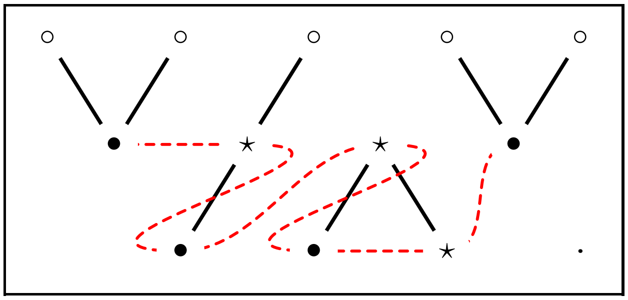
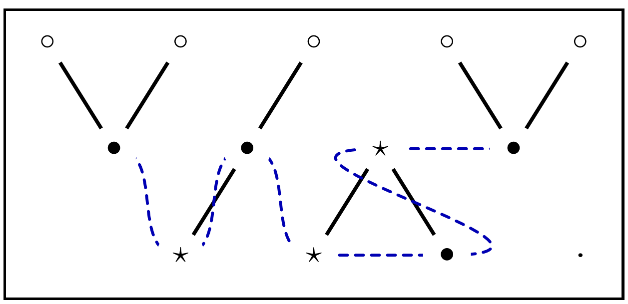
Gauss Sum Combinatorics



& Multiple Dirichlet Series



work in progress by

Brubaker, Bump & Friedberg

Happy Birthday

Dorian

And Thank You 10^6

References

- **WDM2:** Brubaker, Bump and Friedberg, Weyl Group Multiple Dirichlet Series II: The Stable Case. *Invent. Math.*, 165(2):325–355, 2006.
- **WMD3:** Brubaker, Bump, Friedberg and Hoffstein: Weyl group multiple Dirichlet series III: Eisenstein series and twisted unstable A_r. *Annals of Mathematics, to appear*, 2007.
- **WMD4:** Brubaker, Bump, and Friedberg. Weyl group multiple Dirichlet series: The stable twisted case. In *Eisenstein series and applications*. Birkhäuser.
- **WMD4$\frac{1}{2}$**: Brubaker, Bump, and Friedberg. Gauss Sum Combinatorics and Metaplectic Eisenstein Series, submitted to the proceedings of the Gelbart Conference.
- These notes, and other papers on this same subject are available at `http://math.stanford.edu/~bump`.
- Other references, such as the papers of Tokuyama, Casselman-Shalika, Kirillov-Berenstein, Schützenberger and Chinta-Gunnells may be found in the bibliographies of these papers.

Weyl Group Multiple Dirichlet Series

- Weyl Group multiple Dirichlet series are under development by Brubaker, Bump, Chinta, Friedberg, Hoffstein and Gunnells, among others.
- Data: a root system Φ and a totally complex number field F containing the group μ_{2n} of $2n$-th roots of unity. (Though μ_n should suffice, -1 an n-th power is handy.)
- Not an Euler product, yet twisted multiplicative, the coefficients involve n-th order Gauss sums.
- If n is sufficiently large, the theory is complete. The p-part has only $|W|$ nonzero terms, the **stable coefficients**.
- If n is not sufficiently large, other terms will appear inside the convex hull of the stable coefficients. These are more difficult to understand.
- In WMD3 Brubaker, Bump, Friedberg and Hoffstein conjectured coefficients when $\Phi = A_r$. This is the **Gelfand-Tsetlin description.**
- Chinta and Gunnells have a different approach that avoids specifying the unstable coefficients.
- Brubaker, Beineke and Frechette have been studying a **Gelfand-Tsetlin-Proctor** description for Sp_4.
- Recently progress has been made on proving the Gelfand-Tsetlin description. We describe that here.

The Weyl Character Formula

Let G be a complex reductive Lie group, T a maximal torus. Let $\lambda \in X^*(T)$ be a dominant weight.

- The ratio

$$\chi_\lambda = \frac{\sum (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum (-1)^{l(w)} e^{w(\rho)}} \qquad (w \in W, \text{the Weyl group})$$

  is the character of an irreducible representation π_λ of G.

- Here ρ = half the sum of the positive roots – at least modulo the subgroup of characters trivial on the derived group of G, such as the determinant if $G = \mathrm{GL}(n, \mathbb{C})$.
- Notation: $e^\lambda =$ image of λ in the group algebra of $X^*(T)$.
- The denominator factors as a product of $\frac{1}{2}|\Phi|$ linear factors:

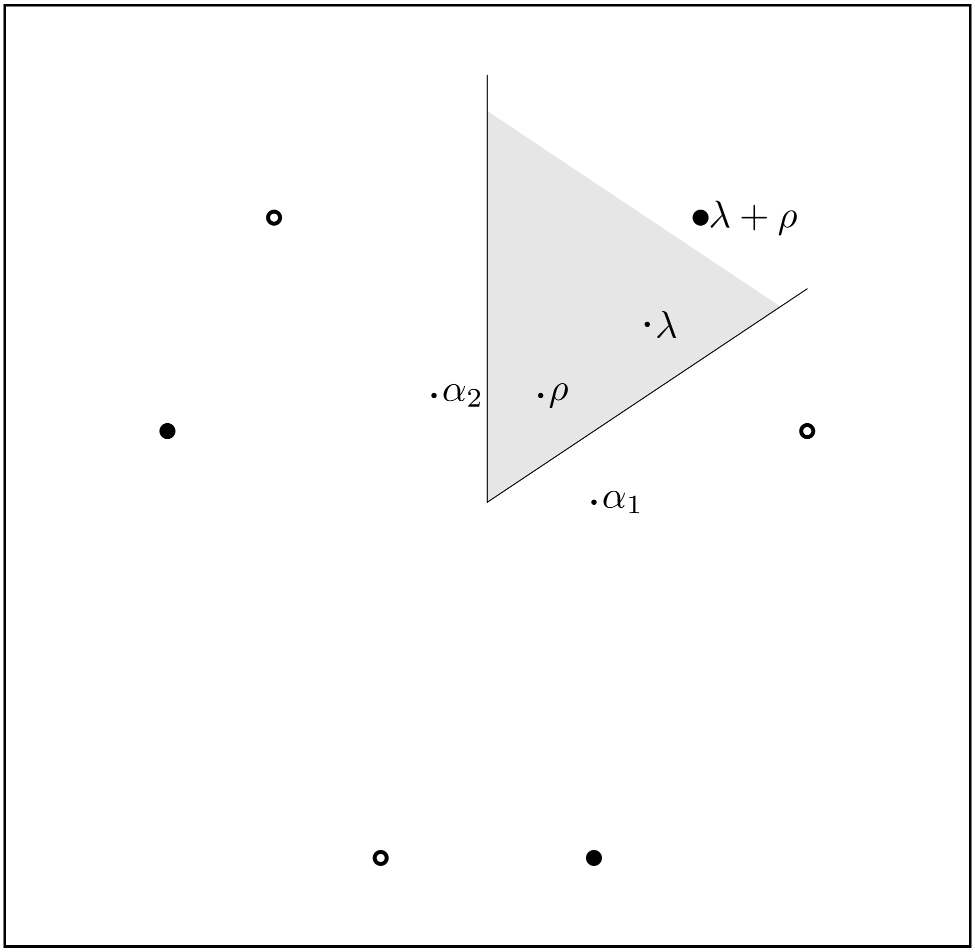
$$\prod_{\alpha \in \Phi^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right).$$

Special case: Let $\mathrm{GL}(r+1, \mathbb{C})$. Identify $\lambda = (\lambda_1, \cdots, \lambda_{r+1})$ with the character $X^*(T) \ni \lambda: t \longmapsto t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_{r+1}^{\lambda_{r+1}}$,

$$t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{r+1} \end{pmatrix}.$$

- λ dominant means $\lambda_1 \geqslant \ldots \geqslant \lambda_{r+1}$ and $\rho = (r, r-1, \cdots, 1, 0)$.
- Numerator is $\det(t_i^{\lambda_j + r - j})$.
- Denominator factorization is Vandermonde identity.
- Thus $\chi_\lambda(t) = s_\lambda(t_1, \cdots, t_{r+1})$ (Schur polynomial).

The Numerator in WCF: A_2 example

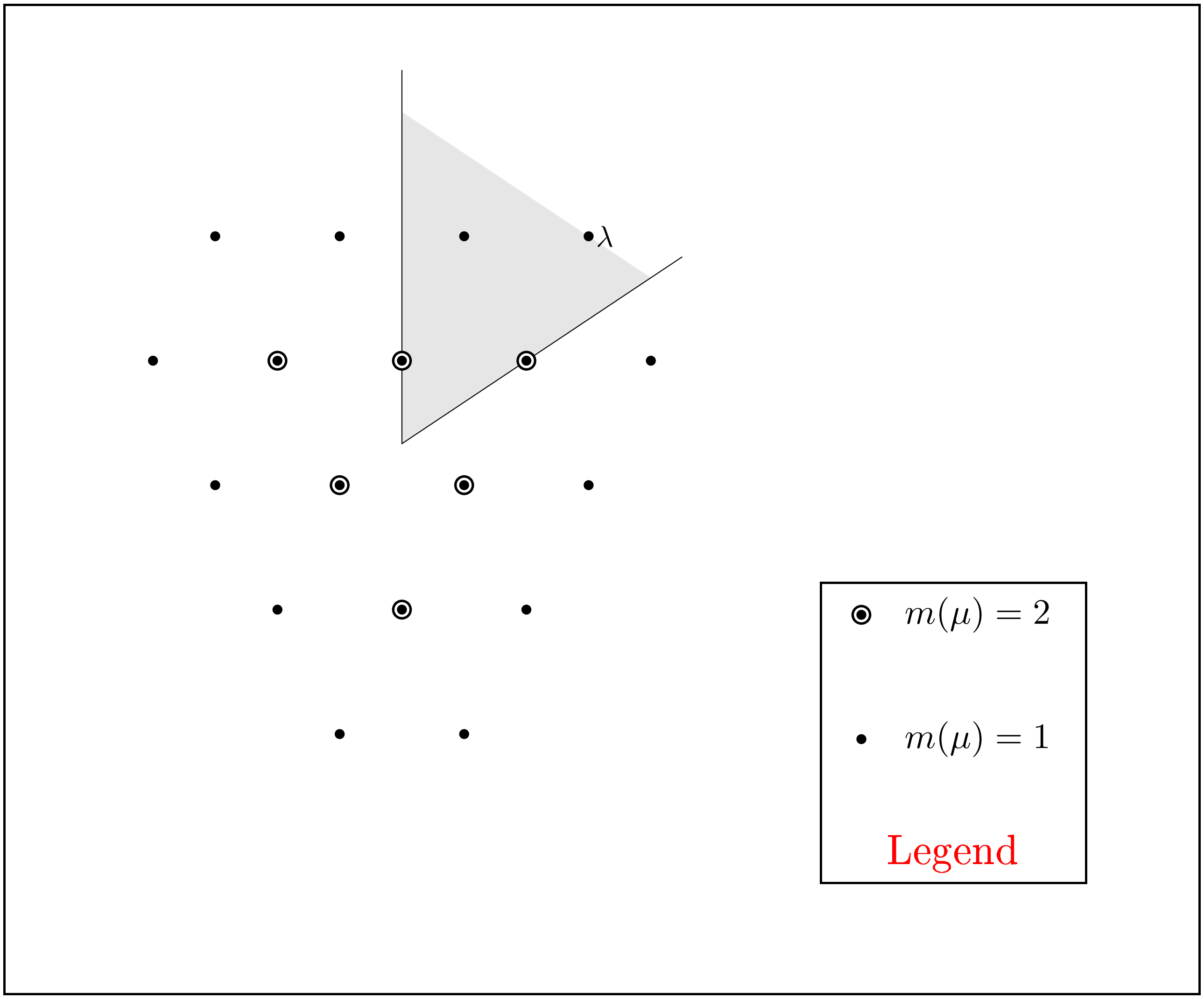


Remember the numerator is $\sum(-1)^{l(w)}e^{w(\lambda+\rho)}$, where the sum is over the Weyl group. Terms with $+$ are indicated with $\bullet$, terms with $-$ are indicated with $-$.

- In this example, $\lambda = (3, 1, 0)$.
- We have drawn the figure two dimensionally. All weights of the representation lie in a plane $\subset X^*(T) \cong \mathbb{Z}^3$.

The Character χ_λ

After dividing by $\sum(-1)^{l(w)}e^{w(\rho)}$ we obtain $\chi_\lambda = \sum m(\mu)e^\mu$.



- The support of χ_λ consists μ inside the convex hull of the $w(\lambda)$ that differ from λ by an element of the root lattice.
- The weight multiplicities $m(\mu)$ can be described by means of Gelfand-Tsetlin patterns. We describe that next for Type A_r.
- In this example $\mathrm{Dim}(\pi_\lambda) = \sum m(\mu) = 24$.

Gelfand-Tsetlin Patterns

A **Gelfand-Tsetlin pattern** is a pattern

$$\left\{\begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & \cdots & & a_{0r} & \\ & a_{11} & & a_{12} & & & a_{1r} & & \\ & & \ddots & & & \iddots & & & \\ & & & & a_{rr} & & & & \end{array}\right\}$$

where the rows are monotone nonincreasing and interlace, i.e. $a_{i,i} \geqslant a_{i+1,i+1} \geqslant a_{i,i+1} \geqslant \cdots$. It is **strict** if the rows are monotone **decreasing**.

- Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{r+1})$ be a highest weight vector. We take it to be the top row of the pattern.
- Then the number of patterns is the degree of the rep'n with character χ_λ.
- Originally Gelfand and Tsetlin parametrized vectors in the irreducible representation with character χ_λ (type A_r) by patterns.
- Dually, we can parametrize weights.

For A_2, $\lambda = (3, 1, 0)$ there are 24 patterns with this top row. Each corresponds to a weight: a pattern

$$\left\{\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & a & & b & \\ & & c & & \end{array}\right\} \quad \Longleftrightarrow \quad \text{weight } \mu = \lambda - k_1\alpha_1 - k_2\alpha_2,$$

where $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$ are the simple roots and

$$k_1 = a + b - \lambda_2 - \lambda_3, \qquad k_2 = c - \lambda_3.$$

Example

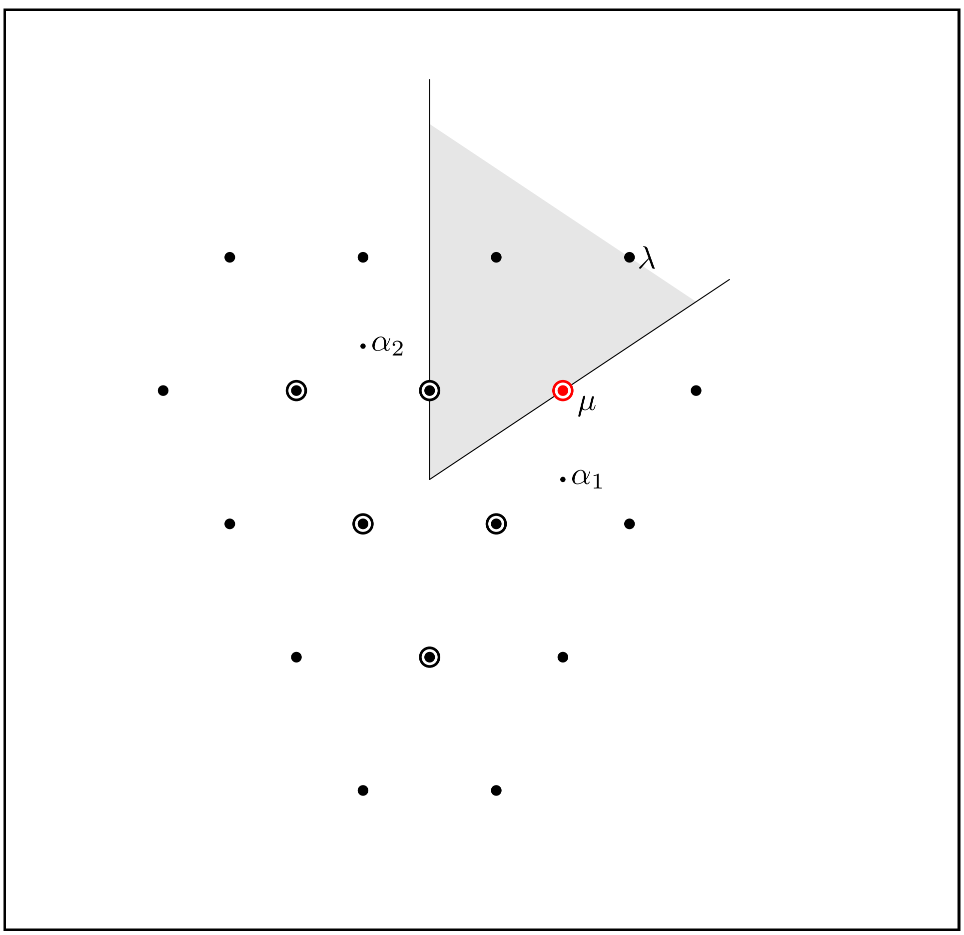
Let $\lambda = (3,1,0)$, $\mu = (2,1,1) = \lambda - \alpha_1 - \alpha_2$. To compute $m(\mu)$ we want to compute the patterns

$$\left\{ \begin{matrix} 3 & & 1 & & 0 \\ & a & & b & \\ & & c & & \end{matrix} \right\} \quad \Longleftrightarrow \quad \begin{aligned} k_1 &= a + b - 1 = 1, \\ k_2 &= c - 0 \qquad = 1. \end{aligned}$$

There are two:

$$\left\{ \begin{matrix} 3 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{matrix} \right\}, \qquad \left\{ \begin{matrix} 3 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{matrix} \right\}$$

so $m(\mu) = 2$. This is the weight marked with red.



The Shintani-Casselman-Shalika formula

The Shintani Casselman-Shalika formula is a purely local statement that implies that the Whittaker coefficients of Eisenstein series are essentially given by the Weil character formula. On GL_{r+1}

$$\int_{F_\infty/\mathfrak{o}} \cdots \int_{F_\infty/\mathfrak{o}} E_{s_1,\cdots,s_r}\begin{pmatrix} 1 & x_{12} & \cdots & x_{1,r+1} \\ & 1 & & \vdots \\ & & \ddots & x_{r,r+1} \\ & & & 1 \end{pmatrix} \times$$
$$\psi(m_1 x_{12} + m_2 x_{23} + \ldots + m_r x_{r,r+1}) dx_{12} \cdots dx_{r,r+1}$$

where $\psi: F_\infty/\mathfrak{o} \longrightarrow \mathbb{C}$ is an additive character is a coefficient

$$c(m_1, \cdots, m_r)$$

times a normalizing factor and a Whittaker coefficient.

- The coefficients are multiplicative due to uniqueness of Whittaker models.
- $\boldsymbol{c(p^{l_1}, \cdots, p^{l_r}) = \chi_\lambda(A_p)}$, where $A_p \in \mathrm{GL}_{r+1}(\mathbb{C})$ is the Langlands-Satake parameter, and the highest weight vector

  $$\lambda = (l_1 + \ldots + l_r, l_2 + \ldots + l_r, \cdots, l_r, 0).$$

- The known proofs of the formula have nothing to do with the Weyl character formula but produce the same formula in the end.

The Eisenstein series involves a product of $\frac{1}{2}|\Phi|$ zeta functions as normalizing factors. These resemble the denominator in the Weyl Character formula in its factored form but they are not the same.

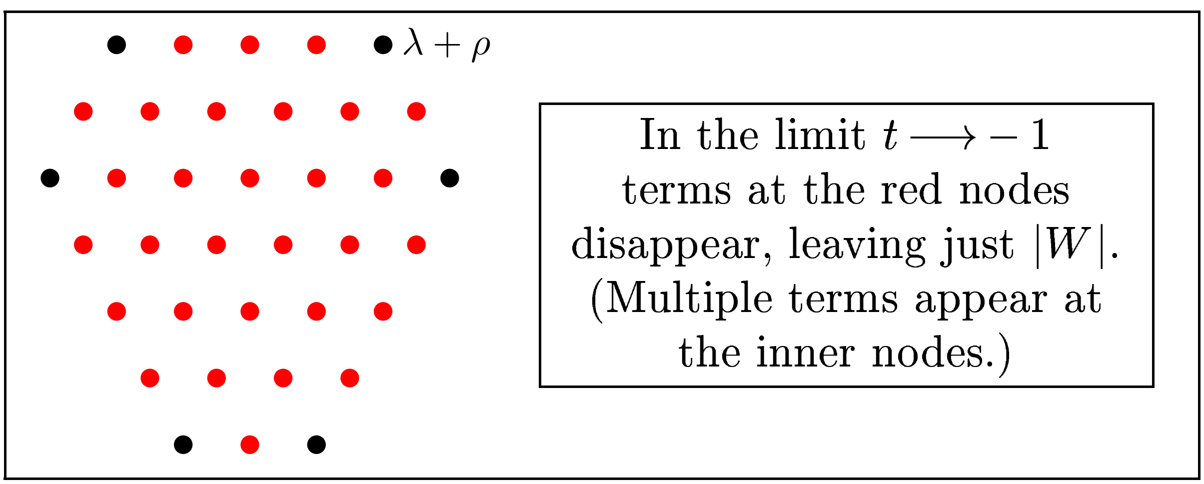
What happens to the Weyl character formula if the denominator is modified to agree with the zeta factor normalizing the Eisenstein series?

Tokuyama's deformation of WCF

For GL_{r+1}, Tokuyama gave a generalization of the Weyl character formula. It is a formula for

$$\left[\prod_{i<j} (x_i + t x_j)\right] s_\lambda(x_1, \cdots, x_{r+1}).$$

- Analogs for classical groups were given by S. Okada and by Hamel and King.
- Taking $t = 0$ the formula is related to a Gelfand and Tsetlin's construction of a basis for π_λ;
- Taking $t = 1$ gives a formula of Stanley.
- Taking $t = -1$ gives Weyl character formula; the product is in brackets is thus a deformation of the **denominator** in the Weyl character formula.
- The terms in the **numerator** in Tokuyama's formula are parametrized by **strict Gelfand-Tsetlin patterns**. In the Weyl limit $t \longrightarrow -1$, all but $|W|$ vanish, leaving the Wey-Weyl numerator.



Some definitions

We will describe the Tokuyama numerator in a different notation from Tokuyama's paper, better adapted for our later discussion. We won't worry about the denominator except to state that it corresponds to the normalizing factor of the Eisenstein series.

Let q be given; in the Eisenstein series application $q = \mathbb{N}\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal. Let $h(0) = 1$ while if $a > 0$,

$$h(a) = q^{a-1}(q-1), \qquad g(a) = -\,q^{a-1} \qquad \text{if } a > 0;$$
$$h(0) = 1, \qquad g(0) \text{ won't occur.}$$

Let $\Lambda = \lambda + \rho = (\lambda_1 + r, \lambda_2 + r - 1, \cdots, \lambda_r - 1, \lambda_{r+1})$.

- Note that adding ρ makes Λ **strictly decreasing**.

Let $\mathfrak{T}$ be a Gelfand-Tsetlin pattern with top row Λ.

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & \cdots & & a_{0r} \\ & a_{11} & & a_{12} & & & a_{1r} & \\ & & \ddots & & & \iddots & & \\ & & & & a_{rr} & & & \end{array} \right\}, \qquad a_{0,i} = \Lambda_{i+1}.$$

We will define two arrays $\Gamma = \Gamma(\mathfrak{T})$ and $\Delta = \Delta(\mathfrak{T})$. They will have one fewer row that $\mathfrak{T}$, but we'll mark the top row with $\circ$. Thus we will write

$$\Gamma(\mathfrak{T}) = \left\{ \begin{array}{ccccccccc} \circ & & \circ & & \circ & \cdots & & \circ \\ & \gamma_{11} & & \gamma_{12} & & & \gamma_{1r} & \\ & & \ddots & & & \iddots & & \\ & & & & \gamma_{rr} & & & \end{array} \right\}.$$

- We will give rules for circling and boxing certain elements of the Γ and Δ arrays.

The Γ array

The Γ and Δ arrays are defined by

$$\gamma_{ij} = \sum_{k=j}^{r} (a_{ik} - a_{i-1,k}), \qquad \delta_{ij} = \sum_{k=i}^{j} (a_{i-1,k-1} - a_{i,k}).$$

Let us explain what this means. If

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} 16 & & 12 & & 8 & & 3 & & 0 \\ & 15 & & 9 & & {}^{-}3 & & {}^{-}1 & \\ & & 10 & & {}^{+}8 & & {}^{+}4 & & \\ & & & 10 & & 5 & & & \\ & & & & 7 & & & & \end{array} \right\},$$

then add the red numbers and subtract the green ones to get 8, which goes at the blue location in the Γ array.

- Add the entries to the right in the same row, and subtract those above and to the right in the next row up.

$$\Gamma = \left\{ \begin{array}{ccccccccc} \circ & & \circ & & \circ & & \circ & & \circ \\ & 5 & & 2 & & \textcircled{1} & & 1 & \\ & & 9 & & 8 & & 3 & & \\ & & & \boxed{3} & & 1 & & & \\ & & & & 2 & & & & \end{array} \right\}.$$

- We **circle** γ_{ij} if $a_{ij} = a_{i-1,j}$ (above and to the right).
- We **box** γ_{ij} if $a_{i,j} = a_{i-1,j-1}$ (above and to the left).

In this example, the 1 in the second row of Γ is circled due to the repeated entry 3 at that location and above and to the right in $\mathfrak{T}$, and similarly the 3 is circled due to the repeated entry 10 in $\mathfrak{T}$.

We call this definition of Γ (including the circling and boxing convention) the **right-hand rule**.

The Δ array

The Δ array is not really needed for Tokuyama's theorem, but will become crucial later. It is defined like Γ but everything is reversed.

To explain Δ, we recolor $\mathfrak{T}$.

$$\mathfrak{T}=\left\{\begin{array}{ccccccccc} 16 & & 12 & & 8 & & 3 & & 0 \\ & {}^{+}15 & & {}^{+}9 & & 3 & & 1 & \\ & & {}^{-}10 & & {}^{-}8 & & 4 & & \\ & & & 9 & & 5 & & & \\ & & & & 7 & & & & \end{array}\right\}.$$

Now adding the red numbers and subtracting the green gives 6 which again goes in the blue spot in the Δ array.

$$\Delta=\left\{\begin{array}{ccccccccc} \circ & & \circ & & \circ & & \circ & & \circ \\ & 1 & & 4 & & \boxed{9} & & 11 & \\ & & 5 & & 6 & & 7 & & \\ & & & \textcircled{1} & & 4 & & & \\ & & & & 2 & & & & \end{array}\right\}.$$

We call this the **left-hand rule**.

- The circling and boxing conventions are reversed from Γ.
- In the Γ array, the rows decrease, while in Δ they increase.

Now define

$$G_\Gamma(\mathfrak{T})=\prod_{i,j}\left\{\begin{array}{ll} q^{\gamma_{ij}} & \text{if } \gamma_{ij} \text{ is circled,} \\ g(\gamma_{ij}) & \text{if } \gamma_{ij} \text{ is boxed,} \\ h(\gamma_{ij}) & \text{if neither (the usual case).} \end{array}\right.$$

We define G_Δ the same way using the Δ array.

The Tokuyama numerator

We now describe the Tokuyama numerator from the viewpoint of WMD3 and WMD4$\frac{1}{2}$. There are two versions, labeled Γ and Δ.

Let

$$k_\Gamma(\mathfrak{T}) = (k_1^\Gamma, \cdots, k_r^\Gamma), \qquad k_\Delta(\mathfrak{T}) = (k_1^\Delta, \cdots, k_r^\Delta),$$

with

$$k_i^\Gamma = k_i^\Gamma(\mathfrak{T}) = \sum_{j=i}^{r} (a_{i,j} - a_{0,j})$$

and

$$k_i^\Delta = k_i^\Delta(\mathfrak{T}) = \sum_{j=r+1-i}^{r} (a_{0,j-r-1+i} - a_{r+1-i,j}).$$

- To reiterate **these depend linearly on the row sums of $\mathfrak{T}$** in slightly different ways. That's all that's important.
- We may associate a weight with $\mathfrak{T}$ by either

$$\mu_\Gamma(\mathfrak{T}) = \lambda - \sum k_i^\Gamma \alpha_i \qquad \text{or} \qquad \mu_\Delta(\mathfrak{T}) = \lambda - \sum k_i^\Delta \alpha_i.$$

Then Tokuyama's numerator can be written either

$$\sum_{\mathfrak{T}} G_\Gamma(\mathfrak{T})\, \mu_\Gamma(\mathfrak{T}) \qquad \text{or} \qquad \sum_{\mathfrak{T}} G_\Delta(\mathfrak{T})\, \mu_\Delta(\mathfrak{T}).$$

Sum is over **strict** Gelfand-Tsetlin patterns with top row $\lambda + \rho$. Recall **strict** means that each row is strictly decreasing.

- The equivalence of the Γ and Δ descriptions follows from the existence of an outer involution $g \mapsto {}^t g^{-1}$ of $\mathrm{GL}_{r+1}(\mathbb{C})$.
- This equivalence fails to be obvious in the metaplectic generalization that we are coming to, an important issue.
- But I'm getting ahead of my story.

Whittaker coefficients of Eisenstein series

Combining the results of Tokuyama and Casselman-Shalika, we may now describe the Whittaker coefficient of the Eisenstein series on GL_{r+1}. It is a multiple Dirichlet series

$$Z(s_1, \cdots, s_r; m_1, \cdots, m_r) = \sum H(c_1, \cdots, c_r; m_1, \cdots, m_r) \mathbb{N}c_1^{-2s_1} \cdots \mathbb{N}c_r^{-2s_r}.$$

- **This multiple Dirichlet series is an Euler product.** It is multiplicative in both the c's and the m's, so the specification of the coefficients is reduced to c_i and m_i being powers of the same prime.
- Tokuyama + Casselman Shalika imply

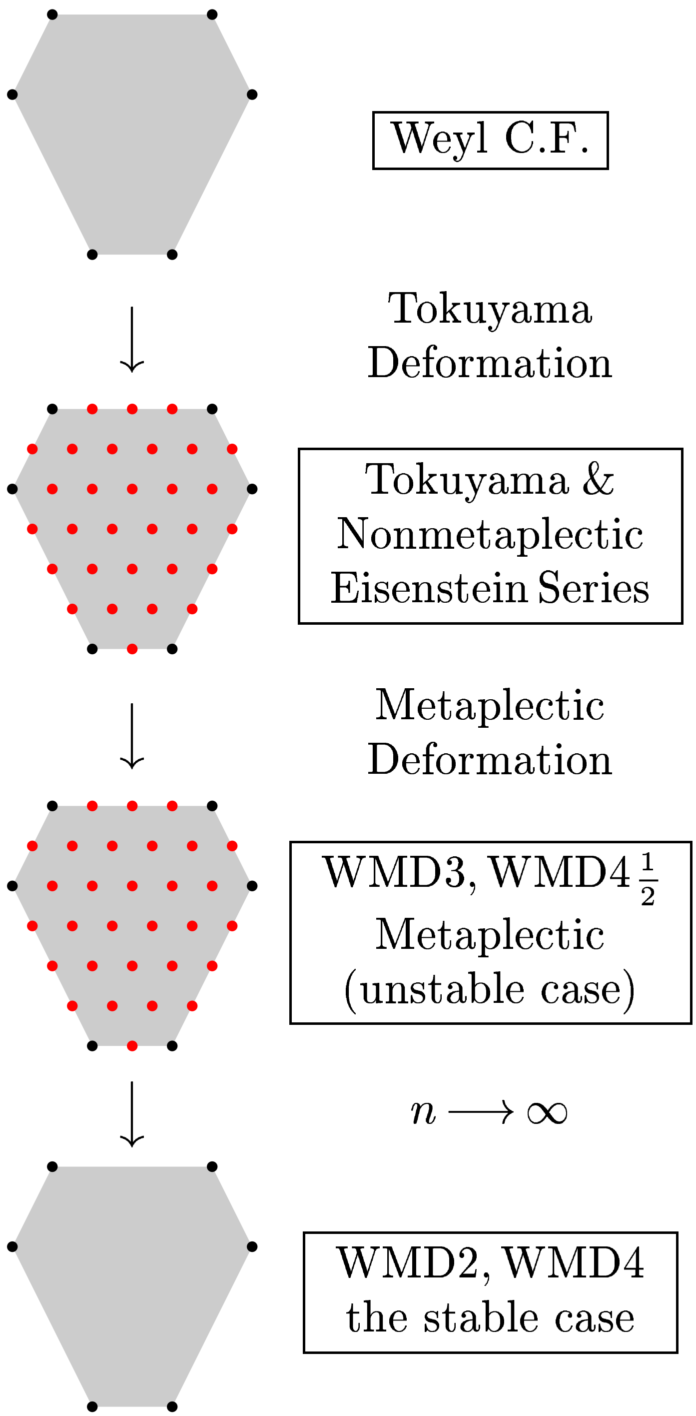
$$H(p^{k_1}, \cdots, p^{k_r}; p^{l_1}, \cdots, p^{l_r}) = \sum_{\substack{\text{top row}=\Lambda \\ k_\Gamma(\mathfrak{T})=(k_1,\cdots,k_r)}} G_\Gamma(\mathfrak{T}) = \sum_{\substack{\text{top row}=\Lambda \\ k_\Delta(\mathfrak{T})=(k_1,\cdots,k_r)}} G_\Delta(\mathfrak{T}).$$

- Here Λ is

$$l_1 + \ldots + l_r + r, l_2 + \ldots + l_r + r - 1, \cdots, l_r - 1, 0.$$

- The Γ formula is equivalent to Tokuyama's statement.
- The Δ formula follows using the outer automorphism of the L-group $\mathrm{GL}(r+1, \mathbb{C})$.

Down the Rabbit Hole



Now we come to another deformation–in the “metaplectic direction. The multiple Dirichlet series from this point on will involve n-th order Gauss sums. (Until now, $n=1$ and the Gauss sums have been explicitly evaluable.) Great complexity ensues but if n is sufficiently big, interior terms introduced in the Tokuyama deformation go away again.

The Gelfand-Tsetlin description

The formulas conjectured in WMD3 for the A_r Weyl group multiple Dirichlet series give formulas identical to what we have already seen, but the meaning of g and h are now changed. We assume that F is a totally complex number field containing the $2n$-th roots of unity. Let S be a finite set of places large enough that the ring $\mathfrak{o}_S$ of S-integers is principal. Let p be a fixed prime, and $q = \mathbb{N}p$. Let ψ be an additive character of $F_S/\mathfrak{o}_S$ and let

$$\mathfrak{g}(m, c) = \sum_{a \bmod c} \left(\frac{a}{c}\right) \psi\left(\frac{am}{c}\right),$$

when m and c are nonzero elements of $\mathfrak{o}_S$.

- **The Weyl group multiple Dirichlet series is not an Euler product**. It is **twisted** multiplicative and therefore determined by its p-part.
- This is defined by the same formulas as previously, but now we let

$$g(a) = \mathfrak{g}(p^{a-1}, p^a), \qquad h(a) = \mathfrak{g}(p^a, p^a).$$

- Every term in the Dirichlet series **except** the $|W|$ terms that are the vertices of the permutahedron involves at least one $h(a)$ with $a > 0$.
- **If a does not divide n then $h(a) = 0$**, which means that if n is sufficiently large, only the $|W|$ stable terms survive, and the theory reduces to that in WMD2.
- The correctness of the Gelfand-Tsetlin description leads to fascinating combinatorial questions which we now discuss.

Example

Let us take $r = 2$ and $(l_1, l_2) = (0, 0)$ so $\Lambda = \rho = (2, 1, 0)$. There are eight patterns, one nonstrict (hence contributing 0).

- First we list the **stable patterns**. A pattern is **stable** if every element equals one of the two above it (except of course in the top row).

stable patterns

$\mathfrak{T}$	Γ	(k_1, k_2)	$G_\Gamma(\mathfrak{T})$
2 1 0 1 0 0	$\circ\ \circ\ \circ$ $\textcircled{0}\ \textcircled{0}$ $\textcircled{0}$	$(0, 0)$	1
2 1 0 1 0 1	$\circ\ \circ\ \circ$ $\textcircled{0}\ \textcircled{0}$ $\boxed{1}$	$(0, 1)$	$g(1)$
2 1 0 2 0 0	$\circ\ \circ\ \circ$ $\boxed{1}\ \textcircled{0}$ $\textcircled{0}$	$(1, 0)$	$g(1)$
2 1 0 2 0 2	$\circ\ \circ\ \circ$ $\boxed{1}\ \textcircled{0}$ $\boxed{2}$	$(1, 2)$	$g(1)g(2)$
2 1 0 2 1 1	$\circ\ \circ\ \circ$ $\boxed{2}\ \boxed{1}$ $\textcircled{0}$	$(1, 2)$	$g(1)g(2)$
2 1 0 2 1 2	$\circ\ \circ\ \circ$ $\boxed{2}\ \boxed{1}$ $\boxed{1}$	$(2, 2)$	$g(1)^2g(2)$

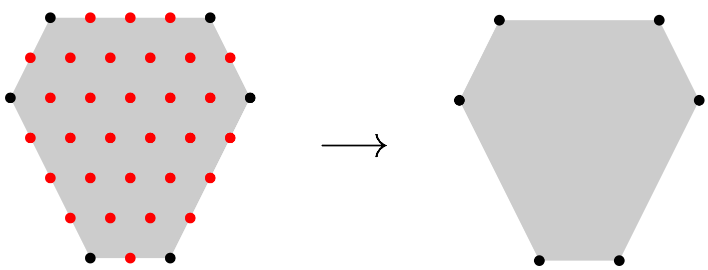
- In the stable patterns, every nonzero entry is boxed, and every 0 is circled. This produces a product of $l(w)$ Gauss sums, where $l(w)$ is the number of boxed sums. The Weyl group element w is explained in WMD4.

Example continued: unstable patterns

There are also two unstable patterns. For the second, $G(\mathfrak{T}) = 0$ since the pattern is nonstrict. The first has a factor of $h(1)$ which is 0 unless n divides 1.

	$\mathfrak{T}$	Γ	(k_1, k_2)	$G_\Gamma(\mathfrak{T})$
unstable patterns	2 1 0 2 0 1	$\circ \quad \circ \quad \circ$ $\boxed{1} \quad \textcircled{0}$ 1	$(0, 0)$	$g(1)h(1)$
	2 1 0 1 1 1	($\mathfrak{T}$ is nonstrict)	$(0, 0)$	0

- If $\mathfrak{T}$ is unstable, $G_\Gamma(\mathfrak{T})$ is divisible by $h(a)$ for some $a > 0$. It is therefore 0 unless $n|a$. This explains why only the unstable patterns remain when $n \longrightarrow \infty$.



First reduction

- In WMD3 it was conjectured that Z has meromorphic continuation and functional equations. The major advance of that paper was giving the conjectural Gelfand-Tsetlin description.
- The goal of proving that Z has meromorphic continuation and functional equations has not been reached but substantial progress has been made.
- The first step was to give the alternative definition in terms of the Δ array (not known when WMD3 was written).

Theorem. *If*

$$\sum_{\mathfrak{T}} G_\Gamma(\mathfrak{T})\,\mu_\Gamma(\mathfrak{T}) = \sum_{\mathfrak{T}} G_\Delta(\mathfrak{T})\,\mu_\Delta(\mathfrak{T})$$

then Z has the conjectured meromorphic continuation and functional equations.

Proof. (sketch) Using the original Γ definition, one may expand $Z(s_1, \cdots, s_r; m_1, \cdots, m_r; A_r)$ in terms of lower rank A_{r-1} series by fixing the first **two** rows of the pattern and summing over the remaining rows. This gives functional equations with respect to the simple reflections $\sigma_2, \cdots, \sigma_r$. Similarly the Δ definition gives functional equations for the simple reflections $\sigma_1, \cdots, \sigma_r$. Combining all information, one may use Bochner's tube domain theorem to obtain analytic continuation to all $\mathbb{C}^r$. □

- When $n = 1$, the equivalence of the Γ and Δ definitions followed from the existence of an outer automorphism of the L-group $\mathrm{GL}_{r+1}(\mathbb{C})$, but that argument is not available and we must therefore substitute combinatorial reasoning.

The Schützenberger involution

- An involution on Young tableaux was described by Schützenberger (1978) in terms of **jeu de taquin**.
- It was adapted to the setting of Gelfand-Tsetlin patterns by Kirillov and Berenstein.
- It is necessary to work with nonstrict patterns since the property of strictness is not preserved by the involution, so we extend the Γ and Δ definitions by $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}) = 0$ if $\mathfrak{T}$ is not strict.

If

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} a_{00} & & a_{01} & & a_{02} & \cdots & & a_{0r} & \\ & a_{11} & & a_{12} & & & a_{1r} & & \\ & & \ddots & & & \iddots & & & \\ & & & & a_{rr} & & & & \end{array} \right\},$$

then a_{ij} is constrained between

$$\min\left(a_{i-1,j-1}, a_{i+1,j}\right) \quad \text{and} \quad \max\left(a_{i-1,j}, a_{i+1,j+1}\right),$$

so let

$$a'_{i,j} = \min\left(a_{i-1,j-1}, a_{i+1,j}\right) + \max\left(a_{i-1,j}, a_{i+1,j+1}\right) - a_{i,j}.$$

Let $t_i\mathfrak{T}$ be the pattern obtained by replacing the i-th row by the elements $a'_{i,j}$. Thus the row is “reflected.”

- The operations t_i do not satisfy the braid relation, so although they **can** be used to define an action of the symmetric group, this is not in an obvious way.
- Let $q = (t_r \cdots t_3 t_2 t_1) \cdots (t_r t_{r-1} t_{r-2})(t_{r-1} t_r) t_r$, the **Schützenberger** involution.
- The involution interchanges the weights k_Γ and k_Δ. If it is well-behaved on patterns, then the conjecture is proved. We are reduced to working with **three rows at a time.**

Short patterns, long definitions

A **short** Gelfand-Tsetlin pattern is one with just three rows:

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} l_1 & & l_2 & & l_3 & & \cdots & & l_{r+1} \\ & a_1 & & a_2 & & & & a_r & \\ & & b_1 & & \cdots & & b_{r-1} & & \end{array} \right\},$$

The rows are nondecreasing and interleave.

- The top and bottom row may be assumed strictly decreasing. (The middle one **may not be**.)
- **Note: l_i has a different meaning from before.**

We will associate with $\mathfrak{t}$ two arrays

$$\Gamma = \Gamma(\mathfrak{t}) = \left\{ \begin{array}{ccccccccc} \circ & & \circ & & \cdots & & \circ & & \circ \\ & \Gamma_{1,1} & & \Gamma_{1,2} & & & & \Gamma_{1,r} & \\ & & \Gamma_{2,1} & & \cdots & & \Gamma_{2,r-1} & & \end{array} \right\}$$

and

$$\Delta = \Delta(\mathfrak{t}) = \left\{ \begin{array}{ccccccccc} \circ & & \circ & & \cdots & & \circ & & \circ \\ & \Delta_{1,1} & & \Delta_{1,2} & & & & \Delta_{1,r} & \\ & & \Delta_{2,1} & & \cdots & & \Delta_{2,r-1} & & \end{array} \right\}.$$

- We use the **right-hand rule** on the **middle row of Γ** and the **bottom row of Δ**.
- We use the **left-hand rule** on the **bottom row of Δ** and the **middle row of Γ**.
- This applies to the circling convention, too.
- Although Γ and Δ have only two rows, we use the term "middle row" instead of "top row" since the $\Gamma_{1,j}$ row corresponds to the middle row of $\mathfrak{T}$.

Γ and Δ for short patterns

For example, suppose

$$\mathfrak{T} = \left\{\begin{array}{ccccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & 20 & & 12 & & 5 & & 4 & & 2 & \\ & & 14 & & 9 & & 5 & & 3 & & \end{array}\right\}.$$

We have

$$\Gamma(\mathfrak{T}) = \left\{\begin{array}{ccccccccccc} \circ & & \circ & & \circ & & \circ & & \circ & & \circ \\ & 9 & & \textcircled{4} & & \textcircled{4} & & 4 & & \boxed{2} & \\ & & 6 & & 9 & & \textcircled{9} & & 10 & & \end{array}\right\}.$$

- Since we are using the right-hand rule on the middle row and the left-hand rule on the bottom one, the middle row is decreasing, and the bottom row increasing from left to right.
- Δ works the same way but with everything reversed.

Momentarily we will define an involution $\mathfrak{T} \longmapsto \mathfrak{T}'$ on patterns, and show that

$$\mathfrak{T}' = \left\{\begin{array}{ccccccccccc} 23 & & 15 & & 12 & & 5 & & 2 & & 0 \\ & 18 & & 14 & & 9 & & 4 & & 0 & \\ & & 14 & & 9 & & 5 & & 3 & & \end{array}\right\}.$$

More relevant than $\Delta(\mathfrak{T})$ is $\Delta(\mathfrak{T}')$, and we find that

$$\Delta(\mathfrak{T}') = \left\{\begin{array}{ccccccccccc} \circ & & \circ & & \circ & & \circ & & \circ & & \circ \\ & 5 & & 6 & & 9 & & 10 & & \boxed{12} & \\ & & \textcircled{4} & & \textcircled{4} & & 4 & & 3 & & \end{array}\right\}.$$

The involution and the cartoon

Now the involution $\mathfrak{T} \longmapsto \mathfrak{T}'$ is simple to describe. It only affects the middle row. Since with

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} l_1 & & l_2 & & l_3 & & \cdots & & l_{r+1} \\ & a_1 & & a_2 & & & & a_r & \\ & & b_1 & & \cdots & & b_{r-1} & & \end{array} \right\}$$

we have

$$\min(l_i, b_{i-1}) \geqslant a_i \geqslant \max(l_{i+1}, b_i)$$

we may "reflect" each a_i in this range, replacing $\mathfrak{T}$ by

$$\mathfrak{T}' = \left\{ \begin{array}{ccccccccc} l_1 & & l_2 & & l_3 & & \cdots & & l_{r+1} \\ & a_1' & & a_2' & & & & a_r' & \\ & & b_1 & & \cdots & & b_{r-1} & & \end{array} \right\},$$

where

$$a_i' = \min(l_i, b_{i-1}) + \max(l_{i+1}, b_i) - a_i.$$

The effect of the involution may be diagramed by use of a **cartoon**. It is a graph whose vertices are in bijection with the places of $\mathfrak{T}$ or $\Gamma(\mathfrak{T})$ or $\Delta(\mathfrak{T})$, and whose edges diagram the above averaging.

- We connect a_i to l_i if $l_i \leqslant b_{i-1}$, and to b_{i-1} if $b_{i-1} \leqslant l_i$.
- We connect a_i to l_{i+1} if $l_{i+1} \geqslant b_i$, and to b_i if $b_i \geqslant l_i$.
- Thus if we have x—a_i—y the involution replaces a_i by

$$a_i' = x + y - a_i.$$

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccccccccccccccc} 23 & & & & 15 & & & & 12 & & & & 5 & & & & 2 & & & & 0 \\ & \diagdown & & \diagup & & & & \diagup & & & & \diagup & & \diagdown & & & & \diagdown & & \diagup & \\ & & 20 & & & & 12 & & & & 5 & & & & 4 & & & & 2 & & \\ & & & & & \diagup & & & & \diagup & & \diagdown & & \diagup & & \diagdown & & & & & \\ & & & & 14 & & & & 9 & & & & 5 & & & & 3 & & & & \end{array} \right\}.$$

The Short Pattern Conjecture

Let $\boldsymbol{l} = (l_1, \cdots, l_{r+1})$ and $\boldsymbol{b} = (b_1, \cdots, b_{r-1})$ be decreasing integer sequences of lengths $r+1$ and $r-1$ respectively. Let k be an integer. The **type** $\mathfrak{S} = \mathfrak{S}(\boldsymbol{l}, \boldsymbol{b}, k)$ is the set of all patterns

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccc} l_1 & & l_2 & & l_3 & & \cdots & & l_{r+1} \\ & a_1 & & a_2 & & & & a_r & \\ & & b_1 & & \cdots & & b_{r-1} & & \end{array} \right\},$$

with prescribed top and bottom rows, middle row sum $\sum_i a_i = k$.

- The cartoon depends only on $\boldsymbol{l}$ and $\boldsymbol{b}$ so all patterns in the type have the same cartoon.
- So does $\mathfrak{T}'$, which lies in a different type with same $\boldsymbol{l}$ and $\boldsymbol{b}$ but different k.

Conjecture. *Let $\mathfrak{S}$ be a type. Then*

$$\sum_{\mathfrak{T} \in \mathfrak{S}} G_\Gamma(\mathfrak{T}) = \sum_{\mathfrak{T} \in \mathfrak{S}} G_\Delta(\mathfrak{T}').$$

Theorem. *This conjecture implies the analytic continuation and functional equations of Z.*

Proof. The Schützenberger involution reduces us to working 3 rows at a time, reducing us to consideration of short patterns. If one thinks carefully about what this entails, the present formulation is arrived at where one uses the right-hand rule for one row and the left-hand row for the other in Γ and Δ. $\square$

The Snake Lemma

- $\mathfrak{T}$ is **superstrict** $l_i, b_{i-1} > a_i > l_{i+1}, b_i$, that is, no entries are circled or boxed. In some sense **most patterns are superstrict**.
- $\mathfrak{T}$ is **stable** if each entry (except the top row ones) equals one of the two above it.
- $\mathfrak{T}$ is **nonresonant** if $l_i \neq b_{i-1}$. (**Nonresonance** depends only on the type or cartoon.) More about resonance later.

Theorem. *If $\mathfrak{T}$ is superstrict, stable or nonresonant the $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}')$.*

- In some sense most patterns are superstrict, so "usually" $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}')$ **but not always!**
- Still we find that we can group the patterns into fairly small sets Π called **packets** such that $\sum_{\mathfrak{T}\in\Pi} G_\Gamma(\mathfrak{T}) = \sum_{\mathfrak{T}\in\Pi} G_\Delta(\mathfrak{T}')$. More about packets later.
- The proof of Theorem 2 contains important ideas so we discuss the superstrict case.

Lemma. (Snake Lemma) *There exists orderings of the $\Gamma_{ij}(\mathfrak{T})$ and $\Delta_{ij}(\mathfrak{T}')$ such that*

$$\{\Gamma_{ij}(\mathfrak{T})\} = \{\gamma_1, \gamma_2, \cdots, \gamma_{2r-1}\}, \qquad \{\Delta_{ij}(\mathfrak{T}')\} = \{\delta'_1, \delta'_2, \cdots, \delta'_{2r-1}\}$$

with the following property. Extend the labelings by letting $\gamma_0 = \gamma_{2r} = 0$. Then

$$\delta'_k = \begin{cases} \gamma_k & \text{if } k \text{ is even,} \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

Example for the Snake Lemma

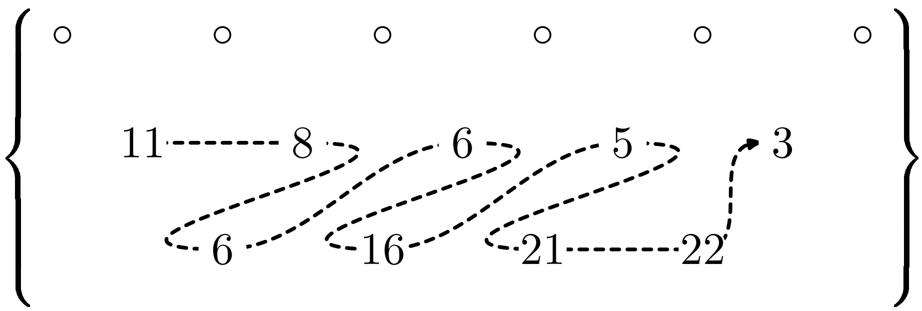
Suppose

$$\mathfrak{T} = \left\{ \begin{array}{ccccccccccc} 45 & & 37 & & 28 & & 14 & & 5 & & 0 \\ & 40 & & 30 & & 15 & & 7 & & 3 & \\ & & 34 & & 20 & & 10 & & 6 & & \end{array} \right\},$$

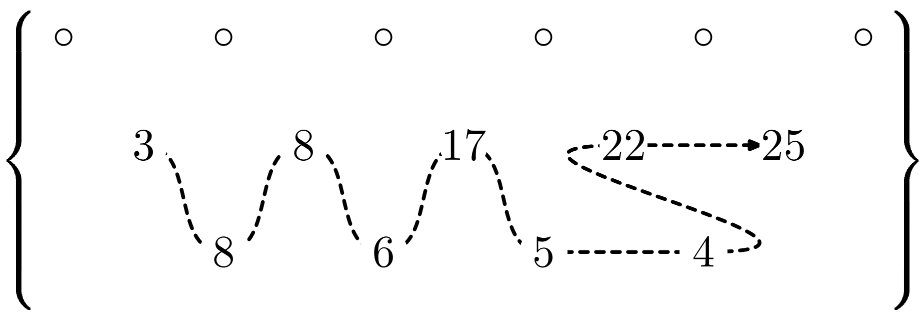
$$\mathfrak{T}' = \left\{ \begin{array}{ccccccccccc} 45 & & 37 & & 28 & & 14 & & 5 & & 0 \\ & 42 & & 32 & & 19 & & 9 & & 2 & \\ & & 34 & & 20 & & 10 & & 6 & & \end{array} \right\}.$$

We can order the $\Gamma(\mathfrak{T})$ and $\Delta(\mathfrak{T}')$ arrays as follows

$\Gamma(\mathfrak{T}) =$



$\Delta(\mathfrak{T}') =$



If we write out the entries in the indicated order:

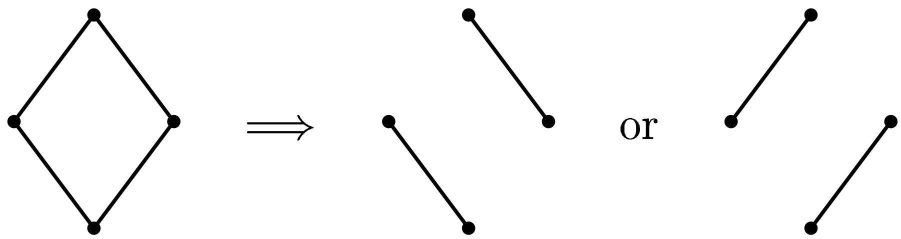
k	0	1	2	3	4	5	6	7	8	9	10
γ_k	0	11	8	6	6	16	5	21	22	3	0
δ_k		3	8	8	6	17	5	4	22	25	

we get (as required)

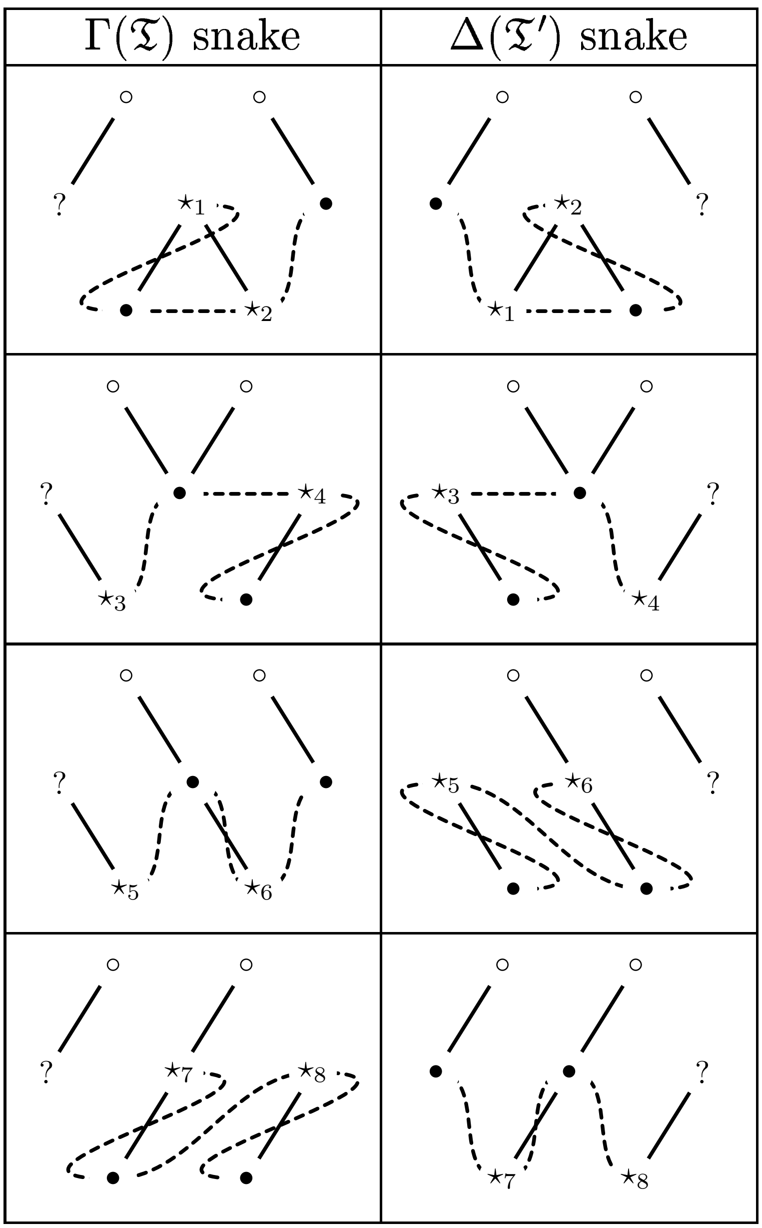
$$\delta'_k = \begin{cases} \gamma_k & \text{if } k \text{ is even,} \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

Proof of the Snake Lemma

If there is resonance, that is, if the cartoon contains "diamonds," replace them (arbitrarily) as follows:



Now read the snakes off from the following table.



First check that these rules are consistent.

$\bullet =$ odd numbered γ_i, δ'_i
$\star =$ odd numbered γ_i, δ'_i

Subscripts on $\star$ show corresponding even γ_i, δ'_i.

The snakes determine the correspondence $\gamma_i \longleftrightarrow \delta'_i$.

Verification that

$$\delta'_k = \begin{cases} \gamma_k & \text{(even)}, \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & \text{(odd)} \end{cases}$$

is then reduced to case-by-case analysis.

Proof of the Theorem

We are assuming $\mathfrak{T}$ is superstrict so there is no circling or boxing.

$$G_\Gamma(\mathfrak{T}) = \prod h(\gamma_i) = \Bigg[\prod_{i \text{ even}} h(\gamma_i)\Bigg]\Bigg[\prod_{i \text{ odd}} h(\gamma_i)\Bigg]$$

By the Snake Lemma

$$\delta'_k = \begin{cases} \gamma_k & k \text{ even}, \\ \gamma_k + \gamma_{k-1} - \gamma_{k+1} & k \text{ odd}, \end{cases}$$

and so

$$G_\Delta(\mathfrak{T}') = \Bigg[\prod_{i \text{ even}} h(\gamma_i)\Bigg]\Bigg[\prod_{i \text{ odd}} h(\gamma_i + \gamma_{i-1} - \gamma_{i+1})\Bigg].$$

Now $h(\gamma_i) = 0$ unless $n|\gamma_i$, so we may assume all the odd γ_i are multiples of n, and we are reduced to proving

$$\prod_{i \text{ odd}} h(\gamma_i + \gamma_{i-1} - \gamma_{i+1}) = \prod_{i \text{ odd}} h(\gamma_i).$$

Since γ_{i-1} and γ_{i+1} are multiples of n, the left-hand side equals

$$\prod_{i \text{ odd}} q^{\gamma_{i-1} - \gamma_{i+1}} h(\gamma_i).$$

The powers of q cancel in pairs, and we are done!

Resonance

Resonance occurs when the cartoon contains a diamond.

- Connected components of the cartoon are called **episodes**.
- Using ideas similar to the Snake Lemma, we can reduce to studying the restriction of $\mathfrak{T}$ to a single episode.
- Thus (oversimplying only slightly) the main task is to study totally resonant types, in which $l_i = b_{i-1}$ for all i.
- We would like to divide each resonant episode into a disjoint union of **packets**, where if Π is a packet

$$\sum_{\mathfrak{T}\in\Pi} G_\Gamma(\mathfrak{T}) = \sum_{\mathfrak{T}\in\Pi} G_\Delta(\mathfrak{T}').$$

Let us consider the case $r = 1$ and $l_2 = b_1$. Thus we are concerned with the following resonant type $\mathfrak{S}$

$$\mathfrak{T} = \left\{ \begin{array}{ccccc} l_1 & & l_2 & & l_3 \\ & x & & y & \\ & & l_2 & & \end{array} \right\}, \qquad x + y = k.$$

There are two distinguished patterns in the type.

- $\mathfrak{T}_1$ in which x is as large as possible,
- $\mathfrak{T}_2$ in which x is as small as possible.

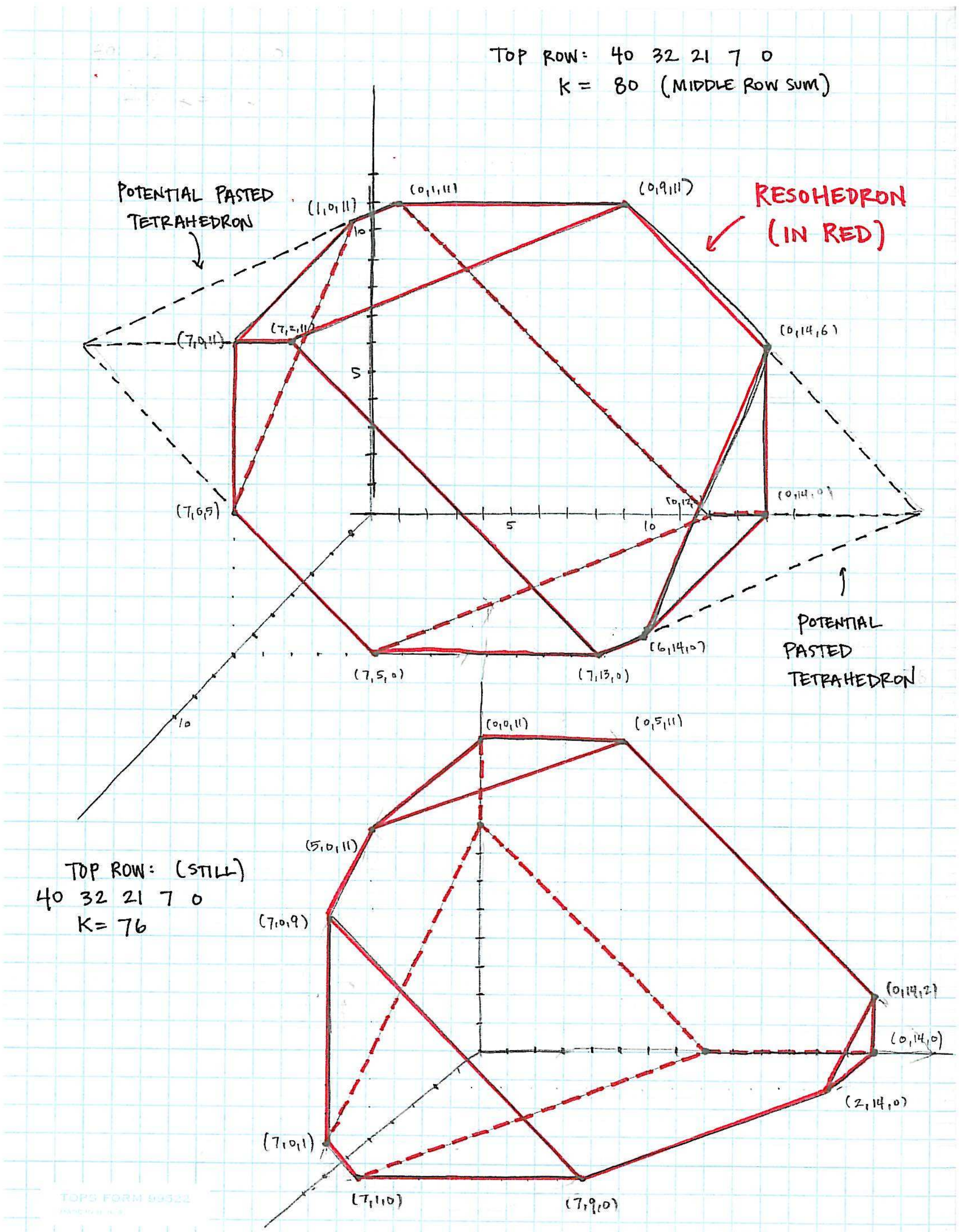
If $\mathfrak{T} \in \mathfrak{S}$ and $\mathfrak{T} \neq \mathfrak{T}_1$, $\mathfrak{T}_2$ then $\mathfrak{T}$ is superstrict and by the Theorem $\{\mathfrak{T}\}$ is a packet.

Theorem. $\{\mathfrak{T}_1, \mathfrak{T}_2\}$ *is a packet.*

The Resohedron

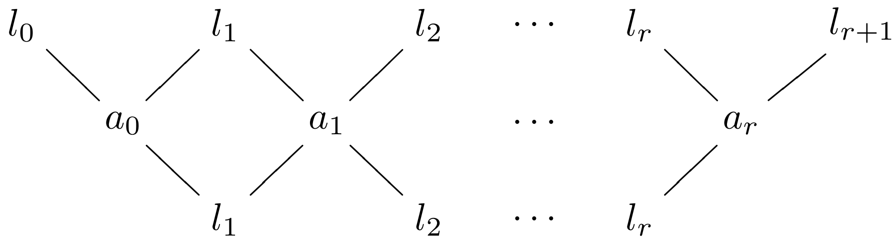
We have mentioned that we may reduce the proof of the conjecture to the case of a totally resonant packet, so from now on we will consider patterns that are totally resonant.

- The patterns in a resonant type form a polyhedron.
- Many phenomena are possible.

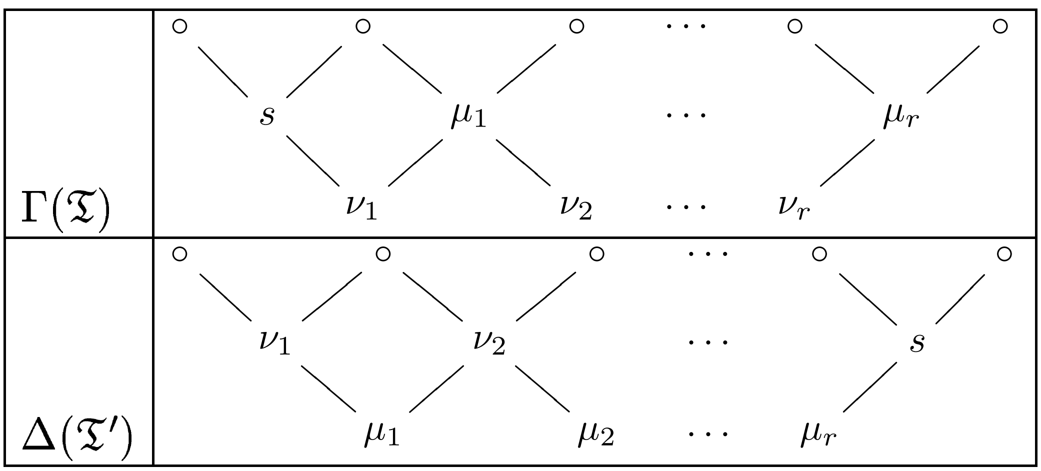


The Simplest Case

We consider Let us examine a type II resonant scheme:



where $a_0 + a_1 + \ldots + a_r = k$. We recall that Γ and Δ' have the following form, with $s = k - l_1 - \ldots - l_{r+1}$.



Assume that $0 < s < \min\,(l_i - l_{i+1})$. Then the resohedron is a **simplex**. Boxing cannot occur.

- If $\mu_i = \mu_{i+1}$ then μ_i and ν_{i+1} are circled in both arrays.
- If $s = \mu_1$ then s is circled in the Γ array, and $\nu_1 = 0$ is circled in both arrays.
- If $\mu_r = 0$ then s is circled in the Δ' array, and μ_r is circled in both arrays.

We have $G_\Gamma(\mathfrak{T}) = G_\Delta(\mathfrak{T}')$ unless s is circled in one array and not the other. These may be grouped into packets of order two.

- Interchange the rows μ_i and ν_i and reverse their orders.
- Combining this pattern with $\mathfrak{T}$ gives a packet.

The Resonant Bestiary

If $r = 2$:

$$\mathfrak{T} = \left\{ \begin{array}{ccccccc} l_0 & & l_1 & & l_2 & & l_3 \\ & x & & y & & z & \\ & & l_1 & & l_2 & & \end{array} \right\}, \qquad x + y + z = k.$$

- If k is small, we just saw the resogon is a triangle (simplex) and the packets have size 1 or 2.
- If k is large enough, the resogon again becomes a triangle but the packets have size 1 or 3, and tricky Gauss sum identities are involved.
- Depending on k and the l_i, the **resogon** may be a triangle, hexagon, pentagon or trapezoid. (Slice a rectangular box.)
- Many phenomena can occur. We look at just one example.
- Assume

  $$2l_1 > l_0 + l_2, \qquad \max(l_0 + l_1 + l_3, l_0 + 2l_2) < k < 2l_1 + l_2.$$

  This makes the resogon into a **trapezoid.**
- Let

  $$\alpha_1 = l_0 - l_1, \qquad \beta_1 = l_0 - l_2, \qquad \gamma_1 = l_1 - l_2, \qquad \delta_1 = l_2 - l_3.$$

- We will also denote

  $$g_\alpha = g(\alpha_1) g(\alpha_2), \qquad h_\alpha = h(\alpha_1) h(\alpha_2),$$

  and similarly define $g_\beta, g_\gamma, g_\delta$ and h_β, h_γ, h_δ.

$\mathfrak{T}$	$G_\Gamma(\mathfrak{T})$	$G_\Delta(\mathfrak{T}')$
$\mathfrak{a}=\left\{\begin{array}{ccccccc} l_0 & & l_1 & & l_2 & & l_3 \\ & l_0 & & l_1 & & k-l_0-l_1 & \\ & & l_1 & & l_2 & & \end{array}\right\}$	$g(u)g_\alpha h_\beta$	$h(u)g_\alpha g_\beta$
$\mathfrak{b}=\left\{\begin{array}{ccccccc} l_0 & & l_1 & & l_2 & & l_3 \\ & l_1 & & l_1 & & k-2l_1 & \\ & & l_1 & & l_2 & & \end{array}\right\}$	0	$h(u)^2 g_\gamma$
$\mathfrak{c}=\left\{\begin{array}{ccccccc} l_0 & & l_1 & & l_2 & & l_3 \\ & l_1 & & k-l_1-l_2 & & l_2 & \\ & & l_1 & & l_2 & & \end{array}\right\}$	$q^u h(u)g_\delta$	$g(u)h(u)h_\delta$
$\mathfrak{d}=\left\{\begin{array}{ccccccc} l_0 & & l_1 & & l_2 & & l_3 \\ & l_0 & & k-l_0-l_2 & & l_2 & \\ & & l_1 & & l_2 & & \end{array}\right\}$	$g(u)h_\alpha g_\delta$	$g(u)g_\alpha h_\delta$
$\mathfrak{e}=\left\{\begin{array}{ccccccc} l_0 & & l_1 & & l_2 & & l_3 \\ & l_0 & & 2l_1-l_0 & & k-2l_1 & \\ & & l_1 & & l_2 & & \end{array}\right\}$	$g(u)h_\alpha h_\gamma$	$h(u)g_\alpha h_\gamma$

- All patterns on the interior are superstrict, hence singleton packets.
- The above five patterns form the "big packet."
- The patterns on the interiors of the edges $[\mathfrak{b},\mathfrak{c}]$ and $[\mathfrak{e},\mathfrak{d}]$ are equal in number and can be paired up into packets of order 2;
- The patterns on the interiors of the edges $[\mathfrak{a},\mathfrak{b}]$, $[\mathfrak{a},\mathfrak{e}]$ and $[\mathfrak{c},\mathfrak{d}]$ are equal in number and can be put into packets of order 3.