Extendability of higher dimensional signotopes*

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Abstract

In 1926, Levi showed that, for any pseudoline arrangement $\mathcal{A}$ and two points in the plane, $\mathcal{A}$ can be extended by a pseudoline which contains the two prescribed points. Later extendability was studied for arrangements of pseudohyperplanes in higher dimensions. While the extendability with d prescribed points in an arrangement of proper hyperplanes in $\mathbb{R}^d$ is trivial, Richter-Gebert (1993) found an arrangement of pseudoplanes in $\mathbb{R}^3$ which cannot be extended with two particular prescribed points.

In this article, we investigate the extendability of signotopes, which are a rich subclass of oriented matroids. Our main result is that signotopes of odd rank are extendable with two prescribed crossing points. Moreover, we conjecture that for all even ranks $r \geq 4$ there exist signotopes which are not extendable for two prescribed points. Our conjecture is supported by examples for rank 4, 6, and 8.

1 Introduction

Given a family of hyperplanes $\mathcal{H}$ in $\mathbb{R}^d$, any d points in $\mathbb{R}^d$, not all on a common hyperplane of $\mathcal{H}$, define a hyperplane which is distinct from the hyperplanes in $\mathcal{H}$. For dimension $d = 2$, Levi [12] proved in his pioneering article on pseudoline arrangements that the fundamental extendability of line arrangements also applies to the more general setting of pseudoline arrangements. A *pseudoline* is a Jordan curve in the Euclidean plane such that its removal from the plane results in two unbounded components, and a *pseudoline arrangement* is a family of pseudolines such that each pair of pseudolines intersects in exactly one point, where the two curves cross.

▶ **Theorem 1.1** (Levi's extension lemma for pseudoline arrangements [12]). *Given an arrangement $\mathcal{A}$ of pseudolines and two points in $\mathbb{R}^2$, not lying on a common pseudoline of $\mathcal{A}$. Then $\mathcal{A}$ can be extended by an additional pseudoline which passes through the two prescribed points.*

Several proofs for Levi's extension lemma are known today (besides [12], see also [1, 8, 16]) and generalizations to higher dimensions have been studied in the context of oriented matroids, which are by the representation theorem of Folkman and Lawrence [9] projective pseudohyperplane arrangements. For more about oriented matroids, see [4].

Goodman and Pollack [11] showed that there is an arrangement of 8 pseudoplanes in $\mathbb{R}^3$ such that three particular prescribed points do not determine a pseudoplane which is compatible with the arrangement. Richter-Gebert [15] then investigated a weaker version with only two prescribed points such that the extending pseudohyperplane contains these

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two points. More specifically, he gave an example of a rank 4 oriented matroid on 8 elements such that even the weaker version does not hold. However, the existence of an extension lemma or counterexamples remains open in higher dimensions/ranks.

In this article, we present a proof of Levi's extension lemma in a purely combinatorial setting which generalizes to higher dimensions. Our proof uses the notion of r-signotopes and applies to even dimensions d, that is, when the rank $r = d + 1$ is odd; see Theorem 1.2. However, there are non-extendable examples for the ranks 4, 6, and 8, and we conjecture that there is no extension lemma for any even rank $r \geq 4$; see Conjecture 1.3.

Before we can formulate our extension lemma for r-signotopes, we have to introduce some notation, discuss the relation between pseudoline arrangements and signotopes (in Section 1.1), and find an appropriate reformulation of Levi's extension lemma which can be investigated in the context of signotopes (in Section 1.2).

1.1 Signotopes

Signotopes are a combinatorial structure generalizing permutations and *simple* pseudoline arrangements (i.e., no three pseudolines cross in a common point). An r-*signotope* ($r \geq 1$) on n elements is a mapping σ from r-element subsets (r-*subsets*) of $[n]$ to $+$ or $-$, i.e., $\sigma : \binom{[n]}{r} \to \{+, -\}$ such that for every $(r+1)$-subset $X = \{x_1, \ldots, x_{r+1}\}$ of $[n]$ with $x_1 < x_2 < \ldots < x_{r+1}$ there is at most one sign change in the sequence

$$\sigma(X \backslash \{x_1\}), \sigma(X \backslash \{x_2\}), \ldots, \sigma(X \backslash \{x_{r+1}\}).$$

Note that this sequence lists the signs of all induced r-subsets of X in reverse lexicographic order. For 3-signotopes, the following 8 sign patterns on 4-subsets are allowed:

$$++++,\ +++-,\ ++--,\ +---,\ ----,\ ---+,\ --++,\ -+++\,.$$

It is well-known that every arrangement of pseudolines is isomorphic to an arrangement of x-monotone pseudolines [10]. If we label the pseudolines from top to bottom on the left by $1, \ldots, n$, we can read its corresponding 3-signotope σ. The sign of $\sigma(a, b, c)$ for $a < b < c$ indicates the orientation of the triangle formed by the pseudolines a, b, c (see Figure 1). If $\sigma(a, b, c) = +$ the crossing of a and c is below b and if $\sigma(a, b, c) = -$ the crossing of a and c is above b. Furthermore, σ gives information about the ordering of the crossings from left to right along each pseudoline. If $\sigma(a, b, c) = +$ it holds $bc \succ ac \succ ab$ and if $\sigma(a, b, c) = -$ it is $bc \prec ac \prec ab$.

Figure 1 Connection between pseudoline arrangements and 3-signotopes.

Felsner and Weil [8] showed that rank 3 signotopes are in correspondence with simple pseudoline arrangements in $\mathbb{R}^2$ with a special top cell related to the cyclic arrangement. For $r \geq 4$, r-signotopes correspond to special pseudohyperplane arrangements in $\mathbb{R}^{r-1}$, i.e., they are a subclass of oriented matroids of rank r. A geometric representation of r-signotopes in the plane is presented in [13] (see also [3] for the rank 3 case).

1.2 An extension lemma for signotopes

In Levi's extension lemma for pseudoline arrangements, each of the two prescribed points can either lie in a cell of the arrangement, on a pseudoline, or be the crossing point of two pseudolines. To formulate an extension lemma in terms of 3-signotopes, which only captures the combinatorics of an arrangement, we restrict ourselves to simple pseudoline arrangements and to prescribed points, which are crossing points. Crossing points in a pseudoline arrangement are subsets of cardinality 2 given by the two crossed elements. Since the extending pseudoline passes through the two prescribed crossing points, the extension yields a non-simple arrangement. However, by perturbing the extending pseudoline at the non-simple crossing points, we end up with a simple arrangement, see Figure 2.

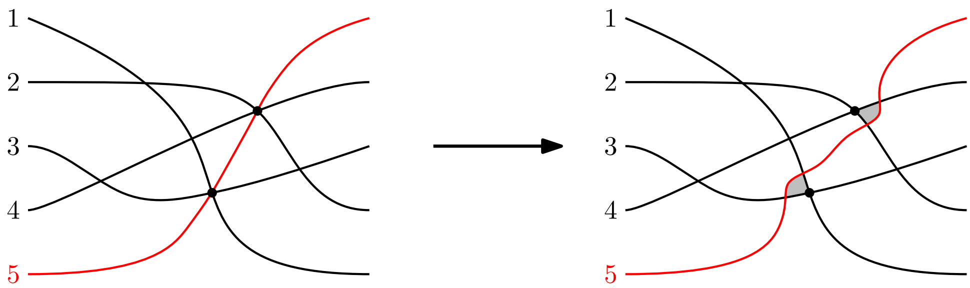

Figure 2 Perturbing an extending pseudoline at the two non-simple crossing points.

A perturbation at a prescribed crossing yields a *triangular cell* incident to the crossing. This cell is bounded by the two pseudolines defining the crossing and the extending pseudoline. Triangular cells play an important role in the study of pseudoline arrangements, since it is possible to change the orientation of a triangle by moving one of its bounding pseudolines over the crossing of the two others. Such a local perturbation is called *triangle flip* and does not change the orientation of any other triangle in the arrangement. For 3-signotopes triangular cells correspond to a 3-subset for which we can exchange the corresponding sign and it remains a signotope. We call such a 3-subset a *fliple*. The notion of fliples generalizes to higher ranks. In an r-signotope σ on $[n]$, an r-subset $X \subseteq [n]$ is a *fliple* if both assignments $+$ and $-$ to $\sigma(X)$ result in a signotope.

When we apply Levi's extension lemma to extend an arrangement of pseudolines, which are ordered from top to bottom on the left, we do not know at which place of the order the new pseudoline will be inserted. In particular, the label of all pseudolines which start below the new one increases by one. To cope with this relabeling-issue in terms of signotopes, we introduce the following notion. For $k \in [n]$ and a subset X of $[n]$, we define

$$X{\downarrow_k} = \{x \mid x \in X, x < k\} \cup \{x - 1 \mid x \in X, x > k\}.$$

For an r-signotope σ on the elements $[n]$, we define the k *deletion* $\sigma{\downarrow_k}$ on $[n-1]$ by $\sigma{\downarrow_k}(X{\downarrow_k}) = \sigma(X)$ for all r-sets $X \subseteq [n]$ with $k \notin X$. This is an r-signotope on $[n-1]$.

An r-signotope σ on n elements is *2-extendable* if for each pair of disjoint $(r-1)$-subsets I, J, there is an r-signotope σ^* on $[n+1]$ with fliples I^*, J^* and an *extending* element $k \in [n+1]$ such that $\sigma^*{\downarrow_k} = \sigma$, $I^*{\downarrow_k} = I$, and $J^*{\downarrow_k} = J$.

Using this notion we are now ready to formulate an extension lemma for r-signotopes.

▶ **Theorem 1.2** (Extension lemma for signotopes of odd rank)**.** *For every odd rank $r \geq 3$, every r-signotope is 2-extendable.*

The statement of Theorem 1.2 only applies to signotopes of odd rank. In fact, for ranks 4, 6, and 8, we found signotopes on 8, 12, and 16 elements, respectively, which are not 2-extendable[1]. Based on these examples, we dare the following conjecture:

▶ **Conjecture 1.3.** *For every even $r \geq 4$, there are r-signotopes which are not 2-extendable.*

Despite the restrictions to simple arrangements and crossing points as prescribed points, Theorem 1.2 implies Levi's extension lemma (Theorem 1.1). Details are deferred to the full version; see Appendix D for a preliminary version.

1.3 Signotopes as a rich subclass of oriented matroids

It is well known that the number of oriented matroids on n elements of rank r is $2^{\Theta(n^{r-1})}$ [4, Corollary 7.4.3]. As shown by Balko [2], r-signotopes are a rich subclass of oriented matroids of rank r.

▶ **Proposition 1.4.** *For $r \geq 3$, the number of r-signotopes on n elements is $2^{\Theta(n^{r-1})}$.*

In ranks 1 and 2 there are 2^n and $n!$ signotopes on $[n]$, respectively. For rank $r \geq 3$, the precise number of r-signotopes on $[n]$ has been computed for small values of r and n; see A6245 (rank 3) and A60595 to A60601 (rank 4 to rank 10) on the OEIS [14].

2 Preliminaries

We now prepare for the proof of Theorem 1.2. In rank 3 the left to right order on each pseudoline yields a partial order of the crossing points of the arrangement. We now define the corresponding partial order on the $(r-1)$-subsets associated with a r-signotope σ. For every r-subset $X = \{x_1, \ldots, x_r\}$ define:

$$X\backslash\{x_1\} \succ X\backslash\{x_2\} \succ \cdots \succ X\backslash\{x_r\} \quad \text{if} \quad \sigma(x_1, \ldots, x_r) = +, \qquad \text{and}$$
$$X\backslash\{x_1\} \prec X\backslash\{x_2\} \prec \cdots \prec X\backslash\{x_r\} \quad \text{if} \quad \sigma(x_1, \ldots, x_r) = -.$$

By taking the transitive closure of all relations obtained from r-subsets, we obtain a partial order on the $(r-1)$-subsets corresponding to σ [8, Lemma 10].

If we rotate an arrangement of pseudolines, i.e., we choose another unbounded cell as the top cell, we get an pseudoline arrangement with the same cell structure. If we only rotate a single pseudoline, then the orientation of the triangle spanned by 3 pseudolines stays the same if and only if the rotated pseudoline is not involved (see for example the triangle spanned by 2,3,4 in the left, resp. 1,2,3 in the right arrangement in Figure 3). In terms of the 3-signotope σ the signs of the rotated signotope σ_{rot} are: $\sigma_{\text{rot}}(a, b, c) = \sigma(a+1, b+1, c+1)$ if $c \neq n$ and $\sigma_{\text{rot}}(a, b, n) = -\sigma(1, a+1, b+1)$.

In general, we define the clockwise rotated signotope σ_{rot} of a given r-signotope σ as:

$$\sigma_{\text{rot}}(x_1, \ldots, x_r) = \begin{cases} -\sigma(1, x_1+1, \ldots, x_{r-1}+1) & \text{if } x_1 < x_2 < \cdots < x_r = n, \\ \sigma(x_1+1, \ldots, x_r+1) & \text{if } x_1 < x_2 < \cdots < x_r < n. \end{cases}$$

Indeed, σ_{rot} is an r-signotope on n elements (see Lemma 3.1 in Appendix C). To keep track of the index shift caused by a clockwise rotation, we define

$$X_{\text{rot}} = \begin{cases} \{x_1 - 1, x_2 - 1, \ldots, x_k - 1\} & \text{if } x_1 \neq 1; \\ \{x_2 - 1, \ldots, x_k - 1, n\} & \text{if } x_1 = 1 \end{cases}$$

[1] The examples and the source code to verify their correctness are available on demand.

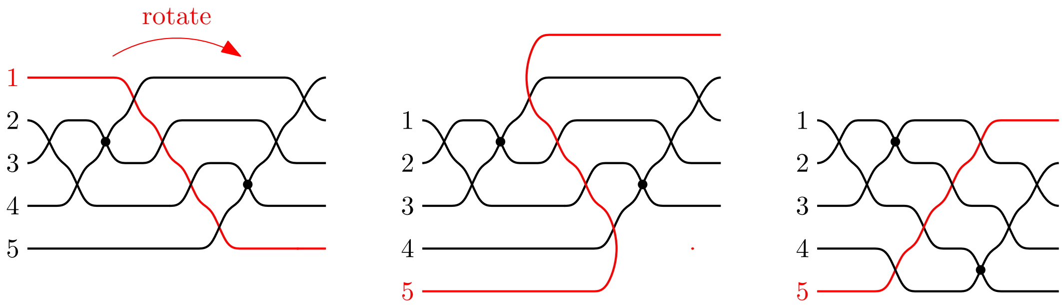

Figure 3 An illustration of a clockwise rotation. The rotated pseudoline is highlighted red.

for any subset $X = \{x_1, \dots, x_k\}$ of $[n]$ with $x_1 < \dots < x_k$.

3 Proof of Theorem 1.2

Using these properties we can give a proof for Levi's extension lemma using only the notation of signotopes and the corresponding partial order as introduced in Section 2.

If incomparable elements in the corresponding order are chosen as prescribed points, an arrangement is extendable by an element which we put in the last position, i.e., the $(n+1)$st element, see Figure 2. More abstractly we can extend the arrangement when the prescribed points are maximal elements of a down-set of the partial order. A *down-set* of a partial order $(P, \prec)$ is a subset $D \subset P$ such that for all $p \in P$ and $d \in D$ with $p \preceq d$ it holds $p \in D$.

▶ **Proposition 3.1.** *Let $(P, \prec)$ be the partial order on $(r-1)$-sets corresponding to an r-signotope σ on $[n]$. For every down-set $D \subseteq P$ there is a signotope σ^* on $[n+1]$ such that all r-subsets of the form $m \cup \{n+1\}$ for a maximal element m of D are fliples.*

Proof. Define the extended r-signotope σ^* on $[n+1]$ as follows:

$$\sigma^*(x_1 \dots, x_r) = \begin{cases} \sigma(x_1, \dots, x_r) & \text{if } x_1, \dots, x_r \in [n]; \\ + & \text{if } x_r = n+1 \text{ and } \{x_1, \dots, x_{r-1}\} \in D; \\ - & \text{if } x_r = n+1 \text{ and } \{x_1, \dots, x_{r-1}\} \notin D. \end{cases}$$

This is indeed an r-signotope and fulfills the conditions mentioned in the statement. Details are deferred to the full version; see Appendix B for a preliminary version. ◀

Note that Proposition 3.1 holds for general rank. For odd rank we can always find a rotation of the corresponding signotope such that the two prescribed $(r-1)$-subsets are incomparable and we can use Proposition 3.1 to define an extension.

▶ **Lemma 3.2.** *Let r be an odd integer, let σ be an r-signotope on $[n]$ and let X, Y be two disjoint $(r-1)$-subsets. After at most n clockwise rotations, σ, X, and Y are transformed into σ', X', and Y', resp., such that X' and Y' are incomparable in the partial order $\prec'$ corresponding to σ'.*

Proof. Assume X and Y are comparable in the partial order $\prec$ corresponding to the r-signotope σ with $X \prec Y$. We show that after n clockwise rotations, all signs of σ are reversed. Hence the partial order $\prec_{\text{rot}}$ is the reversed relation to $\prec$.

The sign of an r-subset $\{z_1, \dots, z_r\}$ changes from $+$ to $-$ or vice versa if and only if the rotated element is contained in $\{z_1, \dots, z_r\}$, i.e., if we rotate z_1. Hence after rotating n times in general every z_i was rotated and thus the sign of an r-subset changes exactly r times. Since r is odd, the sign after rotating n times is opposite. The obtained signotope is the reverse of the original signotope σ and the corresponding partial order is also reversed.

Since we cannot reverse the order of two disjoint $(r-1)$-sets in one rotation (see Corollary 3.8 in Appendix C), there will be a moment where the two disjoint sets are incomparable. ◀

Proposition 3.1 and Lemma 3.2 together imply Theorem 1.2, which completes the proof. The outline is as follows. Using Lemma 3.2, we rotate until the required disjoint $(r-1)$-subsets are incomparable. To extend the signotope we then use the down-set, which consists of all elements smaller than one of the two incomparable $(r-1)$-subsets. In this down-set the two prescribed $(r-1)$-subsets are the maximal elements. Hence we can apply Proposition 3.1 in order to add a new elements as required. Finally, we rotate back so that the original signotope is contained in the new extended signotope. Details are deferred to the full version.

4 Conclusion

Using complete enumeration of small signotopes and a SAT based test of extendability, we found a 4-signotope on 8 elements which is not 2-extendable. Since the number of signotopes explodes as the ranks increases, the complete enumeration was impossible in higher ranks. To still cope with higher ranks, we instead used a SAT based search for signotopes which share structural properties with the rank 4 example. This allowed us to find the examples in rank 6 and 8 which are not 2-extendable.

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A Proof of Proposition 1.4

A.1 Proof of the upper bound

The upper bound follows from the fact, that r-signotopes on n elements are rank r oriented matroids and their number is $2^{\Theta(n^{r-1})}$ [4, Chapter 7.4]. The outline of the proof is as following.

For rank 3, there exists a constant $c > 0$ such that for every n there are at most 2^{cn^2} signotopes on n elements [5]. The currently best bound is provided in [7]; see also the Handbook article [6].

For rank $r \geq 4$, we proceed by induction. Given an r-signotope σ on $[n]$, we compute its projections. For each $i \in [n]$, we project σ to i and obtain an $(r-1)$-signotope $\sigma/_i$ on $n-1$ elements. Formally, $\sigma/_i$ is defined by $\sigma/_i(J \downarrow_i) := \sigma(J)$ for every r-subset J containing i. Since two distinct r-signotopes yield different sequences $(\sigma/_i)_{i\in[n]}$ of projections, we can bound the number of r-signotopes as

$$f_r(n) \leq (f_{r-1}(n-1))^n \leq \left(2^{c(n-1)^{r-2}}\right)^n \leq 2^{cn^{r-1}},$$

where $f_{r-1}(n-1)$ denotes the number of $(r-1)$-signotopes on $n-1$ elements.

A.2 Proof of the lower bound

Suppose $n = rm$ for some $m \in \mathbb{N}$. We partition $[n] = \bigcup_{k=1}^{r} N_k$ with the intervals $N_k = [(k-1)m+1, km]$. We construct an r-signotope σ on the elements $[n]$ with

$$\sigma(x_1, \dots, x_r)$$
$$= \begin{cases} - & \text{if } x_{r-1} \notin N_r,\ x_r \in N_r, \text{ and } \sum_{k=1}^{r-1}(x_k - (k-1)m - 1) > x_r - (r-1)m - 1 \\ \star & \text{if } x_{r-1} \notin N_r,\ x_r \in N_r, \text{ and } \sum_{k=1}^{r-1}(x_k - (k-1)m - 1) = x_r - (r-1)m - 1 \\ + & \text{else,} \end{cases}$$

where all entries marked with a star ($\star$) can be chosen arbitrarily from $\{+,-\}$, i.e., they are fliples.

Let us now count the entries $\{x_1, \dots, x_r\} \in \binom{[n]}{r}$ that fulfill

$$\sum_{k=1}^{r-1}(x_k - (k-1)m - 1) = x_r - (r-1)m - 1 \tag{1}$$

If we further restrict our considerations to entries with $x_k \in N_k$ for all k, then we have a bijection to integer partitions

$$\sum_{k=1}^{r-1} y_k = y_r$$

with $y_1, \dots, y_r \in [0, m-1]$. For fixed $y_r = \ell$, we have $\binom{\ell+r-2}{r-2}$ solutions to $\sum_{k=1}^{r-1} y_k = \ell$, and therefore there are

$$\sum_{\ell=0}^{m-1} \binom{\ell+r-2}{r-2} = \binom{m+r-2}{r-1} = \Theta(n^{r-1})$$

entries $x_1, \dots, x_r \in \binom{[n]}{r}$ with $x_k \in N_k$ for all k that fulfill equation (1). Since each of them can be assigned to both, $+$ or $-$, there are at least $2^{\Omega(n^{r-1})}$ different possibilities to construct an r-signotope on n elements. It remains to show that each so-constructed mapping σ is an r-signotope by checking the signature of all its induced $(r+1)$-subsets.

Before we do so, note that equation (1) can be reformulated as

$$\sum_{k=1}^{r-1} x_k - x_r = m\frac{(r-4)(r-1)}{2} + (r-2), \tag{2}$$

where the right-hand side is only depending on r and m. Also note that the left-hand side $L(x_1,\dots,x_r) := \sum_{k=1}^{r-1} x_k - x_r$ fulfills the following properties:

▶ **Observation 1.1.** *Let $X = (x_1,\dots,x_{r+1}) \in \binom{[n]}{r+1}$ and $1 \le i < j < r+1$. Then*

$$L(X\backslash\{x_i\}) = \sum_{\substack{k\in[r]\\k\neq i}} x_k - x_{r+1} > \sum_{\substack{k\in[r]\\k\neq j}} x_k - x_{r+1} = L(X\backslash\{x_j\})$$

and

$$L(X\backslash\{x_r\}) = \sum_{k\in[r-1]} x_k - x_{r+1} < \sum_{k\in[r-1]} x_k - x_r = L(X\backslash\{x_{r+1}\}).$$

Let us now check the signature of all $(r+1)$-subsets $X = \{x_1,\dots,x_{r+1}\} \in \binom{[n]}{r+1}$, that is, there is at most one sign-change in the sequence

$$\sigma(X\backslash\{x_1\}),\dots,\sigma(X\backslash\{x_r\}),\sigma(X\backslash\{x_{r+1}\}).$$

and the assigned $(\star)$-entries are indeed fliples.

If $x_{r+1} \notin N_r$, we only have plus signs in the signature and therefore there is no sign change. Otherwise, there is some $k \in [r+1]$ such that $x_1,\dots,x_{k-1} \notin N_r$ and $x_k,\dots,x_{r+1} \in N_r$.

If $k < r$, then we have $x_{r-1}, x_r, x_{r+1} \in N_r$ and hence each set $X\backslash\{x_{r+1}\},\dots,X\backslash\{x_1\}$ contains at least two elements from N_r. It follows again that there are only plus signs in the signature and therefore there is no sign change.

If $k = r$, then each set $X\backslash\{x_{r-1}\},\dots,X\backslash\{x_1\}$ contains two elements from N_r and thus maps to plus. Moreover, by Observation 1.1, we have $L(X\backslash\{x_{r+1}\}) > L(X\backslash\{x_r\})$ and hence $X\backslash\{x_{r+1}\}$ and $X\backslash\{x_r\}$ cannot map to plus and minus, respectively. Consequently, there is at most one sign change.

If $k = r+1$, then $X\backslash\{x_{r+1}\}$ maps to plus. By Observation 1.1, L is increasing on $X\backslash\{x_r\},\dots,X\backslash\{x_1\}$. Consequently, there is at most one sign change.

This completes the proof of the lower bound.

B Proof of Proposition 3.1

We show that σ^* as defined in Proposition 3.1 is indeed an r-signotope on $[n+1]$ with the claimed fliples.

First we show that σ^* is a signotope. Consider an $(r+1)$-subset $X = \{x_1,\dots x_{r+1}\}$ with $x_1 < \dots < x_r < x_{r+1}$ of $[n+1]$. We need to show that the sequence

$$\sigma^*(X\backslash\{x_1\}),\sigma^*(X\backslash\{x_2\}),\dots,\sigma^*(X\backslash\{x_{r+1}\})$$

has at most one sign change.

If $x_{r+1} \le n$, then all signs on the considered r-subsets are the same as for σ. Since σ is an r-signotope, there is at most one sign change in the considered sequence.

In the other case, we have $x_{r+1} = n+1$. All r-subsets $X\backslash\{x_i\}$ for $i \in \{1,\dots,x_r\}$ do contain the element $x_{r+1} = n$. Furthermore, since $X\backslash\{x_{r+1}\}$ does not contain $n = x_{r+1}$, it is $\sigma^*(X\backslash\{x_{r+1}\}) = \sigma(X\backslash\{x_{r+1}\})$. We consider the two cases. First, if $\sigma(X\backslash\{x_{r+1}\}) = +$ we have by definition of the partial order

$$X\backslash\{x_{r+1}, x_i\} \succ X\backslash\{x_{r+1}, x_j\} \qquad \text{for } i<j.$$

By the property of a down-set this means whenever $X\backslash\{x_{r+1}, x_i\} \in D$ we also have $X\backslash\{x_{r+1}, x_j\} \in D$ for $i<j$. Let i^* be the smallest integer such that $X\backslash\{x_{r+1}, x_{i^*}\} \in D$. Then it holds $\sigma^*(X\backslash\{x_j\}) = +$ for all $j \geq i^*$. And hence there is at most one sign change between $\sigma^*(X\backslash\{x_{i^*-1}\}) = -$ and $\sigma^*(X\backslash\{x_{i^*}\}) = +$.

Similar arguments apply if $\sigma(X\backslash\{x_{r+1}\}) = -$. Then we have

$$X\backslash\{x_{r+1}, x_i\} \prec X\backslash\{x_{r+1}, x_j\} \qquad \text{for } i<j.$$

Again we have at most one sign change at between $\sigma^*(X\backslash\{x_{j^*}\}) = +$ and $\sigma^*(X\backslash\{x_{j^*+1}\}) = -$, where j^* is the largest index such that $X\backslash\{x_{r+1}, x_{j^*}\} \in D$.

Let m be a maximal element of the down-set. By the analysis above it follows that $m \cup \{n+1\}$ is a fliple.

C Properties of the clockwise rotation of an r-signotope

▶ **Lemma 3.1.** *Let σ be an r-signotope on $[n]$. Then the clockwise rotation σ_{rot} as defined in Section 2 is also an r-signotope on $[n]$.*

Proof. Consider an $(r+1)$-subset $X' = \{x'_1,\dots,x'_{r+1}\}$ with $x'_1 < \dots < x'_{r+1}$. Then there exists $X = \{x_1,\dots,x_{r+1}\}$ of $[n]$ with $x_1 < \dots < x_{r+1}$ such that $X' = X_{\mathrm{rot}}$ since the cyclic rotation gives a bijection on the $(r+1)$-subsets of $[n]$.

If the rotated element is not in X, that is, $x'_{r+1} < n$, then we have $x_i = x'_i + 1$ and the reversed lexicographic ordered sequence of signs given by X' is

$$\begin{array}{lllll} \sigma_{\mathrm{rot}}(X'\backslash\{x'_1\}), & \sigma_{\mathrm{rot}}(X'\backslash\{x'_2\}), & \dots, & \sigma_{\mathrm{rot}}(X'\backslash\{x'_r\}), & \sigma_{\mathrm{rot}}(X'\backslash\{x'_{r+1}\}) \\ = \sigma(X\backslash\{x_1\}), & \sigma(X\backslash\{x_2\}), & \dots, & \sigma(X\backslash\{x_r\}), & \sigma(X\backslash\{x_{r+1}\}), \end{array}$$

and therefore has at most one sign change.

If the rotated element is in X, that is, $x'_{r+1} = n$, then we have $x_1 = 1$ and $x_2 = x'_1 + 1$, $x_3 = x'_2 + 1$, …, $x_{r+1} = x'_r + 1$. The reversed lexicographic ordered sequence of signs given by X' is

$$\begin{array}{lllll} \sigma_{\mathrm{rot}}(X'\backslash\{x'_1\}), & \sigma_{\mathrm{rot}}(X'\backslash\{x'_2\}), & \dots, & \sigma_{\mathrm{rot}}(X'\backslash\{x'_r\}), & \sigma_{\mathrm{rot}}(X'\backslash\{x'_{r+1}\}) \\ = \sigma_{\mathrm{rot}}(X'\backslash\{x_2-1\}), & \sigma_{\mathrm{rot}}(X'\backslash\{x_3-1\}), & \dots, & \sigma_{\mathrm{rot}}(X'\backslash\{x_{r+1}-1\}), & \sigma_{\mathrm{rot}}(X'\backslash\{n\}) \\ = \ -\sigma(X\backslash\{x_2\}), & -\sigma(X\backslash\{x_3\}), & \dots, & -\sigma(X\backslash\{x_{r+1}\}), & \sigma(X\backslash\{1\}) \\ = \ -\sigma(X\backslash\{x_2\}), & -\sigma(X\backslash\{x_3\}), & \dots, & -\sigma(X\backslash\{x_{r+1}\}), & \sigma(X\backslash\{x_1\}) \end{array}$$

and has at most one sign change if and only if the sequence

$$\sigma(X\backslash\{x_1\}), \qquad \sigma(X\backslash\{x_2\}), \qquad \dots, \qquad \sigma(X\backslash\{x_r\}) \qquad \sigma(X\backslash\{x_{r+1}\})$$

has at most one sign change. Finally this sequence has at most one sign change because σ is a signotope and this is a sequence on the $(r+1)$-subset X. ◀

▶ **Lemma 3.2.** *Let σ be an r-signotope and let F be a fliple of σ. Then F_{rot} is a fliple in the clockwise rotated signotope σ_{rot}.*

Proof. To prove whether an r-subset F_{rot} is a fliple, we need to check all $(r+1)$-subsets X containing the r elements of F_{rot}. If X does not contain the rotating element 1, the sequence stays the same and the flipable elements in this sequence are also the same. By a *flipable element in a sequence* we denote an r-subset which is a fliple if restricting to the considered $(r+1)$-elements involved in this sequence. To check the $(r+1)$-subsets containing the rotated element, we look at the proof of Lemma 3.1 in more detail. If the sign change of the sequence

$$\sigma(X\backslash\{1\}), \sigma(X\backslash\{x_2\}), \ldots, \sigma(X\backslash\{x_{r+1}\})$$

is between $\sigma(X\backslash\{1\})$ and $\sigma(X\backslash\{x_2\})$, that is, $\sigma(X\backslash\{x_2\}) = \sigma(X\backslash\{x_{r+1}\})$, then it holds $\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_2-1\}) = \sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_{r+1}-1\})$ and hence

$$\begin{aligned}\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{n\}) &= \sigma(X\backslash\{1\}) = -\sigma(X\backslash\{x_2\}) \\ &= -\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_2-1\}) = \ \sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_{r+1}-1\}).\end{aligned}$$

This shows that the sequence

$$\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_2-1\}),\ \sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_3-1\}),\ \ldots,\ \sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_{r+1}-1\}),\ \sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{n\})$$

consist only of the same signs (only $+$ or only $-$). Hence the first $X_{\mathrm{rot}}\backslash\{x_2-1\}$ and the last element $\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{n\})$ are flipable.

In the other case the sign change is between two elements $\sigma(X\backslash\{x_i\})$ and $\sigma(X\backslash\{x_{i+1}\})$ for $i \in \{2,\ldots,r\}$. This just reverse to a sign change between the two elements $\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_i-1\})$ and $\sigma_{\mathrm{rot}}(X_{\mathrm{rot}}\backslash\{x_{i+1}-1\})$. Hence the flipable elements stay the same. ◀

▶ **Lemma 3.3.** *Let σ be an r-signotope with partial order $\prec$ and σ_{rot} the rotated signotope as defined above with corresponding partial order $\prec_{\mathrm{rot}}$. For two $(r-1)$-subsets X, Y with an intersection $|X \cap Y| = r-2$ and $X \prec Y$ it holds*

$$\begin{aligned} &X_{\mathrm{rot}} \succ_{\mathrm{rot}} Y_{\mathrm{rot}} && \text{if } 1 \in X \cap Y \\ &X_{\mathrm{rot}} \prec_{\mathrm{rot}} Y_{\mathrm{rot}} && \text{if } 1 \notin X \cap Y \end{aligned}$$

Proof. If $1 \notin X, Y$ this follows since the sign on those r-subsets not containing 1 do not change.

If $1 \in X$ but $1 \notin Y$ the sign of the set X is after the sign of Y in the reversed lexicographic ordered sequence corresponding to σ of the $(r-1)$-subsets of $X \cup Y$. By assumption $X \prec Y$ thus $\sigma(X \cup Y) = +$. After rotating, the sign of X_{rot} is before the sign of Y_{rot} in the reversed lexicographic ordered sequence corresponding to σ_{rot} since they overlap in $r-2$ elements. Furthermore the sign of the r-subset changes, i.e., $\sigma(X \cup Y) = -\sigma_{\mathrm{rot}}(X_{\mathrm{rot}} \cup Y_{\mathrm{rot}}) = -$. This means the relation stays the same, i.e., $X_{\mathrm{rot}} \prec_{\mathrm{rot}} Y_{\mathrm{rot}}$. The case $1 \in Y$ but $1 \notin X$ works analogously.

If $1 \in X, Y$ the order of the appearance of X and Y in the reversed lexicographic sequences stays the same but the sign of the signotope is reversed, i.e., $\sigma(X \cup Y) = -\sigma_{\mathrm{rot}}(X_{\mathrm{rot}} \cup Y_{\mathrm{rot}})$. Thus the ordering between $_{\mathrm{rot}}$ and $_{\mathrm{rot}}$ is reversed as claimed. ◀

Furthermore for two arbitrary $(r-1)$-subsets it holds

▶ **Proposition 3.4.** *Let σ be an r-signotope on $[n]$. For two $(r-1)$-subsets X, Y with $X \prec Y$ and $1 \notin X \cap Y$, it holds X_{rot} and Y_{rot} are incomparable in $\prec_{\mathrm{rot}}$ or $X_{\mathrm{rot}} \prec_{\mathrm{rot}} Y_{\mathrm{rot}}$.*

For the proof, we introduce the following two partitions. With respect to the first element 1, we partition the $(r-1)$-subsets $\binom{[n]}{r-1}$ into the following three sets:

$$\begin{aligned}
\mathcal{H}_1^\sigma &= \{\ X \subset [n] : |X| = r-1,\ 1 \in X\ \}\\
\mathcal{U}_1^\sigma &= \{\ X \subset [n] : |X| = r-1,\ 1 \notin X,\ \sigma(X \cup \{1\}) = +\ \}\\
\mathcal{D}_1^\sigma &= \{\ X \subset [n] : |X| = r-1,\ 1 \notin X,\ \sigma(X \cup \{1\}) = -\ \}.
\end{aligned}$$

Similarly, with respect to the last element n, we partition $\binom{[n]}{r-1}$ into the following three sets:

$$\begin{aligned}
\mathcal{H}_n^\sigma &= \{\ X \subset [n] : |X| = r-1,\ n \in X\ \}\\
\mathcal{U}_n^\sigma &= \{\ X \subset [n] : |X| = r-1,\ n \notin X,\ \sigma(X \cup \{n\}) = -\ \}\\
\mathcal{D}_n^\sigma &= \{\ X \subset [n] : |X| = r-1,\ n \notin X,\ \sigma(X \cup \{n\}) = +\ \}.
\end{aligned}$$

Note the sign change in the definition, that is, every $X \in \mathcal{U}_1^\sigma$ fulfills $\sigma(X \cup \{1\}) = +$ while every $X \in \mathcal{U}_n^\sigma$ fulfills $\sigma(X \cup \{n\}) = -$.

▶ **Claim 3.5.** *$\mathcal{U}_1^\sigma$ and $\mathcal{U}_n^\sigma$ are up-sets and $\mathcal{D}_1^\sigma$ and $\mathcal{D}_n^\sigma$ are down-sets of the partial order $\prec$ corresponding to the r-signotope σ.*

Proof of Claim 3.5. In the following we show that $\mathcal{U}_1^\sigma$ is an up-set. Analogous arguments show that $\mathcal{U}_n^\sigma$ is an up-set and that $\mathcal{D}_1^\sigma$ and $\mathcal{D}_n^\sigma$ are down-sets. Let X be an element of $\mathcal{U}_1^\sigma$ and let Y be an $(r-1)$-subset with $Y \succ X$. By definition, we have $\sigma(X \cup \{1\}) = +$.

If the intersection $X \cap Y$ contains $r-2$ elements, we cannot have $1 \in Y$, as otherwise Y was lexicographic smaller than X and thus $- = \sigma(X \cup Y) = \sigma(X \cup \{1\}) = +$, a contradiction. Therefore, $1 \notin Y$ and we have $(r+1)$ elements in $X \cup Y \cup \{1\}$. If X is lexicographic smaller than Y, we have the lexicographical order $X \cup \{1\} \prec_{\text{lex}} Y \cup \{1\} \prec_{\text{lex}} X \cup Y$. Since we have $\sigma(X \cup \{1\}) = +$ and $\sigma(X \cup Y) = +$, it follows $\sigma(Y \cup \{1\}) = +$ and hence $Y \in \mathcal{U}_1^\sigma$. In the other case, if Y is lexicographical smaller than X, we have the lexicographical order $Y \cup \{1\} \prec_{\text{lex}} X \cup \{1\} \prec_{\text{lex}} X \cup Y$. Since we have $\sigma(X \cup \{1\}) = +$ and $\sigma(X \cup Y) = -$, it follows $\sigma(Y \cup \{1\}) = +$ and hence again $Y \in \mathcal{U}_1^\sigma$.

If the intersection $X \cap Y$ contains less than $r-2$ elements, we proceed by induction. There is a chain $X = Z_1 \prec Z_2 \prec \dots \prec Z_k = Y$ such that any two consecutive Z_i have an intersection of $r-2$ elements. For $i = 2, \dots, k$, since $Z_{i-1} \in \mathcal{U}_1^\sigma$, we conclude that $Z_i \in \mathcal{U}_1^\sigma$, and in particular, $Y \in \mathcal{U}_1^\sigma$. This completes the proof that $\mathcal{U}_1^\sigma$ is an upset. △

We store the following observation derived in the proof of Claim 3.5 for later usage.

▶ **Claim 3.6.** *Let $Z_1 \prec \ldots \prec Z_k$ be a chain of the partial order $\prec$ corresponding to σ. If $Z_1 \in \mathcal{U}_1^\sigma$, then $Z_i \in \mathcal{U}_1^\sigma$ for all i. If $Z_k \in \mathcal{D}_1^\sigma$, then $Z_i \in \mathcal{D}_1^\sigma$ for all i.*

We now study the effect of a clockwise rotation to the partial order. In the partial order $\prec_{\text{rot}}$ corresponding to the rotated signotope σ_{rot}, the sets $(\mathcal{U}_1^\sigma)_{\text{rot}}$ and $(\mathcal{D}_1^\sigma)_{\text{rot}}$ remain up-set and down-set, respectively. Here $\mathcal{X}_{\text{rot}} = \{X_{\text{rot}} \colon X \in \mathcal{X}\}$ denotes the clockwise rotated sets of a set-system $\mathcal{X}$.

▶ **Claim 3.7.** *It holds $(\mathcal{H}_1^\sigma)_{\text{rot}} = \mathcal{H}_n^{\sigma_{\text{rot}}}$, $(\mathcal{U}_1^\sigma)_{\text{rot}} = \mathcal{U}_n^{\sigma_{\text{rot}}}$, and $(\mathcal{D}_1^\sigma)_{\text{rot}} = \mathcal{D}_n^{\sigma_{\text{rot}}}$.*

Proof of Claim 3.7. An $(r-1)$-subset X contains the first element 1 if and only if its clockwise rotation X_{rot} contains the last element n. Therefore, we have $(\mathcal{H}_1^\sigma)_{\text{rot}} = \mathcal{H}_n^{\sigma_{\text{rot}}}$ and $(\mathcal{U}_1^\sigma \cup \mathcal{D}_1^\sigma)_{\text{rot}} = \mathcal{U}_n^{\sigma_{\text{rot}}} \cup \mathcal{D}_n^{\sigma_{\text{rot}}}$. To show $(\mathcal{U}_1^\sigma)_{\text{rot}} = \mathcal{U}_n^{\sigma_{\text{rot}}}$ and $(\mathcal{D}_1^\sigma)_{\text{rot}} = \mathcal{D}_n^{\sigma_{\text{rot}}}$, it suffices to prove $(\mathcal{U}_1^\sigma)_{\text{rot}} \subseteq \mathcal{U}_n^{\sigma_{\text{rot}}}$ and $(\mathcal{D}_1^\sigma)_{\text{rot}} \subseteq \mathcal{D}_n^{\sigma_{\text{rot}}}$.

To show $(\mathcal{U}_1^\sigma)_{\mathrm{rot}} \subseteq \mathcal{U}_n^{\sigma_{\mathrm{rot}}}$, let $X \in \mathcal{U}_1^\sigma$, i.e., $\sigma(X \cup \{1\}) = +$. After rotating the element 1, we obtain

$$\sigma_{\mathrm{rot}}((X \cup \{1\})_{\mathrm{rot}}) = -\sigma(X \cup \{1\}) = -.$$

Since $(X \cup \{1\})_{\mathrm{rot}} = X_{\mathrm{rot}} \cup \{n\}$, we have $\sigma_{\mathrm{rot}}(X_{\mathrm{rot}} \cup \{n\}) = -$ and thus $X_{\mathrm{rot}} \in \mathcal{U}_n^{\sigma_{\mathrm{rot}}}$. An analogous argument shows $(\mathcal{D}_1^\sigma)_{\mathrm{rot}} \subseteq \mathcal{D}_n^{\sigma_{\mathrm{rot}}}$. This completes the proof of Claim 3.7. △

Proof of Proposition 3.4. Let X, Y be two $(r-1)$-subsets with $X \prec Y$ and $X_{\mathrm{rot}} \succ_{\mathrm{rot}} Y_{\mathrm{rot}}$.

If $X \in \mathcal{U}_1^\sigma$, then by Claim 3.6, $Y \in \mathcal{U}_1^\sigma$. If $X \in \mathcal{D}_1^\sigma$, then by Claim 3.7, $X_{\mathrm{rot}} \in \mathcal{D}_n^{\sigma_{\mathrm{rot}}}$. By Claim 3.6, $Y_{\mathrm{rot}} \in \mathcal{D}_n^{\sigma_{\mathrm{rot}}}$ and again, by Claim 3.7, $Y \in \mathcal{D}_1^\sigma$. Analogous arguments show that, if $Y \in \mathcal{D}_1^\sigma$ (resp. $Y \in \mathcal{U}_1^\sigma$), then $X \in \mathcal{D}_1^\sigma$ (resp. $X \in \mathcal{U}_1^\sigma$).

If X and Y both lie in $\mathcal{D}_1^\sigma$ (resp. $\mathcal{U}_1^\sigma$), then, by Claim 3.6, there is a chain $X = Z_1 \prec \ldots \prec Z_k = Y$ with $Z_1, \ldots, Z_k \in \mathcal{D}_1^\sigma$ (resp. $\mathcal{U}_1^\sigma$). After a clockwise rotation, we have $X_{\mathrm{rot}} = (Z_1)_{\mathrm{rot}} \prec_{\mathrm{rot}} \ldots \prec_{\mathrm{rot}} (Z_k)_{\mathrm{rot}} = Y_{\mathrm{rot}}$, which is a contradiction to $X_{\mathrm{rot}} \succ_{\mathrm{rot}} Y_{\mathrm{rot}}$. Therefore, X and Y both have to lie in $\mathcal{H}_1^\sigma$, i.e., $1 \in X \cap Y$. ◀

It is worth noting that, for r-subsets X, Y with $1 \in X \cap Y$, we have $X \prec Y$ if and only if $X_{\mathrm{rot}} \succ_{\mathrm{rot}} Y_{\mathrm{rot}}$. For $X, Y \in \mathcal{H}_1^\sigma$ (i.e., $1 \in X \cap Y$) with $X \prec Y$ Claim 3.6 implies that any chain $X = Z_1 \prec \ldots \prec Z_k = Y$ lies entirely in $\mathcal{H}_1^\sigma$ (i.e., $Z_1, \ldots, Z_k \in \mathcal{H}_1^\sigma$). Since a clockwise rotation converts comparability of elements containing the element 1, we have $X_{\mathrm{rot}} = (Z_1)_{\mathrm{rot}} \succ_{\mathrm{rot}} \ldots \succ_{\mathrm{rot}} (Z_k)_{\mathrm{rot}} = Y_{\mathrm{rot}}$,

▶ **Corollary 3.8.** *One clockwise rotation does not reverse the ordering between two disjoint $(r-1)$-subsets.*

D Theorem 1.2 implies Levi's extension lemma (Theorem 1.1)

It is sufficient to prove Levi's extension lemma for simple arrangements of pseudolines and for crossing points as prescribed points. Given a non-simple arrangement, we can slightly perturb the multiple crossing points (as depicted in Figure 2) to obtain a simple arrangement. This simple arrangement can then be extended, and each of the multiple crossing points of the original arrangement can again be obtained by contraction. Also, whenever a prescribed point lies on a pseudosegment or inside a cell, we can extend the arrangement through an adjacent crossing. By perturbing the extending pseudoline, we can ensure that the pseudoline passes through the originally prescribed point.

E The extended signotope contains the original signotope

Although the following lemma is trivial in the geometrical setting of pseudoline arrangements, we need to prove it in the context of general r-signotopes.

▶ **Lemma 5.1.** *Let σ be an r signotope on $[n]$ and $x \in [n]$. Then it is $\sigma_{\mathrm{rot}} \downarrow_{x_{\mathrm{rot}}} = (\sigma \downarrow_x)_{\mathrm{rot}}$.*

Proof. Both mappings are r-signotopes on $[n]$. We need to check whether they map to the same signs. Let X be an r-subset of $[n]$ and let X^* be an r-subset of $[n+1]$ with $x_{\mathrm{rot}} \notin X^*$ and $X^* \downarrow_{x_{\mathrm{rot}}} = X$. We obtain

$$\sigma_{\mathrm{rot}} \downarrow_{x_{\mathrm{rot}}} (X) = \sigma_{\mathrm{rot}}(X^*).$$

We will now rewrite the term to get the statement. Recall that rotating an r-signotope on n elements exactly $2n$ times results in the original signotope. Hence rotating $2n-1$ times corresponds to a backwards rotation, i.e., the inverse operation of a clockwise rotation. We denote this backwards rotation by rot^{-1}. Since $x_{\mathrm{rot}} \notin X^*$, we have $x \notin (X^*)_{\mathrm{rot}^{-1}}$. By definition it is

$$\sigma_{\mathrm{rot}}(X^*) = \varepsilon \cdot \sigma((X^*)_{\mathrm{rot}^{-1}}) = \varepsilon \cdot \sigma \downarrow_x (((X^*)_{\mathrm{rot}^{-1}}) \downarrow_x) = \varepsilon \cdot \sigma \downarrow_x (X_{\mathrm{rot}^{-1}}) = (\sigma \downarrow_x)_{\mathrm{rot}}(X),$$

where the sign $\varepsilon = +$ (resp. $\varepsilon = -$) if $n \in X^*$ (resp. $n \notin X^*$). Note that $n \in X^*$ is equivalent to $1 \in X_{\mathrm{rot}^{-1}}$. This completes the proof of the lemma. ◀

This lemma implies that after extending the signotope and rotating back, the signotope on $[n+1]$ elements contains the original, i.e., deleting the extending element, which is $n+1-k$ if we rotate k times backwards, results in the original signotope.