \#(M'\colon \neg E_\alpha)\#(N'\colon \bigwedge_{\beta\in J} E_\beta) \;\leq\; \sum_{M' N'\in\Omega(i,j)} \#(M')\#(N'\colon \neg E_\alpha \wedge \bigwedge_{\beta\in J} E_\beta)$$

We claim that the last of these inequalities holds component-wise.

Claim. for any pair M', N' of matrices in $\Omega(i,j)$:

$$\#(M'\colon \neg E_\alpha)\#(N'\colon \bigwedge_{\beta\in J} E_\beta) \leq \#(M')\#(N'\colon \neg E_\alpha \wedge \bigwedge_{\beta\in J} E_\beta)$$

$\#(M')$ counts the number of pairs (r_i, r_j) of row vectors that extend $M' \in \Omega(i,j)$ to $M \in \Omega$. The condition on r_i is $\sum_{l=1}^n r_{il} = n-i$, there are $\binom{n}{n-i}$ choices for r_i. The number of choices for r_j is $\binom{n}{n-j}$.

Now consider the pairs (r_i, r_j) counted by $\#(M'\colon \neg E_\alpha)$. To match condition $\neg E_\alpha$ the values $r_{ij} = 0$ and $r_{ji} = 1$ are required. There remain $\binom{n-1}{n-i}$ choices for r_i and $\binom{n-1}{n-j-1}$ choices for r_j.

The number $\#(N'\colon \bigwedge_{\beta\in J} E_\beta)$ really depends on N' respectively on the column vectors s_i and s_j of N'. First consider the choices for r_i. To match the conditions E_β for $\beta \in J$ certain relations between entries of r_i and s_i must hold. Note that due to the choice of I we have $i \leq n/2$ and all pairs containing i in J are of the form (i,k), i.e., $n/2 < k$ and all relations forced between s_i and r_i are of the form $r_{ik} \geq s_{ki}$. Relevant for r_i are only those positions with $s_{ki} = 1$. Let λ_1 be the number of pairs $(i,k) \in J$ with $s_{ki} = 1$, hence, conditions E_β for $\beta \in J$ force exactly λ_1 positions $r_{ik} = 1$. There remain $\binom{n-\lambda_1}{n-i-\lambda_1}$ choices for r_i. For r_j note that all pairs containing j in J are of the form (k,j), i.e., $k \leq n/2 < j$ and all relations forced between s_j and r_j are of the form $r_{kj} \leq s_{jk}$. Define λ_0 as the number of pairs $(k,j) \in J$ with $s_{jk} = 0$. There remain $\binom{n-\lambda_0}{n-j}$ choices for r_j.

Finally, consider $\#(N'\colon \neg E_\alpha \wedge \bigwedge_{\beta\in J} E_\beta)$. Compared to the previous case we have additionally fixed values $r_{ij} = 0$ in r_i and $r_{ji} = 1$ in r_j. Hence, $\binom{n-\lambda_1-1}{n-i-\lambda_1}$ choices for r_i and $\binom{n-\lambda_0-1}{n-j-1}$ choices for r_j. The claim is thus boiled down to the verification of

$$\tbinom{n-1}{n-i}\tbinom{n-1}{n-j-1}\tbinom{n-\lambda_1}{n-i-\lambda_1}\tbinom{n-\lambda_0}{n-j} \leq \tbinom{n}{n-i}\tbinom{n}{n-j}\tbinom{n-\lambda_1-1}{n-i-\lambda_1}\tbinom{n-\lambda_0-1}{n-j-1}.$$

Both of the following inequalities hold separately. Use $\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k}$ and $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ for their proofs.

$$\tbinom{n-1}{n-i}\tbinom{n-\lambda_1}{n-i-\lambda_1} \leq \tbinom{n}{n-i}\tbinom{n-\lambda_1-1}{n-i-\lambda_1}$$

$$\tbinom{n-1}{n-j-1}\tbinom{n-\lambda_0}{n-j} \leq \tbinom{n}{n-j}\tbinom{n-\lambda_0-1}{n-j-1}$$

□

Theorem 2 *The number B_n of arrangements of n pseudolines is at most*

$$\prod_{k=0}^{n-1}\binom{n}{k} \prod_{1\leq i\leq \frac{n}{2}<j\leq n} \Big(1 - \frac{i(n-j)}{n^2}\Big)$$

and hence $b_n \leq 0.6974\, n^2$.

Proof. The above lemmas allows to bound the number R_n of $n \times n$ replace matrices by $|\Omega| \prod_{(i,j)\in I}(1 - \frac{i(n-j)}{n^2})$. Plugging in $|\Omega| = \prod_{k=0}^{n-1}\binom{n}{k}$ and the definition of I bounds R_n by the above formula. By Lemma 1 the bound holds true for the number of arrangements. Taking logarithms we obtain $r_n \leq \log_2(e)(\binom{n+1}{2} - \sum_{(i,j)\in I}\log(1 - \frac{i(n-j)}{n^2}))$. The inner sum is $\sum_{i,j\leq n/2}\log(1-(i/n)(j/n))$ and can (e.g. by `Maple`) be estimated as

$$\int_0^{1/2}\int_0^{1/2}\log(1-xy)\,dx\,dy = -0.01658.$$

altogether $r_n \leq \log_2(e)(1/2 - 0.0165)n^2 = 0.6974\, n^2$. □

Enumeration

$B_{10} = 18{,}410{,}581{,}880$. This is an additional value for the table of Knuth ([9], page 35). This number was obtained by a recursive program. Given an arrangement $\mathcal{A}$ of n pseudo-lines the program generated all cuts from the top to the bottom face. The cuts correspond to all possible ways to thread a $(n+1)$st line into the arrangement. For $n \leq 9$ this resulted in the number B_n given by Knuth.

As a byproduct of the counting algorithm we also found that the maximum number h_{10} of halving-lines a set of 10 points in the plane can have is 13 (Figure 4). This adds a new value to the list $h_4 = 3$, $h_6 = 6$ and $h_8 = 9$. Via the duality between non-vertical lines and points $(y = ax + b) \leftrightarrow (a, b)$ a halving line of point-set P corresponds to a cell c in the arrangement dual to P such that a vertical line through c crosses half of the lines above and the other half below c. We call the set of these cells the *middle-level* of the arrangement. Note that the leftmost and the rightmost cell of the middle-level of an arrangement correspond to the same halving line in the dual. For more on the size of middle-levels and the more general k-set problem see [11], [7] and the references therein.

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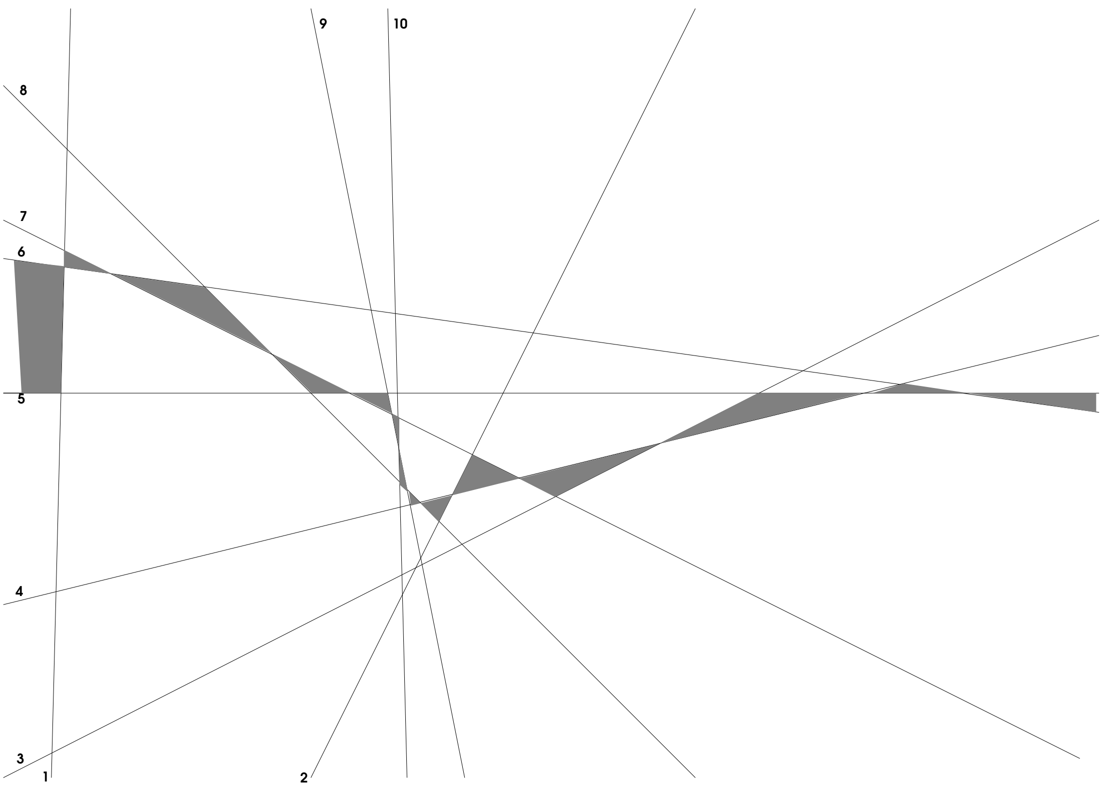

Figure 4. Ten lines with 14 cells in the middle-level.