Averages and Ratios of Characteristic Polynomials

References

1. Keating and Snaith, **RMT and $\zeta\left(\frac{1}{2}+it\right)$**, Comm. Math. Phys. 214 (2000), no. 1, 57–89. Value distribution of ζ mirrors value distribution of characteristic polynomials. Moments computed using Selberg integral.

2. Conrey and Snaith, **Applications of the L-functions ratios conjectures**. Instead of moments, consider ratios. This will have many applications.
   http://arxiv.org/abs/math.NT/0509480

3. Conrey, Farmer and Zirnbauer, **Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups U(N)**, Computes mean values of ratios for classical ensembles using supersymmetry.
   http://arxiv.org/abs/math-ph/0511024

4. Conrey, Forrester and Snaith, **Averages of ratios of characteristic polynomials for the compact classical groups**, Int. Math. Res. Not. 2005, no. 7, 397–431.

5. Bump and Gamburd, **Averages and ratios of characteristic polynomials**, Comm. Math. Phys. (to appear) is the **basis of this talk**.

Papers [4] and [5] reprove results of [3] by different methods. **Only [3] has results for N small.**

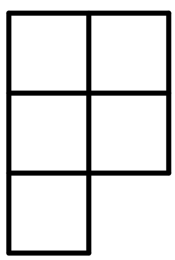
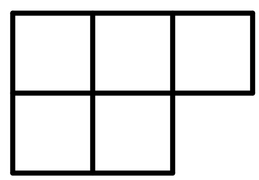
Representation Theory of S_k

Let $\lambda = (\lambda_1, \cdots, \lambda_N)$ be a partition of k. Thus $\lambda_i \in \mathbb{Z}$, $\lambda_1 \geqslant \cdots \geqslant \lambda_N \geqslant 0$ and $\sum \lambda_i = k$. Let

$$S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_N} \subseteq S_k.$$

If λ is a partition represent it by a **Young diagram**. Transposing the diagram of λ gives the diagram of the **conjugate partition** λ'.

Example. $N = 4$, $\lambda = (2, 2, 1, 0) = (2, 2, 1) = (2^2 1)$.

$\lambda =$  $\lambda' =$ 

Thus λ is a partition of 5 of length 3 or into 3 parts.

Theorem. $\mathrm{Ind}_{S_\lambda}^{S_k}(1)$ *and* $\mathrm{Ind}_{S_{\lambda'}}^{S_k}(\mathrm{sgn})$ *have a unique irreducible constituent in common.*

Proof. Mackey theory + combinatorics. $\square$

- Call this $\boldsymbol{s}_\lambda$.
- Thus **the irreducible representations of S_k** are parametrized by partitions of k.

The ring Λ

We define a graded $\mathbb{Z}$-algebra Λ.

- The homogeneous part Λ_k of degree k is the additive group of generalized characters of S_k.
- The multiplication is **induction**. Thus

  $$S_k \times S_l \hookrightarrow S_{k+l}$$

  so if χ, ψ are characters of S_k and S_l then $\chi \otimes \psi$ is a character of $S_k \times S_l$ which we induce to S_{k+l}.
- A $\mathbb{Z}$-basis of Λ_k is $\{\boldsymbol{s}_\lambda | \lambda \text{ is a partition of } k\}$.
- Let $\boldsymbol{h}_k$ and $\boldsymbol{e}_k$ be the trivial and sign characters of S_k. Then $\Lambda = \mathbb{Z}[\boldsymbol{h}_1, \boldsymbol{h}_2, \cdots] = \mathbb{Z}[\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots]$.
- We have

  $$\begin{aligned} \boldsymbol{h}_k &= \boldsymbol{s}_{(k)} & \quad (k) \text{ is the partition } (k, 0, \cdots, 0) \\ \boldsymbol{e}_k &= \boldsymbol{s}_{(1^k)} & \quad (1^k) \text{ is the partition } (1, 1, \cdots, 1) \end{aligned}$$

- Λ has an automorphism ι of degree 2 such that $\iota(\boldsymbol{h}_i) = \boldsymbol{e}_i$ and $\iota(\boldsymbol{e}_i) = \boldsymbol{h}_i$. This is the **involution**.
- On Λ_k the involution ι amounts to tensoring with the sign character.
- The involution corresponds to conjugation of partitions: $\iota(\boldsymbol{s}_\lambda) = \iota(\boldsymbol{s}_{\lambda'})$

The characteristic map

Let

$$s_\lambda(X_1,\cdots,X_N)=\frac{\mathrm{Alt}(X_1^{\lambda_1+N-1}X_2^{\lambda_2+N-2}\cdots X_N^{\lambda_N})}{\mathrm{Alt}(X_1^{N-1}X_2^{N-2}\cdots X_N^0)}$$

be the **Schur polynomial**, where

$$\mathrm{Alt}=\sum_{\sigma\in S_N}(-1)^{\mathrm{sgn}(\sigma)}\sigma.$$

There is a **ring homomorphism** (due to Frobenius)

$$\mathrm{ch}\colon\Lambda\longrightarrow\mathbb{Z}[X_1,\cdots,X_N]^{S_N},$$

$$\begin{aligned}\mathrm{ch}(\boldsymbol{h}_i) &= h_i(X_1,\cdots,X_N)=\sum_{i_1\leqslant\cdots\leqslant i_N}X_{i_1}\cdots X_{i_N},\\ \mathrm{ch}(\boldsymbol{e}_i) &= e_i(X_1,\cdots,X_N)=\sum_{i_1<\cdots<i_N}X_{i_1}\cdots X_{i_N},\\ \mathrm{ch}(\boldsymbol{s}_\lambda) &= s_\lambda(X_1,\cdots,X_N).\end{aligned}$$

(Complete, elementary, Schur symmetric polynomials).

- The homomorphism is surjective but not injective. Thus ι does not induce an involution of symmetric polynomials.
- However it is bijective on Λ_k provided $N\geqslant k$.

Frobenius-Schur duality

Let $V=\mathbb{C}^N$ be the standard module of $U(N)$. We have commuting actions of $U(N)$ and S_k on $\otimes^k V = V \otimes \cdots \otimes V$ (k times):

$$\sigma\colon v_1 \otimes \cdots \otimes v_k \longmapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \qquad \sigma \in S_k.$$

Thus we can decompose $\otimes^k V$ simultaneously:

$$\otimes^k V = \sum_\lambda \pi_\lambda \otimes \boldsymbol{s}_\lambda \text{ as } U(N) \times S_k\text{-modules.}$$

- The irreducible representations of S_k that appear are the $\boldsymbol{s}_\lambda$ where $\lambda = (\lambda_1, \cdots, \lambda_N)$ is a partition of k of length $\leqslant N$.
- The irreducible representations of $U(N)$ that appear are those whose matrix coefficients are homogeneous polynomials of degree k.

If $g \in U(N)$ has eigenvalues $\alpha_1, \cdots, \alpha_N$ and $\boldsymbol{f} \in \Lambda$ maps to $f = \mathrm{ch}(\boldsymbol{f}) \in \mathbb{Z}[X_1, \cdots, X_N]^{S_N}$ then

$$\chi_\lambda(g) = \boldsymbol{f}(\alpha_1, \cdots, \alpha_N), \qquad \chi_\lambda = \text{ character of } \pi_\lambda.$$

Multiplication in Λ corresponds to tensor production of representations of $U(N)$. Thus

$$\boldsymbol{s}_\lambda \boldsymbol{s}_\mu = \sum_\nu c_{\lambda\mu}^\nu \boldsymbol{s}_\nu, \qquad \pi_\lambda \otimes \pi_\mu = \bigoplus_\nu c_{\lambda\mu}^\nu \pi_\nu.$$

The $c_{\lambda\mu}^\nu$ are **Littlewood-Richardson coefficients**.

Weights

We consider integer sequences $(\lambda_1, \cdots, \lambda_N)$.

- $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{Z}^N$ is called a **weight** of N.
- Weights correspond to rational characters of the diagonal torus of $U(N)$:

$$\lambda \colon \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_N \end{pmatrix} \longmapsto t_1^{\lambda_1} \cdots t_N^{\lambda_N}.$$

- If $\lambda_1 \geqslant \cdots \geqslant \lambda_N$ the weight is called **dominant**.
- The **height** of λ is the inner product $\langle \lambda, \rho \rangle$,

$$\rho = (N-1, N-2, \cdots, 0).$$

- If λ is a dominant weight λ, there is a unique irreducible π_λ of $U(N)$ containing λ, no higher weight. **Highest weight module** (Weyl).
- If $\lambda_N \geqslant 0$ the dominant weight $\lambda = (\lambda_1, \cdots, \lambda_N)$ is a **partition** of $k = \sum \lambda_i$ into $\leqslant N$ parts.
- So **dominant weights** parametrize irreducible representations of $U(N)$ ...
- ... and **partitions** of k parametrize irreducible representations of S_k.
- Weights and partitions overlap, hence give a bijection between **some** representations of $U(N)$ and **some** representations of S_k. **This is the bijection we constructed using $\otimes^k \mathbb{C}^N$.**

Generating functions

Remember that

$$h_i(X_1, \cdots, X_N) = \sum_{i_1 \leqslant \cdots \leqslant i_N} X_{i_1} \cdots X_{i_N},$$
$$e_i(X_1, \cdots, X_N) = \sum_{i_1 < \cdots < i_N} X_{i_1} \cdots X_{i_N},$$

are the complete and elementary symmetric poly's.

$$\sum_{t=0}^{\infty} t^k h_k(\alpha_1, \cdots, \alpha_N) = \prod_{j=1}^{N} (1 - t\alpha_j)^{-1}.$$

$$\sum_{t=0}^{\infty} t^k e_k(\alpha_1, \cdots, \alpha_N) = \prod_{j=1}^{N} (1 + t\alpha_j).$$

These may be regarded as generating functions for the symmetric and exterior algebras on $\mathbb{C}^N$.

- The involution ι of Λ interchanges

$$\sum_{t=0}^{\infty} t^k \boldsymbol{h}_k \longleftrightarrow \sum_{t=0}^{\infty} t^k \boldsymbol{e}_k$$

- There is no corresponding correspondence of symmetric functions (since $e_k = 0$ for $k > N$) but **roughly we can think of the involution as transforming the two types of generating functions into one another.**

Correspondences

If G and H are groups, a **correspondence** of representations is a bijection between some of the irreducible representations π_λ of G and some of the irreducible representations ρ_λ of H. We consider the module

$$\omega = \bigoplus \pi_\lambda \otimes \rho_\lambda \text{ of } G \times H.$$

- By assumption there are no repetitions among the π_λ or ρ_λ.
- In important cases, ω has a natural construction.
- If $G = U(N)$ and $H = S_k$ this module is $\otimes^k V$.
- Howe discovered that the Weil representation gives correspondences for reductive dual pairs of subgroups of $\mathrm{Sp}(2n)$ or its double cover (the metaplectic group.

$\mathrm{GL}(N, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})$ duality

Let $\mathrm{GL}(N, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})$ act on $\mathrm{Mat}_{n\times m}(\mathbb{C})$ by left and right multiplication. There is induced an action on the polynomial ring $S(\mathrm{Mat}_{n\times m}(\mathbb{C}))$. This is a correspondence. It induces a correspondence of the maximal compact subgroups $U(N) \times U(M)$ since irreducible rep's of $U(N)$ correspond bijectively to analytic reps of $\mathrm{GL}(N, \mathbb{C})$. This is a Howe correspondence.

Inner product formulas

Theorem. (Peter-Weyl + Schur orthogonality) *Assume G compact. The characters of the irreducible representations span the subspace of $L^2(G)$ consisting of class functions.* ***They are an orthonormal basis.***

Thus given a correspondence:

$$\begin{array}{ccc} G & & H \\ \pi_\lambda & \longleftrightarrow & \rho_\lambda \\ \chi_{\pi_\lambda} \in L^2(G) & \longleftrightarrow & \chi_{\rho_\lambda} \in L^2(H) \end{array} \qquad (\lambda \in \text{indexing set})$$

Let $L^2_\omega(G)$ and $L^2_\omega(H)$ be the span of the characters χ_{π_λ} and χ_{ρ_λ}. The correspondence determines an **isometry**

$$L^2_\omega(G) \longrightarrow L^2_\omega(H).$$

They may allow us to **transfer an inner product computation** from G to H. Thus if $\phi \in L^2_\omega(G)$ is a class function, and $\psi \in L^2_\omega(H)$ is the corresponding class function on H we have

$$\int_G |\phi(g)|^2 dg = \int_H |\psi(h)|^2 dh.$$

We will give some examples where the right-hand side is easier to evaluate than the left-hand side, leading to nontrivial results in RMT.

Diaconis and Shashahani

Let $G = U(N)$, $H = S_k$, $\omega =$ Frobenius-Schur duality

Theorem 1. *If* $N \geqslant k_1 + 2k_2 + \cdots + r\,k_r$ *then*

$$\int_{U(N)} |\mathrm{tr}(g)|^{2k_1} |\mathrm{tr}(g^2)|^{2k_2} \cdots |\mathrm{tr}(g^r)|^{2k_r} d\,g = \prod_{j=1}^{r} j^{k_j} k_j! \quad .$$

Proof. Let $k = k_1 + 2k_2 + \cdots + r\,k_r$, and let λ be the partition of k containing k_1 entries equal to 1, k_2 entries equal to 2, and so forth. Let $\mathcal{C}_\lambda$ be the conjugacy class of permutations $\sigma \in S_k$ of type λ (so σ has k_j cycles of length j in its decomposition to disjoint cycles). Let $\boldsymbol{p}_\lambda$ be the *conjugacy class indicator* on S_k,

$$\boldsymbol{p}_\lambda(g) = \begin{cases} z_\lambda & \\ 0 & \text{otherwise}, \end{cases}$$

where $z_\lambda = \prod_{j=1}^{r} j^{k_j} k_j!$. The class function

$$g \mapsto \mathrm{tr}(g)^{k_1}\,\mathrm{tr}(g^2)^{k_2} \cdots \mathrm{tr}(g^r)^{k_r}$$

in $L^2_\omega(G)$ corresponds to the function $\boldsymbol{p}_\lambda \in L^2_\omega(H)$ and so **its norm is the same as the norm of $\boldsymbol{p}_\lambda$**, i.e. z_λ. $\square$

$$\text{Thus} \quad \int_{U(N)} |\mathrm{tr}(g)|^{2k_1} |\mathrm{tr}(g^2)|^{2k_2} \cdots |\mathrm{tr}(g^r)|^{2k_r} d\,g$$

stabilizes when N is large. Asymptotically the value distribution of $\mathrm{tr}(g), \mathrm{tr}(g^2), \cdots, \mathrm{tr}(g^r)$ is of **independent normal (i.e. Gaussian) random variables**.

Cauchy Identity

$$\sum_\lambda s_\lambda(\alpha_1, \cdots, \alpha_n) s_\lambda(\beta_1, \cdots, \beta_m) = \prod_{i,j} (1 - \alpha_i \beta_j)^{-1}$$

- Sum is over partitions of length $\leqslant N$.
- Fundamental in what we do next.
- Underlying correspondence: $G, H = U(n), U(m)$

Take $n = m$. Consider the action of $U(n) \times U(n)$ on $L^2(G)$:

$$(g, h) f(x) = f(g^{-1} x h)$$

Lemma. *If G is any compact group, $G \times G$ acts on $L^2(G)$ and*

$$L^2(G) = \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes \hat{\pi} \qquad \hat{\pi} = \textit{contragredient rep'n.}$$

(This is again Peter-Weyl theorem). For $G = U(n)$, there is an involution $\iota: G \longrightarrow G$ namely ${}^\iota g = {}^t g^{-1}$ such that $\boldsymbol{g \longmapsto \pi({}^\iota g)}$ **is equivalent to** $\boldsymbol{\hat{\pi}}$. This is because g^{-1} is conjugate to ${}^\iota g$. So we modify the action:

$$(g, h) f(x) = f({}^t g\, x\, h)$$

and in this action:

$$L^2(G) = \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes \pi$$

Polynomial version

In this decomposition

$$L^2(G) = \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes \pi \qquad \text{(Hilbert space } \oplus .)$$

we may restrict ourselves to the subspace of $U(n)$-finite vectors, which form the affine ring

$$A = \mathbb{C}[X_{ij}, \det^{-1}], \qquad X_{ij} = \text{coordinate functions.}$$

(G is compact so continuous functions are L^2.) Then

$$A = \bigoplus_{\pi \in \mathrm{Irr}(G)} \pi \otimes \pi \qquad \text{(Algebraic } \oplus .)$$

- π runs through π_λ where $\lambda = (\lambda_1, \cdots, \lambda_n)$ is a weight, i.e. $\lambda_1 \geqslant \cdots \geqslant \lambda_n$, $\lambda_i \in \mathbb{Z}$
- The weights λ with $\lambda_N \geqslant 0$ are **partitions.**
- Affine ring $\mathbb{C}[X_{ij}]$ of Mat_n is

$$\mathbb{C}[X_{ij}] = \bigoplus_{\text{partitions } \lambda} \pi_\lambda \otimes \pi_\lambda \qquad \text{(Algebraic } \oplus .)$$

- This is because only the matrix coefficients of π_λ where λ is a partition are regular on the determinant locus.

Proof of the Cauchy Identity

We have

$$\mathbb{C}[X_{ij}] = \bigoplus_{\text{partitions } \lambda} \pi_\lambda \otimes \pi_\lambda \qquad (\text{Algebraic } \oplus) \qquad (1)$$

as $U(n) \times U(n)$ modules, where (g, h) act by

$$(g, h) f(x) = f({}^t g x h), \qquad f: \mathrm{Mat}_n(\mathbb{C}) \longrightarrow \mathbb{C}.$$

Let g, h have eigenvalues $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$. Taking the trace in this identity gives

$$\sum_\lambda s_\lambda(\alpha_1, \cdots, \alpha_n) s_\lambda(\beta_1, \cdots, \beta_n) = \prod_{i,j} (1 - \alpha_i \beta_j)^{-1}.$$

- The series is only convergent if $|\alpha_i|, |\beta_j| < 1$. But (1) extends to $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ of which $U(n) \times U(n)$ is a maximal compact, and π_λ extend to analytic reps of $\mathrm{GL}(n, \mathbb{C})$.
- The case $n > m$ can be deduced by specializing

$$b_{m+1}, \cdots, \beta_n \longrightarrow 0.$$

- At the heart of the proof is a correspondence with $G, H = U(n)$ and

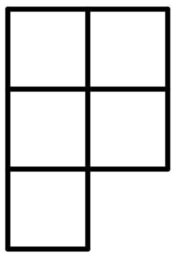
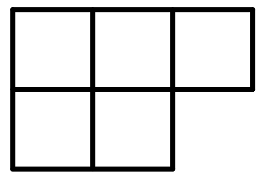
$$\omega = \bigoplus_{\text{partitions } \lambda} \pi_\lambda \otimes \pi_\lambda = \text{polynomials on } \mathrm{Mat}_n(\mathbb{C}).$$

- This a case of the Howe correspondence.

Dual Cauchy Identity

If λ is a partition represent it by a **Young diagram**. Transposing the diagram of λ gives the diagram of the **conjugate partition** λ'.

Example. $N=4$, $\lambda=(2,2,1,0)=(2,2,1)=(2^2 1)$.

$\lambda=$  $\lambda'=$ 

Thus λ is a partition of 5 of length 3 or into 3 parts.
λ' is a partition of 5 of length 2 or into 2 parts.
The ring Λ has a basis $\boldsymbol{s}_\lambda$ which specialize to characters of irreducible rep's of S_k ($k = \sum \lambda_i$) or $U(N)$ $N \geqslant$ length(λ). The map $\boldsymbol{s}_\lambda \longmapsto \boldsymbol{s}_{\lambda'}$ is an **automorphism** of Λ which we will call the **involution**.

Applying the involution to one set of variables in the Cauchy identity

$$\sum_\lambda s_\lambda(\alpha_1, \cdots, \alpha_n) s_\lambda(\beta_1, \cdots, \beta_n) = \prod_{i,j} (1-\alpha_i\beta_j)^{-1}.$$

produces the **dual Cauchy identity**

$$\sum_\lambda s_\lambda(\alpha_1, \cdots, \alpha_n) s_{\lambda'}(\beta_1, \cdots, \beta_n) = \prod_{i,j} (1+\alpha_i\beta_j).$$

Keating and Snaith

The following theorem was very influential in the application of RMT to ζ.

Theorem. *We have*

$$\int_{U(N)} |\det(g-I)|^{2k} dg = \prod_{j=0}^{N-1} \frac{j!(j+2k)!}{(j+k)!^2}.$$

The same constants appear in the (conjectural) $2k$-th moment of ζ.

- The original proof of Keating and Snaith used the Selberg integral.
- We will give another proof (due to Gamburd) that uses $\mathrm{GL}(N) \times \mathrm{GL}(2k)$ duality.
- The two proofs have different generalizations. The proof of Keating and Snaith allows interpolation of k to real numbers, while Gamburd's proof allows more general evaluations such as

  $$\int_{U(N)} |\det(g-I)|^{2k} \chi_\lambda(g)\, dg$$

  where χ_λ is the character of π_λ.

Proof of Keating-Snaith formula

If $\alpha_1, \cdots, \alpha_N$ and $\beta_1, \cdots, \beta_N$ are complex numbers, we will show

$$\int_{U(N)} \prod_{i=1}^{k} \left\{ \det(I + \alpha_i g) \det(I + \beta_i^{-1} g^{-1}) \right\} dg = s_{(N^k)}(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k). \qquad (2)$$

The left-hand side equals

$$\prod \beta_i^{-1} \int_{U(N)} \prod_{i=1}^{k} \{\det(I + \alpha_i g) \det(g\beta_i + I)\} \overline{\det(g)^k} dg.$$

By dual Cauchy id, if $t_1, \cdots, t_N$ are eigenvalues of g,

$$\prod_{i=1}^{k} \{\det(\cdots)\det(\cdots)\} = \sum_{\lambda} s_\lambda(\alpha_1, \cdots, \alpha_k, \beta_1, \cdots, \beta_k) s_{\lambda'}(t_1, \cdots, t_N),$$

Since $\det(g)^k = s_{\lambda'}(t_1, \cdots, t_N)$, integrating over g picks off just one term, with $\lambda' = (k^N)$ and so $\lambda = (N^k)$ so. This proves (2). Taking $\alpha_i = \beta_i = 1$

$$\int_{U(N)} |\det(g - I)|^{2k} dg = s_{(N^k)}(\overbrace{1, \cdots, 1}^{2k \text{ terms}}),$$

the dimension of the rep'n $\pi_{(N^k)}$ of $U(2k)$. This dimension is computed using Weyl's dimension formula, proving the theorem of Keating and Snaith.

Analysis of the proof

- Underlying this computation is the (dual) Cauchy identity.
- The Cauchy identity for $\mathrm{GL}(N) \times \mathrm{GL}(2k)$ amounts to the use of the Howe correspondence for $\mathrm{GL}(N) \times \mathrm{GL}(2k)$.
- In this correspondence, if λ is a dominant weight, $\pi_\lambda^{\mathrm{GL}(N)}$ corresponds to $\pi_\lambda^{\mathrm{GL}(2k)}$.
- But in the Cauchy identity $\pi_\lambda^{\mathrm{GL}(N)}$ corresponds to $\pi_{\lambda'}^{\mathrm{GL}(2k)}$.
- When $\lambda = (N^k)$, $\lambda' = (k^N)$ and the answer turns out to be the dimension of this $\pi_{\lambda'}^{\mathrm{GL}(2k)}$.
- **We used the correspondence to transfer the computation from $\mathrm{GL}(N)$ to $\mathrm{GL}(2k)$.**
- Similarly, moments for classical groups can be expressed in terms of characters of **other groups** parametrized by **rectangular partitions**. Thus a result of Keating and Snaith can be written:

$$\int_{\mathrm{Sp}(2N)} \prod_{j=1}^{k} \det(I - x_j g)\, dg \;=$$

$$(x_1 \ldots x_k)^N \chi_{\langle N^k \rangle}^{\mathrm{Sp}(2k)}(x_1^{\pm 1}, \cdots, x_k^{\pm 1}) \;=$$

$$\sum_{\varepsilon \in \{\pm 1\}} \prod_{j=1}^{k} x_j^{N(1-\varepsilon_j)} \prod_{i \leqslant j} (1 - x_i^{\varepsilon_i} x_j^{\varepsilon_j})^{-1}.$$

Ratios

Let $\Xi_{L,K}$ consist of permutations $\sigma \in S_{K+L}$ such that

$$\sigma(1) < \cdots < \sigma(L), \qquad \sigma(L+1) < \cdots < \sigma(L+K).$$

Theorem. (Conrey, Farmer and Zirnbauer) *If $N \geqslant Q, R$ and $|\gamma_q|, |\delta_r| < 1$ we have*

$$\int_{U(N)} \frac{\prod_{l=1}^{L} \det(I + \alpha_l^{-1} \cdot g^{-1}) \cdot \prod_{k=1}^{K} \det(I + \alpha_{L+k} \cdot g)}{\prod_{q=1}^{Q} \det(I - \gamma_q \cdot g) \prod_{r=1}^{R} \det(I - \delta_r \cdot g^{-1})} dg =$$

$$\sum_{\sigma \in \Xi_{L,K}} \prod_{k=1}^{K} (\alpha_{\sigma(L+k)}^{-1} \alpha_{L+k})^N \times$$

$$\frac{\prod_{q=1}^{Q} \prod_{l=1}^{L} (1 + \gamma_q \alpha_{\sigma(l)}^{-1}) \prod_{r=1}^{R} \prod_{k=1}^{K} (1 + \delta_r \alpha_{\sigma(L+k)})}{\prod_{k=1}^{K} \prod_{l=1}^{L} (1 - \alpha_{\sigma(l)}^{-1} \alpha_{\sigma(K+k)}) \prod_{r=1}^{R} \prod_{q=1}^{Q} (1 - \gamma_q \delta_r)}.$$

- According to CFZ the assumption that N is large can be eliminated.
- Proof will depend on the **generalized Cauchy identity** involving **Littlewood-Schur functions**.
- These were studied by Littlewood and also by Berele and Regev, who apparently rediscovered them.

Generalized Cauchy identity

Define the **Littlewood-Schur polynomial**

$$\mathrm{LS}_\lambda(x_1, \cdots, x_k;\, y_1, \cdots, y_l) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x_1, \cdots, x_k) s_{\nu'}(y_1, \cdots, y_l).$$

The $c^\lambda_{\mu\nu}$ are the Littlewood-Richardson coefficients.

Theorem. (Berele and Regev)

$$\sum_\lambda \mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_m;\, \beta_1, \cdots, \beta_n)\mathrm{LS}_\lambda(\gamma_1, \cdots, \gamma_s;\, \delta_1, \cdots, \delta_t) = \prod_{i,k}(1-\alpha_i\gamma_k)^{-1}\prod_{i,l}(1+\alpha_i\delta_l) \prod_{j,k}(1+\beta_j\gamma_k)\prod_{j,l}(1-\beta_j\delta_l)^{-1}.$$

We will assume it now and discuss the proof later.

Laplace expansion

Let (a_{ij}) be $(L+K)\times(L+K)$. Then $\det(a_{ij}) =$

$$\sum_{\sigma\in\Xi_{L,K}} \mathrm{sgn}(\sigma) \begin{vmatrix} a_{1,\sigma(1)} & \cdots & a_{1,\sigma(L)} \\ \vdots & & \vdots \\ a_{L,\sigma(1)} & \cdots & a_{L,\sigma(L)} \end{vmatrix} \times \begin{vmatrix} a_{L+1,\sigma(L+1)} & \cdots & a_{L+K,\sigma(L+K)} \\ \vdots & & \vdots \\ a_{L+K,\sigma(L+1)} & \cdots & a_{L+K,\sigma(L+K)} \end{vmatrix}.$$

Proof easy.

Laplace expansion for LS_λ

Proposition. *Suppose* λ *of length* $\leqslant K$ *such that* $\lambda_L \geqslant \lambda_{L+1} + Q$*, let* $\lambda = \tau \cup \rho$ *with*

$$\tau = (\lambda_1, \ldots, \lambda_L), \qquad \rho = (\lambda_{L+1}, \cdots, \lambda_{L+K}).$$

Then

$$\mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) = \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1}$$
$$\mathrm{LS}_{\tau + \langle K^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_Q)$$
$$\mathrm{LS}_\rho(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_Q)$$

Proof. Induction on Q. If $Q = 0$, this says

$$s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) = \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1}$$
$$s_{\tau + \langle K^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}) \, s_\rho(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}).$$

This is proved by applying the Laplace expansion to the determinant definition of the Schur function. For $Q > 0$ one adds the γ_i one at a time using Pieri's formula (i.e. the degenerate Littlewood-Richardson rule). $\square$

Proof (sketch) of unitary CFZ

By the dual Cauchy identity,

$$\prod_{l=1}^{L} \det(I+\alpha_l^{-1}\cdot g^{-1})\cdot\prod_{k=1}^{K}\det(I+\alpha_{L+k}\cdot g)$$
$$= \overline{\det(g)^L}\prod_{l=1}^{L}\alpha_l^{-N}\prod_{k=1}^{K+L}\det(I+\alpha_k g)$$
$$= \overline{\det(g)^L}\prod_{l=1}^{L}\alpha_l^{-N}\sum_{\lambda} s_\lambda(\alpha_1,\cdots,\alpha_{K+L})\chi_{\lambda'}(g)$$

On the other hand by the Cauchy identity

$$\prod_{q=1}^{Q}\det(I-\gamma_q g)^{-1}=\sum_{\mu} s_\mu(\gamma_1,\cdots,\gamma_Q)\,\chi_\mu(g)$$

and

$$\prod_{r=1}^{R}\det(I-\delta_r\cdot g^{-1})^{-1}=\sum_{\nu} s_\nu(\delta_1,\cdots,\delta_R)\,\overline{\chi_\nu(g)}.$$

By Schur orthogonality

$$\int_{U(N)}\frac{\prod_{l=1}^{L}\det(I+\alpha_l^{-1}\cdot g^{-1})\cdot\prod_{k=1}^{K}\det(I+\alpha_{L+k}\cdot g)}{\prod_{q=1}^{Q}\det(I-\gamma_q\cdot g)\prod_{r=1}^{R}\det(I-\delta_r\cdot g^{-1})}dg \;=\; \sum_{\lambda,\mu,\nu}\left\langle \chi_{\lambda'}\chi_\mu, \det{}^L\otimes\chi_\nu\right\rangle$$
$$\prod_{l=1}^{L}\alpha_l^{-N}s_\lambda(\alpha_1,\cdots,\alpha_{L+K})s_\mu(\gamma_1,\cdots,\gamma_Q)s_\nu(\delta_1,\cdots,\delta_R).$$

We rewrite this as

$$\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\lambda,\mu,\nu} c^{\tilde{\nu}}_{\lambda'\mu} s_\lambda(\alpha_1, \cdots, \alpha_{L+K}) s_\mu(\gamma_1, \cdots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R) =$$

$$\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} \mathrm{LS}_{\tilde{\nu}}(\gamma_1, \ldots, \gamma_Q; \alpha_1, \ldots, \alpha_{L+K}) s_\nu(\delta_1, \cdots, \delta_R) =$$

$$\prod_{l=1}^{L} \alpha_l^{-N} \sum_{\nu} \mathrm{LS}_{\hat{\nu}}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) s_\nu(\delta_1, \cdots, \delta_R),$$

where $\tilde{\nu} = \nu + \left\langle L^N \right\rangle$ and $\hat{\nu} = \tilde{\nu}' = N^L \cup \nu'$. Using the Laplace expansion for $\mathrm{LS}_{\hat{\nu}}$:

$$\begin{aligned} \mathrm{LS}_{\hat{\nu}}(\alpha_1, \ldots, \alpha_{L+K}; \gamma_1, \ldots, \gamma_Q) \;=\; & \sum_{\sigma \in \Xi_{L,K}} \prod_{\substack{1 \leqslant l \leqslant L \\ 1 \leqslant k \leqslant K}} (\alpha_{\sigma(l)} - \alpha_{\sigma(L+k)})^{-1} \\ & \times \mathrm{LS}_{\langle (N+K)^L \rangle}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(L)}; \gamma_1, \ldots, \gamma_Q) \\ & \times \mathrm{LS}_{\nu'}(\alpha_{\sigma(L+1)}, \cdots, \alpha_{\sigma(L+K)}; \gamma_1, \ldots, \gamma_Q). \end{aligned}$$

Substituting this, using generalized Cauchy identity to evaluate the sum over ν, and Littlewood's formula

$$\mathrm{LS}_{\langle (l+m)^k \rangle}(x_1, \cdots, x_k; y_1, \cdots, y_l) = \left(\prod_{i=1}^{k} x_i \right)^m \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant l}} (x_i + y_j)$$

gives

$$\sum_{\sigma \in \Xi_{L,K}} \prod_{k=1}^{K} (\alpha^{-1}_{\sigma(L+k)} \alpha_{L+k})^N \times$$

$$\frac{\prod_{q=1}^{Q} \prod_{l=1}^{L} (1 + \gamma_q \alpha^{-1}_{\sigma(l)}) \prod_{r=1}^{R} \prod_{k=1}^{K} (1 + \delta_r \alpha_{\sigma(L+k)})}{\prod_{k=1}^{K} \prod_{l=1}^{L} (1 - \alpha^{-1}_{\sigma(l)} \alpha_{\sigma(K+k)}) \prod_{r=1}^{R} \prod_{q=1}^{Q} (1 - \gamma_q \delta_r)}.$$

Remarks on the proof

- There may be more than one way to proceed once we have an adequate set of tools.
- The tools, mainly the generalized Cauchy identity, Laplace expansion and Littlewood's identity are themselves of considerable interest.

We will concentrate on ideas around the generalized Cauchy identity and Laplace expansion.

$U(p+q) \longrightarrow U(p) \times U(q)$ branching

As before, χ_λ = character of π_λ, λ a dominant weight. We assume λ is a partition, so π_λ is a polynomial rep'n.

Theorem. *(i) We have*

$$\chi_\nu^{(p+q)}|_{U(p)\times U(q)}\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} = \sum_{\lambda,\mu} c^\nu_{\lambda\mu}\chi_\lambda(g_1)\chi_\mu(g_2),$$

$$\pi_\nu^{(p+q)}\begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} = \bigoplus_{\lambda,\mu} c^\nu_{\lambda\mu}\pi_\lambda(g_1)\otimes\pi_\mu(g_2).$$

(ii) Let $x_1, \cdots, x_p$ and $y_1, \cdots, y_q$ be two sets of variables.

$$s_\lambda(x_1,\cdots,x_p,y_1,\cdots,y_q) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x_1,\cdots,x_p)s_\nu(y_1,\cdots,y_q).$$

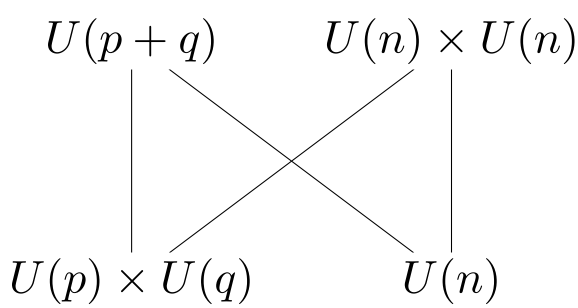
The two statements are equivalent. (Take x_i, y_i to be eigenvalues of g_1 and g_2.)

Proof of unitary branching

In the context of the unitary groups the Littlewood-Richardson rules occur in 2 distinct ways:

- **Clebsch-Gordan coef:** $c^{\lambda}_{\mu\nu}$ is the multiplicity of π_λ in $\pi_\mu \otimes \pi_\nu$ reps of $U(N)$ or $\mathrm{GL}(N, \mathbb{C})$.
- **Unitary branching rule:** $c^{\lambda}_{\mu\nu}$ is the multiplicity of $\pi_\mu^{\mathrm{GL}(p)} \otimes \pi_\nu^{\mathrm{GL}(q)}$ in the restriction of $\pi_\lambda^{\mathrm{GL}(p+q)}$.

The See-Saw:



- Vertical lines are inclusions
- Diagonal lines are correspondences
- Let $\omega =$ action of $U(p+q)$, $U(n)$ on symmetric algebra of $\mathrm{Mat}_{(p+q)n}(\mathbb{C})$ (left, right translation)
- $\omega = \bigoplus_\lambda \pi_\lambda^{U(p+q)} \otimes \pi_\lambda^{U(N)}$.
- Alternatively we have action ω of $U(p) \times U(q)$ and $U(n) \times U(n)$ on same symmetric algebra.
- $\omega = \bigoplus_{\mu,\nu} (\pi_\mu^{U(p)} \otimes \pi_\nu^{U(q)}) \otimes (\pi_\mu^{U(n)} \otimes \pi_\nu^{U(n)})$

Unitary branching, continued

The representation ω is the action of $U((p+q)n)$ on the symmetric algebra on $\mathrm{Mat}_{(p+q)\times n}(\mathbb{C})$. Both dual pairs can be embedded

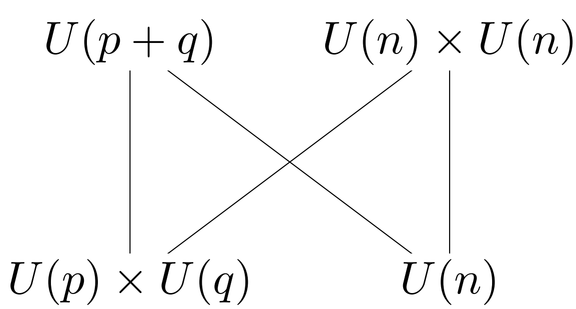
$$\begin{array}{ccc} U(p+q)\times U(n) & & \\ & \searrow & \\ & & U((p+q)n) \\ & \nearrow & \\ \begin{array}{c}(U(p)\times U(q)) \\ \times \\ (U(n)\times U(n))\end{array} & & \end{array}$$

The actions are as follows. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathrm{Mat}_{(p+q)\times n}(\mathbb{C}), \qquad X_1 \in \mathrm{Mat}_{p\times n}(\mathbb{C}),$$

$X_2 \in \mathrm{Mat}_{q\times n}(\mathbb{C})$.

- Action of $U(p+q)$ is by left multiplication.
- $U(n) \times U(n)$ is by right multiplication on X_1 and X_2 individually.



- The unitary branching rule now follows ...

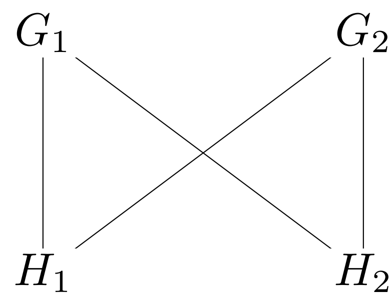
See-Saw

Let ω be a representation of Ω. Let $G_1 \subseteq \Omega$, and H_2 be its centralizer. Assume

$$\omega|_{G_1 \times H_2} = \bigoplus_{i \in I} \pi_i^{(1)} \otimes \sigma_i^{(2)}$$

where $\pi_i^{(1)}$ and $\sigma_i^{(2)}$ are irreducible rep's of G_1 and H_2, and $\pi_i^{(1)} \longleftrightarrow \sigma_j^{(2)}$ is the graph of a correspondence.

Let $H_1 \subseteq G_1$. The centralizer G_2 of H_1 contains H_2.



Assume $\omega|_{H_1 \times G_2}$ is also a correspondence.

$$\omega|_{H_1 \times G_2} = \bigoplus_{j \in J} \sigma_j^{(1)} \otimes \pi_j^{(2)}.$$

Lemma 2. *Assume the branching rules*

$$\pi_i^{(1)} = \sum_{j \in J} c_{ij} \sigma_j^{(1)}, \qquad \pi_j^{(2)} = \sum_{i \in I} d_{ji} \sigma_i^{(2)} \tag{3}$$

Then the $c_{ij} = d_{ij}$.

Proof. Both c_{ij} and $d_{ij} =$ multiplicity of $\sigma_j^{(1)} \times \sigma_i^{(2)}$ in ω as $H_1 \times H_2$ modules. $\square$

Proof of generalized Cauchy

Recall that the involution ι "roughly" interchanges the two generating functions:

$$\sum_{t=0}^{\infty} t^k h_k(\alpha_1, \cdots, \alpha_N) = \prod_{j=1}^{N} (1 - t\alpha_j)^{-1}.$$

$$\sum_{t=0}^{\infty} t^k e_k(\alpha_1, \cdots, \alpha_N) = \prod_{j=1}^{N} (1 + t\alpha_j).$$

Start with Cauchy identity, apply unitary branching:

$$\prod_{i,k} (1-\alpha_i\gamma_k)^{-1} \prod_{i,l} (1-\alpha_i\delta_l)^{-1} \prod_{j,k} (1-\beta_j\gamma_k)^{-1} \prod_{j,l} (1-\beta_j\delta_l)^{-1} =$$

$$\sum_{\lambda} s_\lambda(\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n) s_\lambda(\gamma_1, \cdots, \gamma_s, \delta_1, \cdots, \delta_t) =$$

$$\sum_{\lambda} \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(\alpha_1, \cdots, \alpha_m) s_\nu(\beta_1, \cdots, \beta_n)$$

$$\sum_{\sigma,\tau} c^\lambda_{\sigma\tau} s_\sigma(\gamma_1, \cdots, \gamma_s) s_\tau(\delta_1, \cdots, \delta_t).$$

Now apply ι in variables β and δ:

$$\prod_{i,k} (1-\alpha_i\gamma_k)^{-1} \prod_{i,l} (1+\alpha_i\delta_l) \prod_{j,k} (1+\beta_j\gamma_k) \prod_{j,l} (1-\beta_j\delta_l)^{-1} =$$

$$\sum_{\lambda} \mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n) \mathrm{LS}_\lambda(\gamma_1, \cdots, \gamma_s; \delta_1, \cdots, \delta_t).$$

Thus we obtain the Generalized Cauchy identity.

Hopf algebra structure for Λ

The Generalized Cauchy identity is equivalent to an important fact. The ring Λ is a graded algebra.

$$\Lambda = \bigoplus_k \Lambda_k, \qquad \Lambda_k = \text{gen. characters of } S_k.$$

The multiplication (**induction**) is a bilinear map $\Lambda \longrightarrow \Lambda$ that induces a homomorphism $\Lambda \otimes \Lambda \longrightarrow \Lambda$. In degree k this is a map

$$\bigoplus_{p+q=k} \Lambda_p \otimes \Lambda_q \longmapsto \Lambda_k.$$

On $\Lambda_p \times \Lambda_q$ this is **induction** of chars $S_p \times S_q \longrightarrow S_{p+q}$. There is a dual operation, namely **restriction** of chars $S_{p+q} \longrightarrow S_p \times S_q$. This gives a homomorphism of graded rings $\Lambda \longrightarrow \Lambda \otimes \Lambda$ called **comultiplication**.

Theorem. (Geissinger) *The two operations of multiplication and comultiplication make Λ a Hopf algebra.*

This means that comultiplication is a homomorphism of graded algebras, or (equivalently) that multiplication is a homomorphism of graded coalgebras.

- The Hopf algebra structure was popularized by Zelevinsky.
- We will show that the theorem is **equivalent** to the Generalized Cauchy identity!

The Hopf axiom

Geissinger's theorem boils down to the commutativity of the following diagram:

$$\begin{array}{ccccc} \Lambda \otimes \Lambda & \xrightarrow{m^* \otimes m^*} & \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda & \xrightarrow{1 \otimes \tau \otimes 1} & \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \\ \downarrow m & & & & \downarrow m \otimes m \\ \Lambda & & \xrightarrow{\qquad m^* \qquad} & & \Lambda \otimes \Lambda \end{array}$$

$m = \text{multiplication}, \qquad m^* = \text{comult.}, \qquad \tau(x \otimes y) = y \otimes x.$

- Start with a character of in $\Lambda_p \otimes \Lambda_q$ and push it forward to $\Lambda_r \otimes \Lambda_s$, where $p + q = r + s = k$.
- Thus we are inducing a character from $S_p \times S_q$ to S_k, then restricting to $S_r \times S_s$.

Mackey theory

If $G \supset H_1, H_2$ (finite groups) there are two ways we can get from characters χ of H_1 to characters of H_2. We can

Induce then restrict or **restrict then induce**

And these are the **same**. More exactly

$$\mathrm{Res}_{H_2}\mathrm{Ind}_{H_1}^{G}(\chi) = \bigoplus_{\gamma \in H_2 \backslash G / H_1} \mathrm{Ind}_{H^\gamma}^{H_2}\mathrm{Res}_{H^\gamma}({}^\gamma\chi)$$

where $H^\gamma = H_2 \cap \gamma H_1 \gamma^{-1}$ and ${}^\gamma\chi(h) = \chi(\gamma^{-1} h \gamma)$. For symmetric groups this gives the **Hopf axiom**.

Hopf Axiom = Generalized Cauchy

The Hopf axiom reduces to the formula

$$\sum_{\lambda} c^{\lambda}_{\mu\nu} c^{\lambda}_{\sigma\tau} = \sum_{\varphi,\eta} c^{\sigma}_{\varphi\eta} c^{\tau}_{\psi\xi} c^{\mu}_{\varphi\xi} c^{\nu}_{\psi\eta}, \tag{4}$$

since if we apply $m^* \circ m$ to $s_\mu \otimes s_\nu$, the coefficient of $s_\sigma \otimes s_\tau$ is the left side, $(m \otimes m) \circ (1 \otimes \tau \otimes 1) \circ (m^* \otimes m^*)$ gives the right side.

To deduce (4) from the generalized Cauchy identity we note that (in obvious notation) the right side of

$$\prod_{i,k} (1 - \alpha_i \gamma_k)^{-1} \prod_{i,l} (1 + \alpha_i \delta_l) \prod_{j,k} (1 + \beta_j \gamma_k) \prod_{j,l} (1 - \beta_j \delta_l)^{-1} = \sum_{\lambda} \mathrm{LS}_\lambda(\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n) \mathrm{LS}_\lambda(\gamma_1, \cdots, \gamma_s; \delta_1, \cdots, \delta_t).$$

is

$$\sum_{\lambda} c^{\lambda}_{\mu\nu} s_\mu(\alpha) s_{\nu'}(\beta) c^{\lambda}_{\sigma\tau} s_\sigma(\gamma) s_{\tau'}(\delta)$$

while the left side is

$$\begin{aligned} & \sum s_\varphi(\alpha) s_\varphi(\gamma) s_{\psi'}(\beta) s_{\psi'}(\delta) s_\xi(\alpha) s_{\xi'}(\delta) s_{\eta'}(\beta) s_\eta(\gamma) \\ = & \sum c^{\sigma}_{\varphi\eta} c^{\tau}_{\psi\xi} s_\varphi(\alpha) s_\xi(\alpha) s_{\psi'}(\beta) s_{\eta'}(\beta) s_\sigma(\gamma) s_{\tau'}(\delta) \\ = & \sum c^{\sigma}_{\varphi\eta} c^{\tau}_{\psi\xi} c^{\mu}_{\varphi\xi} c^{\nu}_{\psi\eta} s_\mu(\alpha) s_{\nu'}(\beta) c^{\lambda}_{\sigma\tau} s_\sigma(\gamma) s_{\tau'}(\delta). \end{aligned}$$

Comparing, we obtain the result.