

Unique Functionals and Representations of Hecke Algebras

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To the memory of Jonathan David Rogawski

Abstract

In [24], Rogawski used the affine Hecke algebra to model the intertwining operators of unramified principal series representations of p -adic groups. On the other hand, a representation of this Hecke algebra in which the standard generators act by Demazure-Lusztig operators was introduced by Lusztig [18] and applied by Kazhdan and Lusztig [17] to prove the Deligne-Langlands conjecture. These operators appear in various other contexts. Ion [11] used them to express matrix coefficients of principal series representations in terms of non-symmetric Macdonald polynomials, while Brubaker, Bump and Licata [3] found essentially the same operators underlying recursive relationships for Whittaker functions. In this article, we will explain the role of unique functionals and Hecke algebras in these contexts and revisit the results of Ion [11] from the point of view of [3].

Keywords: Hecke algebra, unramified principal series, Demazure-Lusztig operator, unique functional.

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1 Introduction

One of the innovations in Rogawski [24] was the use of the Hecke algebra to model the intertwining operators of unramified principal series representations of p -adic groups. His goal was the classification of the irreducible

representations of the Hecke algebra, or equivalently, the irreducible representations of a p -adic group having an Iwahori fixed vector. These had already been classified in the case of GL_n by Zelevinsky [27]. It was known (Zelevinsky [28]) that there were analogies between this problem and the decomposition of Verma modules of a semisimple Lie algebra into irreducibles, where deep connections with the theory of Hecke algebras had been found by Kazhdan and Lusztig [15]. Rogawski sought to clarify the relationship between p -adic representation theory and Kazhdan-Lusztig theory. As part of this effort, he expressed intertwining integrals between principal series representations in terms of Kazhdan-Lusztig elements of the Hecke algebra.

Among the tools that had been brought to bear on the study of Verma modules, Jantzen [14] had introduced a filtration of the Verma module based on the Shapovalov bilinear form. With the analogy between p -adic groups and Verma modules in mind, Rogawski gave an analog of the Jantzen filtration, and also reproved the results of Zelevinsky [27] for GL_n using Hecke algebra methods.

After Rogawski's paper was written, Zelevinsky's results were generalized to arbitrary p -adic groups by Kazhdan and Lusztig [16, 17], who proved the Langlands-Deligne conjecture classifying the irreducible representations of a p -adic group G that have Iwahori fixed vectors. They made use of a representation of the affine Hecke algebra \mathcal{H}_J on the ring of rational functions on the maximal torus \hat{T} of the L-group \hat{G} . In this action, the generators of the Hecke algebra act by certain operators known as *Demazure-Lusztig* operators, which resemble the well-known Demazure operators that occur in the cohomology of line bundles over Schubert varieties. Given data consisting of a pair (s, u) of elements of \hat{G} such that s is semisimple, u unipotent, and $sus^{-1} = u^q$, where q is the cardinality of the residue field, a subquotient of this representation can be found that gives an irreducible \mathcal{H}_J -module. This module may be identified with the space of Iwahori fixed vectors in an irreducible representation of $G(F)$. To prove that this gives every irreducible representation of $G(F)$ uniquely, thus proving the Deligne-Lusztig conjecture, Kazhdan and Lusztig made use of the "coincidence" that the same representation of the Hecke algebra by Demazure-Lusztig operators also occurs in the K-theory of flag varieties, allowing statements about representations to be translated into algebraic geometry, where suitable methods are available.

The representation of the Hecke algebra by Demazure-Lusztig operators comes up in yet another context, namely the study of special functions on a p -adic group realized as matrix coefficients involving Iwahori fixed vectors.

Our first task, after some preliminaries, will be to briefly retrace Rogawski's steps and to discuss the relationship between the Hecke algebra and the intertwining operators. The intertwining operators involve different principal series representations, which must be taken together to obtain a representation of the Hecke algebra. We express this by saying that the principal series representation is *variable* in this representation of the Hecke algebra.

As we will explain in Section 3, this representation may be converted into something concrete by introducing a family of functionals on the principal series representations. We will consider two particular such families: the Whittaker functionals, and the spherical functionals. The role of the unique functional is that it converts the Iwahori fixed vectors in a variable principal series representation into a family of regular functions on \hat{T} . The problem of variability of the principal series representation disappears, and the action of the Hecke algebra generators is by some variant (depending on the functional) of the Demazure-Lusztig operator. The regularity of the functions obtained this way is related to Bernstein's method of showing that a unique functional defined (typically by a suitable integral) on an open subspace extends to a meromorphic function for all Langlands parameters. (See, for example, Banks [1] or Sakellaridis [25], Sect. 7 for Bernstein's method.)

Both the spherical and the Whittaker functionals are defined and non-zero on a Zariski dense set of \hat{T} . By contrast, one may consider functionals defined and non-zero on only a subset of \hat{T} that is not Zariski dense. An example is the Shalika functional on representations of GL_n , proved unique by Jacquet and Rallis [13], with a Casselman-Shalika type formula established by Sakellaridis [25]. The Shalika functional, like the Bessel models for classical groups, exhibits characteristics of both the spherical and Whittaker functionals and can be studied using our methods. We hope to return to this in a later paper.

Shortly before Rogawski's paper [24], intertwining operators had been used for another purpose, namely the computation of the spherical vector in each of our two archetypal models. For the spherical model, Casselman [4] used intertwining operators to give a new proof of the Macdonald formula which expresses the values of the spherical function as specializations of Macdonald symmetric functions. This specialization is the Hall-Littlewood polynomial in the case of GL_n . Later nonsymmetric Macdonald polynomials were defined in Macdonald [20], Opdam [21] and Cherednik [6]. Generalizing the Macdonald formula, Ion [11] showed that the Iwahori-fixed vectors in the model are expressible in terms of nonsymmetric Macdonald polynomials,

making use of recursions that they satisfy involving Demazure-Lusztig operators. See also Cherednik and Ostrik [7] for earlier hints of this connection.

For the Whittaker functional, Casselman and Shalika [5] used the intertwining operators to show that the values of the spherical Whittaker function are the irreducible characters of \hat{G} , multiplied by a factor which is a deformation of the denominator in the Weyl character formula. Regarding the more general Iwahori-fixed vectors in the model, Reeder [22, 23] used the Hecke algebra action to give recursions between these, usable for explicit computation. These relations can be understood in terms of Demazure-Lusztig operators as proved by Brubaker, Bump and Licata [3].

In Section 2 we review the relation between the Iwahori Hecke algebra and the intertwining operators for principal series. Then in Section 3 we show that, given a unique functional on a Zariski dense subset of \hat{T} , to each generator T_i of the Iwahori Hecke algebra we may attach a difference operator on a suitable ring of regular functions that is similar to a Demazure operator. This extends to an action of the Hecke algebra. In the case of the spherical functional, this gives a new perspective on the work of Ion [11].

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2 Hecke algebras and intertwiners of principal series

Let G be a reductive algebraic group defined and split over a nonarchimedean local field F with ring \mathfrak{o} of integers and prime \mathfrak{p} . We may regard G as defined over \mathfrak{o} in such a way that $K = G(\mathfrak{o})$ is a special maximal compact subgroup. Let $q = |\mathfrak{o}/\mathfrak{p}|$. Let T be a split maximal torus contained in Borel subgroup B , and let $W_0 = N(T)/T$ be the Weyl group. We will always choose representatives for W_0 from $N(T) \cap K$.

The connected L-group \hat{G} is an algebraic group defined over \mathbb{C} with a maximal torus \hat{T} that is in duality with T in the sense that elements of $\hat{T}(\mathbb{C})$ are in bijection with the unramified characters of $T(F)$, that is, those that are trivial on $T(F) \cap K = T(\mathfrak{o})$. Let J be the Iwahori subgroup, which is the

preimage in K of $B(\mathbb{F}_q)$ under the canonical homomorphism $G(\mathfrak{o}) \rightarrow G(\mathbb{F}_q)$.

Let s_1, \dots, s_r be the simple reflections in the Weyl group W_0 . The affine Weyl group W_{aff} is obtained by adjoining one more ‘‘affine’’ simple reflection s_0 (Bourbaki [2], Section VI.2). It is the semidirect product of W_0 by the root lattice Q^\vee of \hat{T} , which is the coroot lattice of T . The groups W_0 and W_{aff} are Coxeter groups. The extended affine Weyl group W_{ext} is a slightly larger group that is the semidirect product of W_0 by the weight lattice P^\vee of \hat{T} , which is the coweight lattice of T , isomorphic to $T(F)/T(\mathfrak{o})$. In this presentation, P^\vee is a normal subgroup. We will denote by $\lambda \mapsto a_\lambda$ a map sending $\lambda \in P^\vee$ to a representative of the corresponding coset in $T(F)/T(\mathfrak{o})$.

The group W_{ext} is not a Coxeter group, but like Coxeter groups it has a length function. The finite subgroup Ω of elements of length 0 is isomorphic to P^\vee/Q^\vee . (For example, if G is semisimple of adjoint type, then Ω is isomorphic to the fundamental group of G .) The group W_{ext} is the semidirect product of W_{aff} by Ω , with W_{aff} being a normal subgroup. Conjugation by an element of Ω permutes the s_i .

The (affine) Iwahori Hecke algebra \mathcal{H}_J is the convolution algebra of compactly supported J -biinvariant functions on $G(F)$. Let \mathcal{H}_0 be the subring of function with support in K . Then \mathcal{H}_0 and \mathcal{H}_J have the following explicit description due to Iwahori and Matsumoto [12]. If r is the rank of G , the ring \mathcal{H}_0 is generated by T_1, \dots, T_r where each T_i is the characteristic function of Js_iJ . These T_i then satisfy quadratic relations

$$T_i^2 = (q - 1)T_i + q \tag{1}$$

and the braid relations

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

where the number of terms on each side is the order of $s_i s_j$. The affine Hecke algebra \mathcal{H}_{aff} is obtained by adjoining an element T_0 satisfying the same quadratic and braid relations but allowing $i = 0$ corresponding to the affine simple reflection s_0 . The algebra \mathcal{H}_J is slightly larger than \mathcal{H}_{aff} , and is isomorphic as a vector space to $\mathcal{H}_{\text{aff}} \otimes \mathbb{C}[\Omega]$. The elements of Ω act on \mathcal{H}_{aff} by conjugation, and this action corresponds to permuting the T_i , just as in W_{ext} conjugation by elements of Ω permutes the s_i .

The algebra \mathcal{H}_J also has a presentation analogous to the presentation of W_{ext} as the semidirect product of P^\vee by W_0 . This presentation, sometimes known as the *Bernstein presentation* was developed but not published by

Bernstein and Zelevinsky. Possibly its first use in a published paper was in Rogawski [24], and a full treatment was given by Lusztig [18]. In the Bernstein presentation \mathcal{H}_0 is supplemented by a ring homomorphism $\theta : \mathbb{C}[P^\vee] \rightarrow \mathcal{H}_J$. As a vector space $\mathcal{H}_J = \mathcal{H}_0 \otimes \mathbb{C}[P^\vee]$. To describe the ring structure, it is sufficient to give one relation. Let $1 \leq i \leq r$ and let $\lambda \in P^\vee$. Then

$$\theta(\lambda)T_i - T_i\theta(s_i\lambda) = (q - 1) \frac{\theta(\lambda) - \theta(s_i\lambda)}{1 - \theta(-\alpha_i^\vee)},$$

where α_i^\vee is the coroot corresponding to i .

Let $\mathbf{z} \in \hat{T}(\mathbb{C})$, and let $\tau = \tau_{\mathbf{z}} : T(F) \rightarrow \mathbb{C}^\times$ be the corresponding unramified character. We extend it to $B(F)$ by letting the unipotent radical $N(F)$ be in the kernel. The principal series representation $M(\tau)$ consists of all locally constant maps $f : G(F) \rightarrow \mathbb{C}$ such that $f(bg) = (\delta^{1/2}\tau)(b) f(g)$ for $b \in B(F)$, $g \in G(F)$. The action of $G(F)$ is by right translation. The module $M(\tau)$ is irreducible if τ is in general position.

If (π, V) is an irreducible representation having a J -fixed vector, then V^J is a finite-dimensional irreducible \mathcal{H}_J module and its isomorphism class determines π . Any such (π, V) with a J -fixed vector is a subquotient of $M(\tau)$ for some τ , and the category of smooth representations of finite length all of whose composition factors have J -fixed vectors is equivalent to the category of finite-dimensional \mathcal{H}_J -modules.

The Weyl group W_0 acts on T and hence on unramified characters. We will make this a right action, so $\tau w(a) = \tau(waw^{-1})$ for $a \in T(F)$. If $w \in W_0$, the modules $M(\tau)$ and $M(\tau w)$ are isomorphic if irreducible, and in any case have isomorphic semisimplifications. In order to see this, one may construct homomorphisms $A_w : M(\tau) \rightarrow M(\tau w)$ by means of *intertwining integrals*. By definition

$$A_w f(g) = \int_{N \cap w^{-1}N_-w} f(wng) dn \quad (2)$$

where N_- is the unipotent radical of the negative Borel. The integral is convergent if $|\tau(a_\lambda)| < 1$ for dominant weights λ . By the *singular set* we mean the union of hyperplanes in $\hat{T}(\mathbb{C})$ that are the kernels of the coroots α^\vee (regarded as characters of \hat{T}). For arbitrary τ , the intertwining operator may be defined by analytic continuation, except that the A_w can have poles in the singular set.

Now we return to the point in question: why may Hecke algebras be used to model intertwining operators? The basic insight is that, to any ring R

regarded as a left R -module, a left R -module homomorphism $\lambda : R \rightarrow R$ is necessarily of the form $\lambda(x) = x \cdot a$ for some $a \in R$. This is trivial to prove with $a = \lambda(1)$.

The first way of applying this is to note that the space $M(\tau)^J$ of J -invariants is $|W_0|$ -dimensional. It has several natural bases indexed by Weyl group elements. A particular one is the basis $\Phi_w^\tau = \Phi_w$ ($w \in W_0$) defined by

$$\Phi_w^\tau(bk) = \begin{cases} \delta^{1/2}\tau(b) & \text{if } k \in BwJ, \\ 0 & \text{otherwise} \end{cases}$$

for $b \in B(F)$ and $k \in K$.

We see that \mathcal{H}_0 and $M(\tau)^J$ are both $|W_0|$ -dimensional \mathcal{H}_0 -modules, and in fact they are isomorphic as left \mathcal{H}_0 -modules. A particular isomorphism $\varrho_\tau : M(\tau)^J \rightarrow \mathcal{H}_0$ is given by $\varrho_\tau(f) = F$ where

$$F(g) = \begin{cases} f(g^{-1}) & \text{if } g \in K, \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to check that this is an isomorphism of left \mathcal{H}_0 -modules.

Composing with this isomorphism, the intertwining integral A_w thus gives rise to a homomorphism $\mathcal{H}_0 \rightarrow \mathcal{H}_0$. This can have poles (in the singular set) or zeros (if $\tau(a_{\alpha^\vee}) = q^{\pm 1}$ for some coroot α^\vee) but if τ is in general position it is an isomorphism and so it agrees with right multiplication by a particular element F_w of \mathcal{H}_0 , which was identified by Rogawski [24]. It is sufficient to describe it when $w = s_i$ is a simple reflection, and in this case

$$F_{s_i} = \frac{1}{q}(T_i + 1) - C_{\alpha_i}(\tau)$$

where $C_{\alpha_i}(\tau)$ with $\tau = \tau_{\mathbf{z}}$ is the ubiquitous rational function

$$C_{\alpha_i}(\tau) = \frac{1 - q^{-1}\mathbf{z}^{\alpha_i^\vee}}{1 - \mathbf{z}^{\alpha_i^\vee}}. \quad (3)$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} M(\tau)^J & \xrightarrow{\varrho_\tau} & \mathcal{H}_0 \\ \downarrow A_{s_i} & & \downarrow \cdot F_{s_i} \\ M(\tau s_i)^J & \xrightarrow{\varrho_{\tau s_i}} & \mathcal{H}_0 \end{array} \quad (4)$$

See Brubaker, Bump and Licata [3], Lemma 24 for a proof.

Instead of \mathcal{H}_0 one may also use \mathcal{H}_J to model the intertwining integrals. The factor $C_{\alpha_i}(\tau)$, which depends on the spectral parameter, may then be replaced by an element of $\theta(P^\vee)$. This point of view is taken in Haines, Kottwitz and Prasad [9]. In order to see how this should work, consider that since the intertwining operator permutes the spaces $M(\tau)^J$ one might consider each A_w to be an endomorphism of

$$\bigoplus_{w \in W_0} M(\tau w)^J. \quad (5)$$

The characters τ of $B(F)$ that are induced all have a common trivial restriction to the subgroup $B_0 = T(\mathfrak{o})N(F)$, so by Frobenius reciprocity the module $M = \text{ind}_{B_0}^{G(F)}(1)$ is the direct integral of the spaces (5), with τ running over \hat{T} modulo the action of W_0 . This is the *universal principal series*. It is almost true that the intertwining integrals are endomorphisms of $\text{ind}_{B_0}^{G(F)}(1)$; the difficulty is that the operators are polar in the singular set, so one must restrict to the orthogonal complement in M of the spaces (5) with singular τ . Alternatively, one may consider the compact induction $M_c = \text{c-ind}_{B_0}^{G(F)}(1)$. Although this is no longer closed under the intertwining operators, at least for f in M_c the intertwining integral (2) is always convergent, and A_w is realized as a map $M_c \rightarrow M$.

We now come to the point, which is that as an \mathcal{H}_J -module M_c is a free module of rank one. This is to be expected from the Bruhat decomposition because every coset in $B_0 \backslash G(F)/J$ has a unique representative in the extended affine Weyl group, and this is also in bijection with $J \backslash G(F)/J$. Thus the extended affine Weyl group parametrizes both a basis of M_c and a basis of \mathcal{H}_J . For a proof that the module M_c is one dimensional, see Chriss and Khuri-Makdisi [8] or Haines, Kottwitz and Prasad [9], Lemma 1.6.1.

If A_w were a map $M_c \rightarrow M_c$ we could then transfer A_w to a map $\mathcal{H}_J \rightarrow \mathcal{H}_J$ and conclude that it agreed with right multiplication by some element. Due to the poles of the intertwining operators, this does not quite work. What may be done is to consider a somewhat larger Hecke algebra. In the Bernstein presentation $\mathcal{H}_J \cong \mathcal{H}_0 \otimes \mathbb{C}[P^\vee]$, we can enlarge $\mathbb{C}[P^\vee]$ to any ring \mathfrak{R} such that $\mathbb{C}[P^\vee] \subseteq \mathfrak{R} \subseteq \mathfrak{M}$, where \mathfrak{M} is the field of fractions of $\mathbb{C}[P^\vee]$. We take \mathfrak{R} to be the localization at the set of singular hyperplanes, that is, the ring obtained by adjoining $1 - \alpha^\vee$ for all coroots α^\vee . Let $\mathcal{H}'_J = \mathcal{H}_0 \otimes \mathfrak{R}$. Since the poles of the intertwining operators are contained in the singular

locus, if M' denotes the submodule of M generated by the image of the A_w , then we have a commutative diagram

$$\begin{array}{ccc}
M_c \cong \mathcal{H}_J & \longrightarrow & \mathcal{H}'_J \\
A_w \downarrow & & \downarrow \\
M' & \longrightarrow & \mathcal{H}'_J \quad .
\end{array} \tag{6}$$

The top arrow is the natural inclusion and the bottom arrow extends the injection of M_c into \mathcal{H}_J . The vertical arrows are \mathcal{H}'_J -module homomorphisms. As before, if R is a ring, then a left R -module homomorphism $R \rightarrow R$ is multiplication by some element, and therefore A_w may be realized as multiplication by some element of \mathcal{H}'_J .

3 Hecke algebra modules from unique functionals

There are two different kinds of actions of the affine Hecke algebra that we need to consider. First, in any smooth representation, the Hecke algebra acts by convolution. The Hecke algebra action on the Iwahori fixed vectors in an induced representation is an action of this type.

Second, with the notation as in the prior section, the affine Hecke algebra acts on the ring $\mathcal{O}(\hat{T})$ of rational functions on the maximal torus \hat{T} in the L-group. This ring is isomorphic to the group algebra $\mathbb{C}[P^\vee]$ of the coweight lattice P^\vee of T , which may be identified with the weight lattice of \hat{T} . The generators T_i of the finite Hecke algebra \mathcal{H}_0 will act by so-called *Demazure-Lusztig operators*. This action was introduced by Lusztig [19] and has far-reaching consequences.

We have seen in the previous section how the Hecke algebra can be used to model intertwining integrals. In this section we will show how we may translate this action of the Hecke algebra into an action on rational functions. We described two interpretations of intertwining operators via Hecke algebras in the diagrams (4) and (6); the simpler point of view in (4) will be sufficient for our purposes, as it was for Rogawski.

Let us consider, for every \mathbf{z} in some Zariski-dense subset of \hat{T} , a linear functional $L_{\mathbf{z}}$ on $M(\tau)$. We shall suppose that $L_{\mathbf{z}}$ arises from a multiplicity-free representation in the following way. Let H be a subgroup of $G(F)$

and η a character of H such that the induced representation $\text{ind}_H^{G(F)}(\eta)$ is multiplicity-free. Then by Frobenius reciprocity, a functional $L_{\mathbf{z}}$ (if it exists) is characterized up to scalar multiple by the property that $L_{\mathbf{z}}(\pi(g)\phi) = \eta(g)L_{\mathbf{z}}(\phi)$ for $g \in H$. As noted above, using the uniqueness of the functional it is often possible to show by a method of Bernstein that $L_{\mathbf{z}}(\phi)$ is a rational function of \mathbf{z} on this set. Alternatively, this rationality may be proved for one specific vector, together with recursions that imply the rationality for all ϕ .

As particular examples, $L_{\mathbf{z}}$ could be the spherical functional

$$\mathcal{S}_{\mathbf{z}}(f) = \int_K f(k) dk$$

or the Whittaker functional

$$\mathcal{W}_{\mathbf{z}}(f) = \int_{N(F)} f(w_0ng) \psi(n) dn$$

where ψ is a nondegenerate character of $N(F)$, the unipotent radical of the Borel, and w_0 is the long element of the Weyl group W . The spherical and Whittaker functionals are both characterized up to scalar multiple by the uniqueness property described above. Indeed, since the spherical Hecke algebra of compactly supported K -binvariant functions is commutative, K is a Gelfand subgroup of $G(F)$ so the functional $L_{\mathbf{z}}$ is determined (up to scalar) by the fact that $\mathcal{S}_{\mathbf{z}}(\pi(k)\phi) = \mathcal{S}_{\mathbf{z}}(\phi)$ for all $k \in K$. For the Whittaker functional, the corresponding uniqueness result was obtained by Gelfand and Graev, Piatetski-Shapiro and Shalika [26]. The Whittaker integral, like the intertwining integral, is convergent for \mathbf{z} in an open set, and has analytic continuation to all \mathbf{z} .

It follows from these uniqueness results that for every pair \mathbf{z} and w the functional $L_{\mathbf{z}w} \circ A_w$ is a constant multiple of $L_{\mathbf{z}}$. For the two examples above, these constants were computed by Casselman [4] and Casselman and Shalika [5], who found that for the spherical functional

$$\mathcal{S}_{\mathbf{z}w} \circ A_w = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} \left(\frac{1 - q^{-1}\mathbf{z}^{\alpha^\vee}}{1 - \mathbf{z}^{\alpha^\vee}} \right) \mathcal{S}_{\mathbf{z}} = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} C_{\alpha}(\tau) \mathcal{S}_{\mathbf{z}} \quad (7)$$

with $C_\alpha(\tau)$ as in (3) and for the Whittaker functional

$$\mathcal{W}_{\mathbf{z}w} \circ A_w = \prod_{\substack{\alpha \in \Delta^+ \\ w^{-1}\alpha \in \Delta^-}} \left(\frac{1 - q^{-1}\mathbf{z}^{-\alpha^\vee}}{1 - \mathbf{z}^{\alpha^\vee}} \right) \mathcal{W}_{\mathbf{z}}. \quad (8)$$

Here Δ^+ and Δ^- are the positive and negative roots, and if α is root then α^\vee is the associated coroot.

Our goal is to describe a Hecke algebra action on $M(\tau)^J$ arising from each of these functionals and explain how this action gives a recursion for $L_{\mathbf{z}}(\pi(g)\Phi_w)$ for any standard basis element $\Phi_w \in M(\tau)^J$. To describe the function $L_{\mathbf{z}}(\pi(g)\Phi_w)$, it suffices to choose g from a set of representatives for $H \backslash G(F) / J$, where $H = K$ when $L = \mathcal{S}$ and $H = N$ when $L = \mathcal{W}$. This means that we may choose $g = a_\lambda$, where $\lambda \in P^\vee$, and in the Whittaker case we may assume λ is dominant, since otherwise $\mathcal{W}(\pi(a_\lambda)\phi) = 0$ for any $\phi \in M(\tau)$ (cf. [3], Lemma 5).

For both the spherical and Whittaker functionals, there is a standard basis vector Φ_w for which $L_{\mathbf{z}}(\pi(a_\lambda)\Phi_w)$ has a particularly simple form. In the Whittaker case, for any dominant weight λ we have

$$\mathcal{W}_{\mathbf{z}}(\pi(a_\lambda)\Phi_{w_0}^\tau) = \delta^{1/2}(a_\lambda)\mathbf{z}^{w_0\lambda}.$$

(See [3] Proposition 6.) In the spherical case, we have the following statement.

Proposition 1. *Let $\lambda \in P^\vee$. Then $\mathcal{S}_{\mathbf{z}}(\pi_{\mathbf{z}}(a_\lambda)\Phi_1^\tau) = c(\lambda)\mathbf{z}^\lambda$ where the constant $c(\lambda)$ is independent of \mathbf{z} .*

Proof. We have

$$\mathcal{S}_{\mathbf{z}}(\pi_{\mathbf{z}}(a_\lambda)\Phi_1^\tau) = \int_K \Phi_1(ka_\lambda) dk = (\delta^{1/2}\tau)(a_\lambda) \int_K \Phi_1(a_\lambda^{-1}ka_\lambda) dk.$$

The support of Φ_1 is $B(F)J$. It is easy to see that if $k \in K$ and $a_\lambda^{-1}ka_\lambda \in B(F)J$ then $a_\lambda^{-1}ka_\lambda \in B_0J$ where B_0 is the kernel of $\tau_{\mathbf{z}} : B(F) \rightarrow \mathbb{C}$. Hence the integral is a constant independent of \mathbf{z} . \square

For either functional, this choice of standard basis vector will be the starting point for our recursion. It remains to define the Hecke action. From Rogawski's perspective, $M(\tau)^J$ is understood abstractly as a Hecke algebra module via the regular representation, which is determined by:

$$T_w \Phi_1 = \Phi_w, \quad \text{and} \quad T_w \Phi_y = T_w T_y \Phi_1 \quad \text{for all } y, w \in W.$$

Remembering the quadratic relation given in (1), we see that for a simple reflection s ,

$$T_s \Phi_w = \begin{cases} \Phi_{sw} & \text{if } sw > w, \\ q\Phi_{sw} + (q-1)\Phi_w & \text{if } sw < w. \end{cases}$$

But (4) gives a relation between the intertwining operator A_s and T_s . So combining these two ingredients and manipulating the terms we obtain the following result.

Proposition 2. *Let $\tau = \tau_{\mathbf{z}}$ and let $w \in W_0$. Let $s = s_i$ be a simple reflection. Then*

$$A_s \Phi_w^{\tau s} + C_\alpha(\tau) \Phi_w^\tau = \begin{cases} \Phi_w^\tau + \Phi_{sw}^\tau & \text{if } sw < w, \\ q^{-1}(\Phi_w^\tau + \Phi_{sw}^\tau) & \text{if } sw > w. \end{cases} \quad (9)$$

Proof. This is an easy consequence of Casselman [4], Theorem 3.4; see Proposition 8 of [3]. \square

Thus it is reasonable to expect that we may obtain an action of the Hecke algebra on the ring $\mathcal{O}(\hat{T})$ of regular functions on \hat{T} via the regular representation by applying the functional to both sides of (9).

The case where $L = \mathcal{W}$ is treated in detail in [3], so we will focus on the case $L = \mathcal{S}$, and return at the end to make some remarks about the difference between the two cases. For $L = \mathcal{S}$, we intend to start our recursion at Φ_1 and move up in the Bruhat order. So let us rewrite (9) in the case $sw > w$:

$$qA_s \Phi_w^{\tau s} + (qC_\alpha(\tau) - 1)\Phi_w^\tau = \Phi_{sw}^\tau. \quad (10)$$

We will find that applying $\mathcal{S}_{\mathbf{z}}$ to both sides gives a recursive identity for the matrix coefficient

$$F_{\lambda, w}(\mathbf{z}) := \mathcal{S}_{\mathbf{z}}(\pi(a_\lambda)\Phi_w^\tau),$$

of the principal series representation. To state this more precisely, let us introduce the *Demazure-Lusztig* operator \mathcal{T}_i defined on an arbitrary function F on the dual torus \hat{T} by

$$\mathcal{T}_i F(\mathbf{z}) := qC_{-\alpha_i}(\tau)F(\mathbf{z}s_i) + (qC_{\alpha_i}(\tau) - 1)F(\mathbf{z}), \quad (11)$$

Here $1 \leq i \leq r$ corresponds to a simple reflection s_i of the finite Weyl group. After some algebra, this is equivalent to

$$\mathcal{T}_i F(\mathbf{z}) = (\mathbf{z}^{\alpha_i} - 1)^{-1}(F(\mathbf{z}) - F(\mathbf{z}s_i) - qF(\mathbf{z}) + q\mathbf{z}^{\alpha_i}F(\mathbf{z}s_i)),$$

which is precisely the operator defined in Lusztig [19], equation (8.1). (This can also be defined if $i = 0$, but for this discussion we are excluding this case.)

The following result is equivalent to a result of Ion [11], Proposition 5.8.

Theorem 1. *The Demazure-Lusztig operators satisfy the quadratic and braid relations, hence generate a ring isomorphic to the Hecke algebra \mathcal{H}_0 . If $w \in W_0$ and $s_i w > w$, then*

$$F_{\lambda, s_i w} = \mathcal{T}_i F_{\lambda, w}. \quad (12)$$

The fact that the Demazure-Lusztig operators satisfy the quadratic and braid relations is due to Lusztig [19], Section 8. However checking the braid relations directly depends on a tedious computation for rank 2 root systems, so it may be of interest that we can avoid such computations using our methods.

Proof. Assume that $sw > w$. Apply $\pi(a_\lambda)$ to both sides of (10) and then apply \mathcal{S}_z . We obtain

$$F_{\lambda, sw}(\mathbf{z}) = q\mathcal{S}_z^\tau A_s(\pi(a_\lambda)\Phi_w^{\tau s}) + (qC_\alpha(\tau) - 1)F_{\lambda, w}(\mathbf{z}). \quad (13)$$

Now we use (7), replacing \mathbf{z} by $\mathbf{z}s$ and remembering that $C_\alpha(\tau_{zs}) = C_{-\alpha}(\tau)$. Comparing with (11), the right-hand side of (13) is just $\mathcal{T}_i F_{\lambda, w}^\tau$ as desired.

We turn to the fact that the \mathcal{T}_i satisfy the generating relations of \mathcal{H}_0 . Let m be the order of $s_i s_j$ with $1 \leq i, j \leq r$, $i \neq j$. To show that

$$\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \cdots = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \cdots \quad (m \text{ factors on both sides})$$

it is sufficient to show that they have the same effect on \mathbf{z}^λ , where $\lambda \in P^\vee$. By Proposition 1, it is thus sufficient to show that

$$\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \cdots F_{a_\lambda, 1}(\mathbf{z}) = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \cdots F_{a_\lambda, 1}(\mathbf{z}).$$

But applying (12), both sides equal $F_{a_\lambda, w}(\mathbf{z})$ where $w = s_i s_j s_i \cdots = s_j s_i s_j \cdots$ is the longest element of the dihedral group generated by s_i and s_j .

We next prove the quadratic relation. Assume now that $s_i w < w$. Then applying (12) to $F_{\lambda, s_i w}$ we have $F_{\lambda, s_i w} = \mathcal{T}_i^{-1} F_{\lambda, w}$. Now we can compute this by the first case of (9), and we find

$$\mathcal{T}_i^{-1} F(\mathbf{z}) = C_{-\alpha} F(\mathbf{z}s_i) + (C_\alpha(\tau) - 1) F(\mathbf{z}).$$

This means that $\mathcal{T}_i^{-1} = q^{-1}\mathcal{T}_i + q^{-1} - 1$, which is equivalent to the quadratic relation $\mathcal{T}_i^2 = (q - 1)\mathcal{T}_i + q$. \square

This theorem guarantees that, given any reduced decomposition for the Weyl group element $w = s_{i_1} \dots s_{i_k}$, the operator

$$\mathcal{T}_w := T_{i_1} \cdots T_{i_k}$$

is well defined. As Ion notes in Theorem 3.1 of [10], the Demazure-Lusztig operators applied to \mathbf{z}^λ for λ dominant give a recursive definition for a certain limit of nonsymmetric Macdonald polynomials with weight $w \cdot \lambda \in P^\vee$. Thus our Iwahori-spherical functions are also limits of these polynomials for λ dominant.

We caution the reader that our conventions for Demazure-Lusztig operators and nonsymmetric Macdonald polynomials differ slightly from those of Ion. Instead they more closely parallel those of Cherednik [6]. In particular, our \mathcal{T}_i essentially match those in [6], equation (3.5), which are then used to construct nonsymmetric Macdonald polynomials.

Returning to the Whittaker case, similar arguments to those presented above were given in [3]. The operators that result are not Demazure-Lusztig operators, but are related in a way that is made precise in Section 5 of [3]. The difference results from the fact that the starting point for the Whittaker recursion is the Iwahori fixed vector Φ_{w_0} , rather than Φ_1 . Furthermore, the constant of proportionality in (8) differs slightly from the spherical case given in (7). The resulting recursive operators for Whittaker functions are Demazure-Lusztig operators conjugated by $\theta(\rho^\vee)$ and with q replaced by q^{-1} . Here $\theta : \mathbb{C}[P^\vee] \rightarrow \mathcal{H}_J$ is as in the previous section and ρ^\vee is half the sum of the positive coroots. In either the spherical or Whittaker cases, the resulting action of the finite Hecke algebra can be generalized to an action of the (extended) affine Hecke algebra where the elements in $\theta(P^\vee)$ act by translation. (See Theorem 28 of [3] for the Whittaker case.)

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