Basic types of non-boolean description logics

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1 Introduction

Description logics are the theoretical basis for the theory of ontologies. They are widely used to store data (knowledge) and to do reasoning and query tasks on that data. A lot of research focused on the development of algorithms that handle those tasks, e.g. [13, 21, 22]. Nevertheless, the management of uncertain or vague data is still not common in ontologies, though uncertain or vague data appears very often in applications [17]. Our aim in this paper is to clarify the enrichment of description logics with uncertainty or vagueness and to precisely describe the process of the involved tasks on those logics. A good overview about the topic of this paper is also given in [25].

The paper is organized in the following way. Each section describes a certain logic. Each section starts with a general introduction and some definitions. Then we demonstrate how knowledge is stored and interpreted in the logic. And at the end we show how algorithms (consistency and inference) work on that particular logic.

We start with classical description logics in the second section. The third section adds uncertainty and the fourth adds vagueness to a classical description logic respectively. In this first section we give some basic definitions and explain the difference between the discussed types of logics.

There are two ways of interpreting data in a non-boolean description logic: In uncertainty theory, statements have a certain probability to be true or false. In vagueness or imprecision theory, a statement is formulated in an inexact way.

Example 1

A weather forecast wants to predict something about the weather situation of tomorrow, rain in particular. Will it rain tomorrow? If so, how much will it rain? And what's the probability and intensity of the rain? The weather forecast wants to store this information in a database (to combine it with other knowledge and gain new knowledge automatically).

Classical description logics only allow the storage of boolean information, such as *tomorrow it will rain* or *tomorrow it will not rain.*

Uncertainty theory allows to state a probability for a piece of information. The probability of rain is 90% would mean that in 9 of 10 comparable cases there will be rain tomorrow. And in 1 of 10 cases it will be dry. In general: The probability of rain is p would mean that in p of 1 comparable cases there will be rain tomorrow and not in the other cases.

Vagueness theory allows to state a degree of truth for a piece of information. The intensity of rain tomorrow is 90% would mean that tomorrow will be a rather strong thunderstorm. But it will rain tomorrow definitely. In general: The intensity of rain is p would state the amount of rainfall compared to the maximum possible.

Statements in vagueness theory are rather hard to describe by values, because if an exact value would be known (such as the amount of rain in mm) one could use classical description logics instead. Therefor usually a fuzzy logic is used instead of a single value. In this paper we will stay with a single value, since our aim is rather the explanation of processes and less the development of applications.

The following definitions are for clarification of notation in the next sections.

Definition 1: power set

Let Δ be an arbitrary set. The power set of Δ is defined as follows.

$$\mathcal{P}(\Delta) = \{X \subseteq \Delta\}$$

Definition 2: cardinality

Let $\Delta = \{x_1, x_2, \ldots, x_n\}$ be an arbitrary finite set. Then $|\Delta| = n$ is called the cardinality of Δ.

2 Description Logic

This section is an introduction to classical description logic (DL). We present a definition for the description logic $\mathcal{ALC}$ and introduce all symbols and wordings, that are necessary to understand the other sections, where we add uncertainty and vagueness to DLs. For further details about classical DLs see [2].

Description logics are designed to represent knowledge. The elementary description consists of three types of atomic expressions. Named individuals denote the most important objects in the world. Named concepts denote a certain group of individuals and named roles denote a type of relation between two individuals.

Definition 3: elementary description

The basis of a description logic are three finite sets N_R, N_C and N_I, that contain named roles, named concepts and named individuals respectively. Each element $R \in N_R$ is called a named role, sometimes also called an atomic role. Each element $C \in N_C$ is called a named concept, sometimes also called an atomic concept. Each element $a \in N_I$ is called a named individual. The triple (N_R, N_C, N_I) is called an elementary description.

A description logic is build upon those three arbitrary sets. The set N_I contains names for single objects, N_C contains names for an elementary group of objects and N_R contains names for elementary relations between two objects. The mapping between these names and the real world objects is made by an interpretation.

Beside the atomic roles and concepts, description logics give the possibility to have composite roles and concepts. The sets of all concepts and roles are denoted with bold letters respectively. Within the description logic $\mathcal{ALC}$, roles are almost equivalent to named roles, but in more complex description logics other types of roles can exist.

Definition 4: set of roles

Let (N_R, N_C, N_I) be an elementary description. The set of roles over that elementary description is denoted by $\mathbf{R}$. It contains expressions (and only those expressions) according to the following rules.

(universal role) $U \in \mathbf{R}$

(named roles) $R \in N_R \implies R \in \mathbf{R}$

Each expression $R \in \mathbf{R}$ is called a role.

Sometimes there is also an empty role (bottom role) E defined separately, but one can define it also by the restrictions of concepts, e.g. $\exists E.\top \sqsubseteq \bot$ (see definition of interpretation).

As for the roles, there are also lots of other concepts in $\mathcal{ALC}$ beside the atomic concepts, we will introduce them in definition 5.

Definition 5: set of concepts

Let (N_R, N_C, N_I) be a elementary description and let $\mathbf{R}$ be the set of roles over that elementary description. The set of concepts over that elementary description is denoted by $\mathbf{C}$. It contains expressions (and only those expressions) according to the following rules.

(top concept) $\top \in \mathbf{C}$

(bottom concept) $\bot \in \mathbf{C}$

(named concept) $C \in N_C \implies C \in \mathbf{C}$

(set of individuals) $A \subseteq N_I \implies A \in \mathbf{C}$

(conjunction) $C, D \in \mathbf{C} \implies C \sqcap D \in \mathbf{C}$

(disjunction) $C, D \in \mathbf{C} \implies C \sqcup D \in \mathbf{C}$

(negation) $C \in \mathbf{C} \implies \neg C \in \mathbf{C}$

(universal restriction) $C \in \mathbf{C}, R \in \mathbf{R} \implies \forall R.C \in \mathbf{C}$

(existential restriction) $C \in \mathbf{C}, R \in \mathbf{R} \implies \exists R.C \in \mathbf{C}$

Each expression $C \in \mathbf{C}$ is called a concept.

There is no separate set denoted with bold letter for individuals since there are no composite individuals.

And the symbols used in this context are only expressions, they are not calculable. An interpretation is necessary to enrich these expressions with meaning.

Definition 6: interpretation

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description.
A function

$$\cdot^{\mathcal{I}} : \mathbf{R} \cup \mathbf{C} \cup N_I \to \mathcal{P}\left(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\right) \cup \mathcal{P}\left(\Delta^{\mathcal{I}}\right) \cup \Delta^{\mathcal{I}}$$

together with an arbitrary set $\Delta^{\mathcal{I}}$ is called an interpretation if and only if

- $R^{\mathcal{I}} \in \mathcal{P}\left(\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}\right)$ for all $R \in \mathbf{R}$
- $C^{\mathcal{I}} \in \mathcal{P}\left(\Delta^{\mathcal{I}}\right)$ for all $C \in \mathbf{C}$
- $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for all $a \in N_I$

and the following conditions hold for every $R \in \mathbf{R}$, $C, D \in \mathbf{C}$ and $A = \{a_1, \ldots, a_n\} \subseteq N_I$.

(universal role) $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

(top concept) $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$

(bottom concept) $\bot^{\mathcal{I}} = \emptyset$

(set of individuals) $A^{\mathcal{I}} = \left\{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\right\}$

(conjunction) $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$

(disjunction) $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$

(negation) $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$

(universal restriction) $(\forall R.C)^{\mathcal{I}} = \left\{x \in \Delta^{\mathcal{I}} \mid \forall y \notin C^{\mathcal{I}} \implies (x,y) \notin R^{\mathcal{I}}\right\}$

(existential restriction) $(\exists R.C)^{\mathcal{I}} = \left\{x \in \Delta^{\mathcal{I}} \mid \exists y \in C^{\mathcal{I}} \implies (x,y) \in R^{\mathcal{I}}\right\}$

The tuple $\mathcal{I} = \left(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\right)$ is called an interpretation over (N_R, N_C, N_I) with the universe of discourse $\Delta^{\mathcal{I}}$. And $\cdot^{\mathcal{I}}$ is called interpretation function.

Sometimes the universe of discourse $\Delta^{\mathcal{I}}$ is also called domain. We will give a short example for a universe of discourse. In later examples we reduce the attention from the concrete universe of discourse and will assume that the universe of discourse consists of named individuals only.

Example 2

Given the following elementary description.

$$N_R = \{\texttt{r}\}$$
$$N_C = \{\texttt{A}, \texttt{B}\}$$
$$N_I = \{\texttt{a}, \texttt{b}, \texttt{c}\}$$

An interpretation $\mathcal{I} = \left(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\right)$ over (N_R, N_C, N_I) with the universe of discourse $\Delta^{\mathcal{I}} = \{1, 2, 3\}$ could be defined by the following mappings.

$$\texttt{r}^{\mathcal{I}} = \{(1,3), (3,3)\}$$
$$\texttt{A}^{\mathcal{I}} = \{1, 3\}$$
$$\texttt{B}^{\mathcal{I}} = \{1\}$$
$$\texttt{a}^{\mathcal{I}} = 1$$
$$\texttt{b}^{\mathcal{I}} = 2$$
$$\texttt{c}^{\mathcal{I}} = 3$$

It is sufficient to define an interpretation only by its mapping of the elementary description. For all other roles and concepts, the interpretation is determined by this. We will show it in the following theorem.

Theorem 1

Let $\mathcal{I} = \left(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\right)$ be an interpretation over the elementary description (N_R, N_C, N_I) with a universe of discourse $\Delta^{\mathcal{I}}$. Let $\mathcal{J} = \left(\Delta^{\mathcal{J}}, \cdot^{\mathcal{J}}\right)$ be another interpretation over the same elementary description with the same universe of discourse $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ such that the following equations hold.

- $R^{\mathcal{I}} = R^{\mathcal{J}}$ for all $R \in N_R$

- $C^{\mathcal{I}} = C^{\mathcal{J}}$ for all $C \in N_C$
- $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all $a \in N_I$

Then $R^{\mathcal{I}} = R^{\mathcal{J}}$ and $C^{\mathcal{I}} = C^{\mathcal{J}}$ for all $R \in \mathbf{R}$ and $C \in \mathbf{C}$.

Nevertheless, we will give now a few examples for the interpretation of composite concepts.

Example 3

Let $\mathcal{I} = \left(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\right)$ be the interpretation from example 2 with universe of discourse $\Delta^{\mathcal{I}}$ and elementary description (N_R, N_C, N_I). Then these are some examples for mappings of composite concepts.

$$\begin{aligned}(\mathtt{A} \sqcap \mathtt{B})^{\mathcal{I}} &= \{1\}\\ (\mathtt{B} \sqcup \neg\mathtt{A})^{\mathcal{I}} &= \{1, 2\}\\ (\exists \mathtt{r}.\mathtt{A})^{\mathcal{I}} &= \{1, 3\}\\ (\forall \mathtt{r}.(\{\mathtt{a}\} \sqcap \mathtt{B}))^{\mathcal{I}} &= \{2\}\end{aligned}$$

Next, we will define a knowledge base (KB). A KB consists of axioms. Terminological axioms express general rules that say something about the general relation of concepts and roles, whereas assertional axioms state which named individuals belong to a certain concept or role.

Definition 7: axioms

Let $\mathbf{C}$ be a set of concepts. For every two concepts $C, D \in \mathbf{C}$, the expression $C \sqsubseteq D$ is called a general concept inclusion (GCI). A TBox, denoted by $\mathcal{T}$, is a finite set of GCIs, i.e. $\mathcal{T} \subset \{C \sqsubseteq D \mid C, D \in \mathbf{C}\}$. If $C \sqsubseteq D \in \mathcal{T}$ and $D \sqsubseteq C \in \mathcal{T}$ for two concepts $C, D \in \mathbf{C}$, we can shortly denote this by $C \equiv D \in \mathcal{T}$. Each axiom $\Phi \in \mathcal{T}$ is called a terminological axiom.

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles and $\mathbf{C}$ be the set of concepts over that signature. For every individual $a \in N_I$ and every concept $C \in \mathbf{C}$, the expression $a : C$ is called a concept assertion. For every two individuals $a, b \in N_I$ and every role $R \in \mathbf{R}$, the expression $(a, b) : R$ is called a role assertion. An ABox, denoted by $\mathcal{A}$, is a finite set of concept assertions and role assertions, i.e. $\mathcal{A} \subset \{a : C \mid a \in N_I, C \in \mathbf{C}\} \cup \{(a, b) : R \mid a, b \in N_I, R \in \mathbf{R}\}$. Each axiom $\phi \in \mathcal{A}$ is called an assertional axiom.

A tuple $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is called a knowledge base (KB), if and only if it consists of a set of terminological axioms (TBox) $\mathcal{T}$ and a set of assertional axioms (ABox) $\mathcal{A}$.

We will now give an example for a knowledge base. Notice, that the knowledge base is independent of an interpretation.

Example 4

Given the following elementary description.

$$N_R = \{\mathtt{r}\}$$
$$N_C = \{\mathtt{A}, \mathtt{B}\}$$
$$N_I = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$$

Then the following is a knowledge base over this elementary description.

$$\begin{aligned}
\mathcal{T} = \{ & \\
& \neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top, \\
& \exists\mathtt{r}.\{\mathtt{a}, \mathtt{b}\} \sqsubseteq \bot \\
\} & \\
\mathcal{A} = \{ & \\
& \mathtt{a} : (\mathtt{A} \sqcap \mathtt{B}), \\
& \mathtt{b} : (\mathtt{A} \sqcup \mathtt{B}), \\
& \mathtt{c} : \forall\mathtt{r}.\mathtt{A}, \\
& (\mathtt{a}, \mathtt{c}) : \mathtt{r}, \\
& (\mathtt{c}, \mathtt{c}) : \mathtt{r} \\
\} & \\
\mathcal{K} = (\mathcal{T}, \mathcal{A}\ ) &
\end{aligned}$$

A knowledge base is designed to store knowledge about a specific real world domain. To use this knowledge for computational purposes, it should be free of contradictions. If a knowledge base is free of contradictions, we call it consistent. We can not be sure that every information in a consistent knowledge base is correct, but we can say that there must be something wrong in an inconsistent knowledge base. Definition 8 describes the relation between the axioms and the interpretation.

Definition 8: model

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation over (N_R, N_C, N_I).

- Let $C, D \in \mathbf{C}$ be two concepts.
  In $\mathcal{I}$ holds the general concept inclusion $C \sqsubseteq D$ (denoted $\mathcal{I} \models C \sqsubseteq D$) if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

- Let $\mathcal{T} \subset \{C \sqsubseteq D \mid C, D \in \mathbf{C}\}$ be a TBox.
  In $\mathcal{I}$ holds the TBox $\mathcal{T}$ (denoted $\mathcal{I} \models \mathcal{T}$)
  if and only if $\mathcal{I} \models \Phi$ for every GCI $\Phi \in \mathcal{T}$.

- Let $a \in N_I$ be an individual and let $C \in \mathbf{C}$ be a concept.
  In $\mathcal{I}$ holds the concept assertion $a : C$ (denoted $\mathcal{I} \models a : C$)
  if and only if $a^{\mathcal{I}} \in C^{\mathcal{I}}$.

- Let $a, b \in N_I$ be two individuals and let $R \in \mathbf{R}$ be a role.
  In $\mathcal{I}$ holds the role assertion $(a, b) : R$ (denoted $\mathcal{I} \models (a, b) : R$)
  if and only if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.

- Let $\mathcal{A} \subset \{a : C \mid a \in N_I, C \in \mathbf{C}\} \cup \{(a, b) : R \mid a, b \in N_I, C \in \mathbf{C}\}$ be an ABox.
  In $\mathcal{I}$ holds the ABox $\mathcal{A}$ (denoted $\mathcal{I} \models \mathcal{A}$)
  if and only if $\mathcal{I} \models \phi$ for every assertion $\phi \in \mathcal{A}$.

- Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a knowledge base.
  In $\mathcal{I}$ holds the knowledge base $\mathcal{K}$ (denoted $\mathcal{I} \models \mathcal{K}$)
  if and only if $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I} \models \mathcal{A}$.

If $\mathcal{I} \models \mathcal{K}$, then $\mathcal{I}$ is called a model of $\mathcal{K}$. If a knowledge base $\mathcal{K}$ is given and there exists at least one interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$, then $\mathcal{K}$ is called consistent.

The interpretation $\mathcal{I}$ from example 2 and 3 is not a model of the knowledge base $\mathcal{K}$ from example 4, i.e. $\mathcal{I} \not\models \mathcal{K}$. We can see this by looking at the axiom $\mathtt{b} : (\mathtt{A} \sqcup \mathtt{B})$. For the interpretation $\mathcal{I}$ is defined $\mathtt{b}^{\mathcal{I}} = 2$ and

$$(\mathtt{A} \sqcup \mathtt{B})^{\mathcal{I}} = \mathtt{A}^{\mathcal{I}} \cup \mathtt{B}^{\mathcal{I}} = \{1, 3\} \cup \{1\} = \{1, 3\}$$

does not contain 2, so $\mathtt{b}^{\mathcal{I}} \notin (\mathtt{A} \sqcup \mathtt{B})^{\mathcal{I}}$.

We will now give an example of another interpretation, that is a model of the knowledge base from example 4. From now on, we will no longer explicitly state the universe of discourse and will always assume that it is equal to the named individuals $\Delta^{\mathcal{I}} = N_I$.

Example 5

Let (N_R, N_C, N_I) be the elementary description and $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be the knowledge base from example 4. Let now $\mathcal{I}$ be an interpretation over (N_R, N_C, N_I) with the following mappings.

$$\mathtt{r}^{\mathcal{I}} = \{(\mathsf{a}, \mathsf{c}), (\mathsf{c}, \mathsf{c})\}$$
$$\mathtt{A}^{\mathcal{I}} = \{\mathsf{a}, \mathsf{c}\}$$
$$\mathtt{B}^{\mathcal{I}} = \{\mathsf{a}, \mathsf{b}\}$$

We will now show that $\mathcal{I} \models \mathcal{K}$ by showing that $\mathcal{I} \models \Phi$ for each axiom $\Phi \in \mathcal{T}$ and then showing that $\mathcal{I} \models \phi$ for each axiom $\phi \in \mathcal{A}$.

$$
\begin{aligned}
(\neg \mathtt{B})^{\mathcal{I}} = \{\mathtt{c}\} \subseteq \{\mathtt{a}, \mathtt{c}\} &= (\exists \mathtt{r}.\top)^{\mathcal{I}} \\
(\exists \mathtt{r}.\{\mathtt{a}, \mathtt{b}\})^{\mathcal{I}} = \emptyset \subseteq \emptyset &= (\bot)^{\mathcal{I}} \\
\mathtt{a} \in \{\mathtt{a}\} &= (\mathtt{A} \sqcap \mathtt{B})^{\mathcal{I}} \\
\mathtt{b} \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} &= (\mathtt{A} \sqcup \mathtt{B})^{\mathcal{I}} \\
\mathtt{c} \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} &= (\forall \mathtt{r}.\mathtt{A})^{\mathcal{I}} \\
(\mathtt{a}, \mathtt{c}) \in \{(\mathtt{a}, \mathtt{c}), (\mathtt{c}, \mathtt{c})\} &= \mathtt{r}^{\mathcal{I}} \\
(\mathtt{c}, \mathtt{c}) \in \{(\mathtt{a}, \mathtt{c}), (\mathtt{c}, \mathtt{c})\} &= \mathtt{r}^{\mathcal{I}}
\end{aligned}
$$

Notice that we didn't show how we found this interpretation that models the given knowledge base. In general it is easier to just determine if a given KB is consistent or not than to explicitly find a concrete interpretation that models it [1].

There exist several algorithms to either determine if an interpretation models a knowledge base or to compute an interpretation from a given knowledge base that is a model for it. With those algorithms, i.e. tableau algorithm, the consistency of a knowledge base is determinable. For further details about programs see [9].

Another important task for knowledge bases is to infer new knowledge, that is stored only implicitly in the KB. As for the consistency, there exist a lot of algorithms for inference. We will not explain these algorithms here and refer to [27].

Definition 9: inference

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a consistent knowledge base.

A terminological axiom $\Phi \in \{C \sqsubseteq D \mid C, D \in \mathbf{C}\}$ is a logical consequence of $\mathcal{K}$, denoted by $\mathcal{K} \models \Phi$, if and only if for all $\mathcal{I} \models \mathcal{K}$ also $\mathcal{I} \models (\mathcal{T} \cup \{\Phi\}, \mathcal{A})$.

An assertional axiom $\phi \in \{a : C \mid a \in N_I, C \in \mathbf{C}\} \cup \{(a, b) : R \mid a, b \in N_I, C \in \mathbf{C}\}$ is a logical consequence of $\mathcal{K}$, denoted by $\mathcal{K} \models \phi$, if and only if for all $\mathcal{I} \models \mathcal{K}$ also $\mathcal{I} \models (\mathcal{T}, \mathcal{A} \cup \{\phi\})$.

Notice that you could infer anything from an inconsistent knowledge base, because there is no interpretation that models the knowledge base, thus there is no interpretation that models the knowledge base with any additional axiom. Instead, one can choose consistent subsets of knowledge bases and infer on them.

We will now give some examples of inferred knowledge that could be gained from the knowledge base introduced in example 4. We will not do this in an algorithmic way, but only explain the plausibility of those logical consequences.

Example 6

Let (N_R, N_C, N_I) be the elementary description and $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be the knowledge base from example 4. Then the following axioms are inferable.

$$\begin{aligned} \mathtt{a} : (\mathtt{A} \sqcap \mathtt{B}) &\models \mathtt{a} : \mathtt{A} \\ \mathtt{a} : (\mathtt{A} \sqcap \mathtt{B}) &\models \mathtt{a} : \mathtt{B} \\ \{\mathtt{c} : \forall \mathtt{r}.\mathtt{A}, (\mathtt{c}, \mathtt{c}) : \mathtt{r}\} &\models \mathtt{c} : \mathtt{A} \end{aligned}$$

The first two logical consequences are obvious. The third requires a bit more consideration. The axiom $\mathtt{c} : \forall \mathtt{r}.\mathtt{A}$ states that for all tuples $(\mathtt{c}, x) : \mathtt{r}$, it must also be $x : \mathtt{A}$. And hence, since $(\mathtt{c}, \mathtt{c}) : \mathtt{r}$, it must also be $\mathtt{c} : \mathtt{A}$. There are other consequences derivable from the given knowledge base. Since $\exists \mathtt{r}.\{\mathtt{a}, \mathtt{b}\} \sqsubseteq \bot$, there can not exist a tuple $(x, y) : \mathtt{r}$ with $y \neq \mathtt{c}$. We will show more logical consequences of this knowledge base in example 8.

This section gave a short overview about classical description logics. The next sections will extend this approach with the management of uncertainty and vagueness respectively.

3 Uncertainty Theory

Description logics can be enriched with the management of uncertainty. An axiom is uncertain, if it is unknown whether the axiom is correct or wrong, but one can state a certain probability or at least a range of trust. There are a lot of different modeling approaches for uncertainty theory such as probabilistic logic and possibilistic logic. And there are also different algorithms to optimize reasoning or other tasks in specific cases. [15] We will not take into account each of them, but sometimes remark a hint to other possibilities of handling a certain problem.

In this section, we give an introduction only to probabilistic logics with a probabilistic distribution of worlds. That is, each axiom gets assigned a certain probability to be true (and false otherwise).

Definition 10: possible worlds

Let (N_R, N_C, N_I) be an elementary description. In uncertainty theory, for a fixed universe of discourse (we will assume N_I as the universe of discourse), an interpretation is called a possible world. The set of all possible worlds is denoted by $\mathbf{I}$.

$$\mathbf{I} := \{\mathcal{I} \mid \mathcal{I} \text{ is an interpretation over } (N_R, N_C, N_I)\}$$

A mapping $\pi : \mathbf{I} \to [0, 1]$ with

$$\sum_{\mathcal{I} \in \mathbf{I}} \pi(\mathcal{I}) = 1$$

is called a probability distribution of possible worlds over (N_R, N_C, N_I).

Notice that the definition of π is not dependent on a given interpretation or knowledge base, only an elementary description is necessary. We will later introduce an uncertain knowledge base to restrict π.

Sometimes π is not defined as a probability distribution and the constraint $\sum_{\mathcal{I} \in \mathbf{I}} \pi(\mathcal{I}) = 1$ is replaced by $\max_{\mathcal{I} \in \mathbf{I}} \pi(\mathcal{I}) = 1$. With that constraint, one gets a possibility value for each world to be the true one [8].

Notice also that the amount of possible worlds is increasing exponentially and even for the very small example from the previous section, there are 2^{15}. But at least it is a finite number. Nevertheless, the computation of all worlds is not possible for practical issues. For optimization tasks we refer to [10].

Theorem 2

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{I}$ be the set of all possible worlds.

$$|\mathbf{I}| = 2^{|N_I| \cdot |N_I| \cdot |N_R|} \cdot 2^{|N_I| \cdot |N_C|}$$

This theorem is easy to validate, because for each named concept there are $|N_I|$ individuals that could either be part of the concept or not. And for each named role there are $|N_I| \cdot |N_I|$ pairs of individuals that could either be part of the role or not.

In uncertainty theory, axioms are affixed with a value that describes the probability of the axiom to be true. In probabilistic logic, this is only a single value. In possibilistic logic, this is a range of values. And there exist other many-valued logics to denote different notions of probability [16]. In this paper, we restrain uncertainty theory to probabilistic logic, where only probabilistic constraints are added to axioms. Such a constraint states that an axiom is true with a certain probability and false otherwise.

Definition 11: uncertain axioms

Let $\mathbf{C}$ be a set of concepts. For every two concepts $C, D \in \mathbf{C}$ and every $p \in [0,1]$, the expression $Pr(C \sqsubseteq D) = p$ is called a probabilistic concept inclusion (PrCI). An uncertain TBox, denoted by $\mathcal{T}_\mathcal{U}$, is a finite set of PrCIs.

$$\mathcal{T}_\mathcal{U} \subset \{Pr(C \sqsubseteq D) = p \mid C, D \in \mathbf{C}, p \in [0,1]\}$$

Each axiom $\Phi \in \mathcal{T}_\mathcal{U}$ is called an uncertain terminological axiom.

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles and $\mathbf{C}$ be the set of concepts over that elementary description. For every individual $a \in N_I$, every concept $C \in \mathbf{C}$ and every $p \in [0,1]$, the expression $Pr(a : C) = p$ is called a probabilistic concept assertion. For every two individuals $a, b \in N_I$, every role $R \in \mathbf{R}$ and every $p \in [0,1]$, the expression $Pr((a,b) : R) = p$ is called a probabilistic role assertion. An uncertain ABox, denoted by $\mathcal{A}_\mathcal{U}$, is a finite set of probabilistic concept and role assertions.

$$\begin{aligned}\mathcal{A}_\mathcal{U} \subset &\{Pr(a : C) = p \mid a \in N_I, C \in \mathbf{C}, p \in [0,1]\} \cup \\ &\{Pr((a,b) : R) = p \mid a, b \in N_I, R \in \mathbf{R}, p \in [0,1]\}\end{aligned}$$

Each axiom $\phi \in \mathcal{A}$ is called an uncertain assertional axiom.

A tuple $\mathcal{K}_\mathcal{U} = (\mathcal{T}_\mathcal{U}, \mathcal{A}_\mathcal{U})$ is called an uncertain knowledge base (KB), if and only if it consists of a set of uncertain terminological axioms $\mathcal{T}_\mathcal{U}$ and a set of uncertain assertional axioms $\mathcal{A}_\mathcal{U}$.

We will now reuse the knowledge base from example 4, re-define it under uncertainty theory and add some further axioms.

Example 7

Let (N_R, N_C, N_I) be the following elementary description.

$$\begin{aligned} N_R &= \{\mathtt{r}\} \\ N_C &= \{\mathtt{A}, \mathtt{B}\} \\ N_I &= \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} \end{aligned}$$

Then this is an uncertain knowledge base.

$$\begin{aligned} \mathcal{T}_\mathcal{U} = \{ & \\ & Pr\left(\neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top\right) = 1, \\ & Pr\left(\exists\mathtt{r}.\{\mathtt{a}, \mathtt{b}\} \sqsubseteq \bot\right) = 1, \\ & Pr\left(\mathtt{A} \sqsubseteq \mathtt{B}\right) = 0.2 \end{aligned}$$

$$
\begin{aligned}
& \quad \} \\
\mathcal{A}_{\mathcal{U}} = \{ & \\
& Pr\,(\texttt{a} : (\texttt{A} \sqcap \texttt{B})) = 1, \\
& Pr\,(\texttt{b} : (\texttt{A} \sqcup \texttt{B})) = 1, \\
& Pr\,(\texttt{c} : \forall \texttt{r}.\texttt{A}) = 1, \\
& Pr\,(\texttt{c} : \texttt{B}) = 0.3, \\
& Pr\,((\texttt{a},\texttt{c}) : \texttt{r}) = 1, \\
& Pr\,((\texttt{b},\texttt{c}) : \texttt{r}) = 0.1, \\
& Pr\,((\texttt{c},\texttt{c}) : \texttt{r}) = 1 \\
& \} \\
\mathcal{K}_{\mathcal{U}} = (\, & \mathcal{T}_{\mathcal{U}}, \mathcal{A}_{\mathcal{U}} \,)
\end{aligned}
$$

So as in classical description logic, we want to know now, if a given uncertain knowledge base is consistent. Therefore we have to find a probability distribution of the possible worlds that models all the given axioms. In a probability distribution of possible worlds holds an axiom, if the sum of the probabilities for the worlds that model the axiom is equal to the probability stated in the axiom.

Definition 12: uncertain model

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description. Let $\pi : \mathbf{I} \to [0,1]$ be a probability distribution of worlds over (N_R, N_C, N_I).

- Let $C, D \in \mathbf{C}$ be two concepts and $p \in [0,1]$.
  In π holds a PrCI (denoted $\pi \models Pr(C \sqsubseteq D) = p$)
  if and only if $\sum_{\mathcal{I} \models C \sqsubseteq D} \pi(\mathcal{I}) = p$.
- Let $\mathcal{T}_{\mathcal{U}}$ be an uncertain TBox.
  In π holds $\mathcal{T}_{\mathcal{U}}$ (denoted $\pi \models \mathcal{T}_{\mathcal{U}}$)
  if and only if $\pi \models \Phi$ for every probabilistic concept inclusion $\Phi \in \mathcal{T}_{\mathcal{U}}$.
- Let $a \in N_I$ be an individual, $C \in \mathbf{C}$ be a concept and $p \in [0,1]$.
  In π holds a pr. concept assertion (denoted $\pi \models Pr(a : C) = p$)
  if and only if $\sum_{\mathcal{I} \models a:C} \pi(\mathcal{I}) = p$.
- Let $a, b \in N_I$ be two individuals, $R \in \mathbf{R}$ be a role and $p \in [0,1]$.
  In π holds a pr. role assertion (denoted $\pi \models Pr((a,b) : R) = p$)
  if and only if $\sum_{\mathcal{I} \models (a,b):R} \pi(\mathcal{I}) = p$.
- Let $\mathcal{A}_{\mathcal{U}}$ be an uncertain ABox.
  In π holds $\mathcal{A}_{\mathcal{U}}$ (denoted $\pi \models \mathcal{A}_{\mathcal{U}}$)

if and only if $\pi \models \phi$ for every probabilistic concept and role assertion $\phi \in \mathcal{A}_\mathcal{U}$.

- Let $\mathcal{K}_\mathcal{U} = (\mathcal{T}_\mathcal{U}, \mathcal{A}_\mathcal{U})$ be an uncertain knowledge base.
In π holds the uncertain knowledge base (denoted $\pi \models \mathcal{K}_\mathcal{U}$)
if and only if $\pi \models \mathcal{T}_\mathcal{U}$ and $\pi \models \mathcal{A}_\mathcal{U}$.

If $\pi \models \mathcal{K}_\mathcal{U}$, then π is called an uncertain model of $\mathcal{K}_\mathcal{U}$. If an uncertain knowledge base $\mathcal{K}_\mathcal{U}$ is given and there exists at least one probability distribution of worlds π such that $\pi \models \mathcal{K}_\mathcal{U}$ and $\sum_{\mathcal{I}\in\mathbf{I}} \pi(\mathcal{I}) = 1$, then $\mathcal{K}_\mathcal{U}$ is called consistent.

For other logics, such as probabilistic logic, this definition is slightly different. For example, $\sum_{\mathcal{I}\models C\sqsubseteq D} \pi(\mathcal{I}) \geq p$ or even $\max_{\mathcal{I}\models C\sqsubseteq D}\pi(\mathcal{I}) \geq p$ could be alternative conditions for concept inclusions.

We will now show that the uncertain knowledge base from example 7 is consistent. We will do this by stating a probability distribution of worlds and showing that it models the uncertain knowledge base. We will explain how to find such a probability distribution afterwards.

Example 8

Let (N_R, N_C, N_I) be the elementary description and $\mathcal{K}_\mathcal{U} = (\mathcal{T}_\mathcal{U}, \mathcal{A}_\mathcal{U})$ be the knowledge base from example 7. Let $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ be three interpretations over (N_R, N_C, N_I) with the following mappings.

$$\begin{aligned}
\mathtt{r}^{\mathcal{I}_1} &= \{(\mathtt{a},\mathtt{c}),(\mathtt{b},\mathtt{c}),(\mathtt{c},\mathtt{c})\}\\
\mathtt{A}^{\mathcal{I}_1} &= \{\mathtt{a},\mathtt{b},\mathtt{c}\}\\
\mathtt{B}^{\mathcal{I}_1} &= \{\mathtt{a},\mathtt{c}\}\\
\\
\mathtt{r}^{\mathcal{I}_2} &= \{(\mathtt{a},\mathtt{c}),(\mathtt{c},\mathtt{c})\}\\
\mathtt{A}^{\mathcal{I}_2} &= \{\mathtt{a},\mathtt{c}\}\\
\mathtt{B}^{\mathcal{I}_2} &= \{\mathtt{a},\mathtt{b},\mathtt{c}\}\\
\\
\mathtt{r}^{\mathcal{I}_3} &= \{(\mathtt{a},\mathtt{c}),(\mathtt{c},\mathtt{c})\}\\
\mathtt{A}^{\mathcal{I}_3} &= \{\mathtt{a},\mathtt{b},\mathtt{c}\}\\
\mathtt{B}^{\mathcal{I}_3} &= \{\mathtt{a},\mathtt{b}\}
\end{aligned}$$

We let $\pi : \mathbf{I} \to [0,1]$ now be the distribution of worlds with $\pi(\mathcal{I}_1) = 0.1$, $\pi(\mathcal{I}_2) = 0.2$, $\pi(\mathcal{I}_3) = 0.7$ and $\pi(\mathcal{I}) = 0$ for all other interpretations $\mathcal{I} \in \mathbf{I} \setminus \{\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3\}$.

We will show that $\pi \models \mathcal{K}_{\mathcal{U}}$. First of all, π is a probability distribution since $0.1 + 0.2 + 0.7 = 1$, i.e. $\sum_{\mathcal{I} \in \mathbf{I}} \pi(\mathcal{I}) = 1$. And now we will show that in π holds each axiom of the given uncertain knowledge base $\mathcal{K}_{\mathcal{U}}$, i.e. the sum of probabilities of π for all the interpretations that model the axiom must be equal to the probability p of the axiom. We will only explain the details for the uncertain axioms with a probability $p < 1$, where we really have an uncertainty in the axiom. We will ignore all the possible worlds with $\pi(\mathcal{I}) = 0$ since they don't contribute to the sums. The previous section shows how to prove if an interpretation models a (not uncertain) axiom, so we won't explain this again for each axiom, but only for those with a probability $p < 1$.

$$\begin{aligned} \mathcal{I}_1 \models \neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top, \quad \mathcal{I}_2 \models \neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top, \quad \mathcal{I}_3 \models \neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top \\ \Longrightarrow \sum_{\mathcal{I} \models \neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1 \end{aligned}$$

$$\begin{aligned} \mathcal{I}_1 \models \exists\mathtt{r}.\{\mathtt{a},\mathtt{b}\} \sqsubseteq \bot, \quad \mathcal{I}_2 \models \exists\mathtt{r}.\{\mathtt{a},\mathtt{b}\} \sqsubseteq \bot, \quad \mathcal{I}_3 \models \exists\mathtt{r}.\{\mathtt{a},\mathtt{b}\} \sqsubseteq \bot \\ \Longrightarrow \sum_{\mathcal{I} \models \exists\mathtt{r}.\{\mathtt{a},\mathtt{b}\} \sqsubseteq \bot} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1 \end{aligned}$$

$$\begin{aligned} \mathtt{A}^{\mathcal{I}_1} = \{\mathtt{a},\mathtt{b},\mathtt{c}\} \not\subseteq \{\mathtt{a},\mathtt{c}\} = \mathtt{B}^{\mathcal{I}_1} &\Longrightarrow \mathcal{I}_1 \not\models \mathtt{A} \sqsubseteq \mathtt{B} \\ \mathtt{A}^{\mathcal{I}_2} = \{\mathtt{a},\mathtt{c}\} \subseteq \{\mathtt{a},\mathtt{b},\mathtt{c}\} = \mathtt{B}^{\mathcal{I}_2} &\Longrightarrow \mathcal{I}_2 \models \mathtt{A} \sqsubseteq \mathtt{B} \\ \mathtt{A}^{\mathcal{I}_3} = \{\mathtt{a},\mathtt{b},\mathtt{c}\} \not\subseteq \{\mathtt{a},\mathtt{b}\} = \mathtt{B}^{\mathcal{I}_3} &\Longrightarrow \mathcal{I}_3 \not\models \mathtt{A} \sqsubseteq \mathtt{B} \\ &\Longrightarrow \sum_{\mathcal{I} \models \mathtt{A} \sqsubseteq \mathtt{B}} \pi(\mathcal{I}) = \pi(\mathcal{I}_2) = 0.2 \end{aligned}$$

$$\begin{aligned} \mathcal{I}_1 \models \mathtt{a} : (\mathtt{A} \sqcap \mathtt{B}), \quad \mathcal{I}_2 \models \mathtt{a} : (\mathtt{A} \sqcap \mathtt{B}), \quad \mathcal{I}_3 \models \mathtt{a} : (\mathtt{A} \sqcap \mathtt{B}) \\ \Longrightarrow \sum_{\mathcal{I} \models \mathtt{a}:(\mathtt{A} \sqcap \mathtt{B})} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1 \end{aligned}$$

$$\begin{aligned} \mathcal{I}_1 \models \mathtt{b} : (\mathtt{A} \sqcup \mathtt{B}), \quad \mathcal{I}_2 \models \mathtt{b} : (\mathtt{A} \sqcup \mathtt{B}), \quad \mathcal{I}_3 \models \mathtt{b} : (\mathtt{A} \sqcup \mathtt{B}) \\ \Longrightarrow \sum_{\mathcal{I} \models \mathtt{b}:(\mathtt{A} \sqcup \mathtt{B})} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1 \end{aligned}$$

$$\begin{aligned} \mathcal{I}_1 \models \mathtt{c} : \forall\mathtt{r}.\mathtt{A}, \quad \mathcal{I}_2 \models \mathtt{c} : \forall\mathtt{r}.\mathtt{A}, \quad \mathcal{I}_3 \models \mathtt{c} : \forall\mathtt{r}.\mathtt{A} \\ \Longrightarrow \sum_{\mathcal{I} \models \mathtt{c}:\forall\mathtt{r}.\mathtt{A}} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1 \end{aligned}$$

$$\begin{aligned}
\mathtt{c} \in \{\mathtt{a}, \mathtt{c}\} = \mathtt{B}^{\mathcal{I}_1} &\implies \mathcal{I}_1 \models \mathtt{c} : \mathtt{B} \\
\mathtt{c} \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\} = \mathtt{B}^{\mathcal{I}_2} &\implies \mathcal{I}_2 \models \mathtt{c} : \mathtt{B} \\
\mathtt{c} \notin \{\mathtt{a}, \mathtt{b}\} = \mathtt{B}^{\mathcal{I}_3} &\implies \mathcal{I}_3 \not\models \mathtt{c} : \mathtt{B} \\
\implies \sum_{\mathcal{I} \models \mathtt{c}:\mathtt{B}} &\pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) = 0.3
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_1 \models (\mathtt{a}, \mathtt{c}) : \mathtt{r}, \quad \mathcal{I}_2 \models (\mathtt{a}, \mathtt{c}) : \mathtt{r}, \quad \mathcal{I}_3 \models (\mathtt{a}, \mathtt{c}) : \mathtt{r} \\
\implies \sum_{\mathcal{I} \models (\mathtt{a},\mathtt{c}):\mathtt{r}} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1
\end{aligned}$$

$$\begin{aligned}
(\mathtt{b}, \mathtt{c}) \in \{(\mathtt{a}, \mathtt{c}), (\mathtt{b}, \mathtt{c}), (\mathtt{c}, \mathtt{c})\} = \mathtt{r}^{\mathcal{I}_1} &\implies \mathcal{I}_1 \models (\mathtt{b}, \mathtt{c}) : \mathtt{r} \\
(\mathtt{b}, \mathtt{c}) \notin \{(\mathtt{a}, \mathtt{c}), (\mathtt{c}, \mathtt{c})\} = \mathtt{r}^{\mathcal{I}_2} &\implies \mathcal{I}_2 \not\models (\mathtt{b}, \mathtt{c}) : \mathtt{r} \\
(\mathtt{b}, \mathtt{c}) \notin \{(\mathtt{a}, \mathtt{c}), (\mathtt{c}, \mathtt{c})\} = \mathtt{r}^{\mathcal{I}_3} &\implies \mathcal{I}_3 \not\models (\mathtt{b}, \mathtt{c}) : \mathtt{r} \\
\implies \sum_{\mathcal{I} \models (\mathtt{b},\mathtt{c}):\mathtt{r}} &\pi(\mathcal{I}) = \pi(\mathcal{I}_1) = 0.1
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_1 \models (\mathtt{c}, \mathtt{c}) : \mathtt{r}, \quad \mathcal{I}_2 \models (\mathtt{c}, \mathtt{c}) : \mathtt{r}, \quad \mathcal{I}_3 \models (\mathtt{c}, \mathtt{c}) : \mathtt{r} \\
\implies \sum_{\mathcal{I} \models (\mathtt{c},\mathtt{c}):\mathtt{r}} \pi(\mathcal{I}) = \pi(\mathcal{I}_1) + \pi(\mathcal{I}_2) + \pi(\mathcal{I}_3) = 1
\end{aligned}$$

This example proves the correctness of the model, but it doesn't show how to find such a model (probability distribution of worlds), but it is necessary to find such a model to prove the consistency of an uncertain knowledge base. The brute force method to solve the task of a consistency check for an uncertain knowledge base would be to set up a huge system of linear equations

$$A \cdot \pi = p$$

where π is the probability distribution of worlds we are looking for, A is a matrix with $|\mathbf{I}|$ columns and one row for each axiom. Vector p contains the probability values for each axiom. Each entry of the matrix A is 1 if and only if the corresponding world $\mathcal{I}$ is a model of the axiom denoted by the row, the entry of the matrix is 0 otherwise. Since the matrix is a boolean matrix, there exist algorithms for optimization. And there are several possibilities to exclude some worlds and axioms from the beginning.

Example 9

We will take the setting of example 7 again. Let (N_R, N_C, N_I) be the elementary interpretation and let $\mathcal{K}_\mathcal{U} = (\mathcal{T}_\mathcal{U}, \mathcal{A}_\mathcal{U})$ be the knowledge base from example 7. There are more possible worlds than the three we mentioned in example 8. All possible worlds that are not excluded by taking into account the inference of the not uncertain axioms (i.e. those with a probability of $p = 1$), we denote by $\mathcal{I}_1, \mathcal{I}_2, \ldots \mathcal{I}_{10}$ according to the following table.

	$(\mathtt{b}, \mathtt{c}) \in \mathtt{r}^{\mathcal{I}}$	$(\mathtt{b}, \mathtt{c}) \notin \mathtt{r}^{\mathcal{I}}$
$\mathtt{b} \in \mathtt{A}^{\mathcal{I}}, \mathtt{b} \in \mathtt{B}^{\mathcal{I}}, \mathtt{c} \in \mathtt{B}^{\mathcal{I}}$	$\mathcal{I}_4$	$\mathcal{I}_9$
$\mathtt{b} \notin \mathtt{A}^{\mathcal{I}}, \mathtt{b} \in \mathtt{B}^{\mathcal{I}}, \mathtt{c} \in \mathtt{B}^{\mathcal{I}}$	$\mathcal{I}_5$	$\mathcal{I}_2$
$\mathtt{b} \in \mathtt{A}^{\mathcal{I}}, \mathtt{b} \notin \mathtt{B}^{\mathcal{I}}, \mathtt{c} \in \mathtt{B}^{\mathcal{I}}$	$\mathcal{I}_1$	impossible
$\mathtt{b} \in \mathtt{A}^{\mathcal{I}}, \mathtt{b} \in \mathtt{B}^{\mathcal{I}}, \mathtt{c} \notin \mathtt{B}^{\mathcal{I}}$	$\mathcal{I}_6$	$\mathcal{I}_3$
$\mathtt{b} \notin \mathtt{A}^{\mathcal{I}}, \mathtt{b} \in \mathtt{B}^{\mathcal{I}}, \mathtt{c} \notin \mathtt{B}^{\mathcal{I}}$	$\mathcal{I}_7$	$\mathcal{I}_{10}$
$\mathtt{b} \in \mathtt{A}^{\mathcal{I}}, \mathtt{b} \notin \mathtt{B}^{\mathcal{I}}, \mathtt{c} \notin \mathtt{B}^{\mathcal{I}}$	$\mathcal{I}_8$	impossible

For all these possible worlds is $(\mathtt{a}, \mathtt{a}), (\mathtt{b}, \mathtt{a}), (\mathtt{c}, \mathtt{a}), (\mathtt{a}, \mathtt{b}), (\mathtt{b}, \mathtt{b}), (\mathtt{c}, \mathtt{b}) \notin \mathtt{r}^{\mathcal{I}}$, $(\mathtt{a}, \mathtt{c}), (\mathtt{c}, \mathtt{c}) \in \mathtt{r}^{\mathcal{I}}$, $\mathtt{a}, \mathtt{c} \in \mathtt{A}$ and $\mathtt{a} \in \mathtt{B}$.

These possible worlds are determinable with algorithms for classical description logics, since the axioms with a probability of $p < 1$ are not taken into account yet. We will now formulate the system of equations that correspond to the three probabilistic axioms. For $j \in \{1, \ldots, 10\}$ is $\pi_j = \pi(\mathcal{I}_j)$. Each axiom has the form $Pr(\psi_i) = p_i$ and is represented in one of the rows.

$$\begin{aligned} \psi_1 &= \mathtt{A} \sqsubseteq \mathtt{B} & \quad p_1 &= 0.2 \\ \psi_2 &= \mathtt{c} : \mathtt{B} & \quad p_2 &= 0.3 \\ \psi_3 &= (\mathtt{b}, \mathtt{c}) : \mathtt{r} & \quad p_3 &= 0.1 \end{aligned}$$

An entry of the matrix is $A_{ij} = 1$ if $\mathcal{I}_j \models \psi_i$ and is $A_{ij} = 0$ if $\mathcal{I}_j \not\models \psi_i$. In addition to the axioms we add a last equation at the end of the matrix to make sure that all possible worlds sum up to 1.

$$A \cdot \pi = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \\ \pi_5 \\ \pi_6 \\ \pi_7 \\ \pi_8 \\ \pi_9 \\ \pi_{10} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.1 \\ 1 \end{bmatrix} = p$$

If we take into account that all π_k must be probabilities, i.e. $0 \leq \pi_k \leq 1$, then for all $\alpha \in [0, 0.2]$ and all $\beta \in [0, 0.7]$,

$$\pi = \begin{bmatrix} 0.1 & \alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0.2 - \alpha & 0.7 - \beta \end{bmatrix}$$

is a solution of this linear system of equations. And hence, we can infer for example the following knowledge.

$$Pr\,(\texttt{b} : \texttt{A} \sqcap \texttt{B}) \leq 0.9$$
$$Pr\,(\texttt{b} : \neg\texttt{B}) = 0.1$$
$$Pr\,(\texttt{B} \sqsubseteq \{\texttt{a}\}) = 0$$

Especially the last result is interesting, since it says, that at least either `b` or `c` must be part of the concept `B`. This knowledge could not be inferred by only taking into account the inference of classical description logics.

This brute force method needs too many resources and hence we will show some other possibilities for inference in theorem 3. All rules from classical logics, such as double negation, modus ponens, de morgans laws etc. are still valid in uncertainty theory. But uncertainty theory suffers from the drowning problem [20]. It means, that inferring on an inconsistent knowledge base is not possible in the usual way (like in classical description logics). Since the uncertain model takes into account all possible interpretations at once, there is no way to infer on parts of the axioms without the contingency of destroying the whole intended knowledge. There are only limited ways to infer the borders of possible ranges for axioms.

Theorem 3

Let Δ be a domain. Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles and $\mathbf{C}$ be the set of concepts over that elementary description. Let $\mathcal{K}_\mathcal{U} = (\mathcal{T}_\mathcal{U}, \mathcal{A}_\mathcal{U})$ be an uncertain knowledge base.

For uncertain axioms, the following is inferable.

- For all $C, D \in \mathbf{C}$:
  $Pr(C \sqsubseteq D) \leq \min_{x\in\Delta}(Pr(x : D \sqcup \neg C))$
- For all $C, D \in \mathbf{C}$:
  $Pr(C \sqsubseteq D) \geq \max(0, 1 + \sum_{x\in\Delta}(Pr(x : D \sqcup \neg C) - 1))$
- For all $a, b \in N_I$:
  $Pr((a, b) : U) = 1$
- For all $a \in N_I$:
  $Pr(a : \top) = 1$
- For all $a \in N_I$:
  $Pr(a : \bot) = 0$
- For all $A \subseteq N_I$:
  $Pr(a : A) = \begin{cases} 0 & a \in A \\ 1 & a \notin A \end{cases}$

- For all $a \in N_I, C, D \in \mathbf{C}$:
  $Pr(a : C \sqcap D) = Pr(a : C) + Pr(a : D) - Pr(a : C \sqcup D)$
- For all $a \in N_I, C, D \in \mathbf{C}$:
  $Pr(a : C \sqcap D) \leq \min(Pr(a : C), Pr(a : D))$
- For all $a \in N_I, C, D \in \mathbf{C}$:
  $Pr(a : C \sqcap D) \geq \max(0, Pr(a : C) + Pr(a : D) - 1)$
- For all $a \in N_I, C, D \in \mathbf{C}$:
  $Pr(a : C \sqcup D) = Pr(a : C) + Pr(a : D) - Pr(a : C \sqcap D)$
- For all $a \in N_I, C, D \in \mathbf{C}$:
  $Pr(a : C \sqcup D) \leq \min(1, Pr(a : C) + Pr(a : D))$
- For all $a \in N_I, C, D \in \mathbf{C}$:
  $Pr(a : C \sqcup D) \geq \max(Pr(a : C), Pr(a : D))$
- For all $a \in N_I, C \in \mathbf{C}$:
  $Pr(a : \neg C) = 1 - Pr(a : C)$
- For all $a \in N_I, C \in \mathbf{C}, R \in \mathbf{R}$:
  $Pr(a : \forall R.C) \leq \min_{x \in N_I}(1 + Pr(x : C) - Pr((a, x) : R))$
- For all $a \in N_I, C \in \mathbf{C}, R \in \mathbf{R}$:
  $Pr(a : \forall R.C) \geq 1 + \sum_{x \in N_I} (\max(Pr(x : C), 1 - Pr((a, x) : R)) - 1)$
- For all $a \in N_I, C \in \mathbf{C}, R \in \mathbf{R}$:
  $Pr(a : \exists R.C) = 1 - Pr(a : \forall R.(\neg C))$

For other types of uncertain logics, reasoning is also done in a different way. For further readings see [3].

In the following last example we will show some simple inference for $Pr(\mathtt{b} : \mathtt{B})$ that could be done for example 7 without computing the solution of the linear system of equations.

Example 10

Let (N_R, N_C, N_I) be the elementary description and $\mathcal{K}_\mathcal{U} = (\mathcal{T}_\mathcal{U}, \mathcal{A}_\mathcal{U})$ be the knowledge base from example 7. Then the axiom $Pr(\mathtt{b} : \mathtt{B}) \geq 0.1$ is inferable.

By using the inference rules from theorem 3, we know that

$$\begin{aligned} 0.2 &= Pr(\mathtt{A} \sqsubseteq \mathtt{B}) \\ &\leq \min_{x \in N_I}(Pr(x : \mathtt{B} \sqcup \neg\mathtt{A})) \\ &\leq Pr(\mathtt{b} : \mathtt{B} \sqcup \neg\mathtt{A}) \end{aligned}$$

$$\leq \min(1, Pr(\texttt{b} : \texttt{B}) + Pr(\texttt{b} : \neg\texttt{A}))$$
$$\leq Pr(\texttt{b} : \texttt{B}) + Pr(\texttt{b} : \neg\texttt{A})$$

and on the other hand we also know

$$\begin{aligned} 1 &= Pr(\texttt{b} : \texttt{B} \sqcup \texttt{A}) \\ &\leq \min(1, Pr(\texttt{b} : \texttt{B}) + Pr(\texttt{b} : \texttt{A})) \\ &\leq Pr(\texttt{b} : \texttt{B}) + Pr(\texttt{b} : \texttt{A}) \\ &= Pr(\texttt{b} : \texttt{B}) + 1 - Pr(\texttt{b} : \neg\texttt{A}) \end{aligned}$$

which implies $Pr(\texttt{b} : \neg\texttt{A}) \leq Pr(\texttt{b} : \texttt{B})$. Thus it is $0.2 \leq Pr(\texttt{b} : \texttt{B}) + Pr(\texttt{b} : \neg\texttt{A}) \leq 2 \cdot Pr(\texttt{b} : \texttt{B})$ and this leads to the result $Pr(\texttt{b} : \texttt{B}) \geq 0.1$ by halving both sides.

Notice that by solving the linear system of equations we get $Pr(\texttt{b} : \texttt{B}) = 0.9$, which is much more informative than the presented result in example 10. But the computation that leads to that result is way easier.

4 Vagueness Theory

This section describes a different approach. Again, each axiom gets assigned a value between 0 and 1. But this time it is interpreted in a different way. The value doesn't state the probability of the axiom to be true or false, but the axiom is imprecise itself.

In a vague description logic almost everything can be adapted from classical description logic. There is a set of axioms that is interpreted by an interpretation function which leads to a model of a knowledge base. A knowledge base is consistent, if there exists at least one model for it. And inference is possible if every model of the knowledge base also models the inferred axiom. The only difference is that the interpretation doesn't map to a boolean value, but to a numeric value between 0 and 1.

Therefore it is not intuitively clear how calculations on boolean values could be extended to numeric values. Typically, there are four operators that define a boolean logic.

	$a =$ true $b =$ true	$a =$ true $b =$ false	$a =$ false $b =$ true	$a =$ false $b =$ false
$a \sqcap b$	true	false	false	false
$a \sqcup b$	true	true	true	false
$a \sqsubseteq b$	true	false	true	true
$\neg a$	false	false	true	true

These operators can be extended to numeric values. We denote them with small circles. For each operator, there should at least true correspond to 1 and false correspond to 0, i.e. the following equivalent table should be fulfilled by the operators.

	$a=1$ $b=1$	$a=1$ $b=0$	$a=0$ $b=1$	$a=0$ $b=0$
$a \otimes b$	1	0	0	0
$a \oplus b$	1	1	1	0
$a \rhd b$	1	0	1	1
$\ominus a$	0	0	1	1

There is no way to define such operators in a way that usual behaviors of boolean variables are extendable to the numeric values in-between 0 and 1. For example, it is impossible that $\ominus \ominus x = x$ (Double Negation), $x \leq y \implies \ominus y \leq \ominus x$ (monotonically decreasing Negation), $x \oplus x = x$ (Tautology) and $\ominus x \oplus x = 1$ (Law of Excluded Middle) occur simultaneously. [17]

Each definition for these operators leads to a different logic, there are useful ones for several applications. We will only introduce the Goedel Logic in this paper, which is defined in the following way.

Definition 13: operators

For values $a, b \in [0, 1]$, the Goedel operators $\otimes$, $\oplus$, $\rhd$ and $\ominus$ are defined in the following way.

$$a \otimes b = \min(a, b)$$
$$a \oplus b = \max(a, b)$$
$$a \rhd b = \begin{cases} 1 & a \leq b \\ b & a > b \end{cases}$$
$$\ominus a = \begin{cases} 1 & a = 0 \\ 0 & a > 0 \end{cases}$$

The following definition of a vague interpretation would be the same with other valid operators.

Definition 14: vague interpretation

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description.
A function

$$\cdot^{\mathcal{I}_\mathcal{V}} : \mathbf{R} \cup \mathbf{C} \cup N_I \to \left\{\left(\Delta^{\mathcal{I}_\mathcal{V}} \times \Delta^{\mathcal{I}_\mathcal{V}}\right) \to [0,1]\right\} \cup \left\{\left(\Delta^{\mathcal{I}_\mathcal{V}}\right) \to [0,1]\right\} \cup \Delta^{\mathcal{I}_\mathcal{V}}$$

together with an arbitrary set $\Delta^{\mathcal{I}_\mathcal{V}}$ is called a vague interpretation if and only if

- $R^{\mathcal{I}_\mathcal{V}} \in \left\{\left(\Delta^{\mathcal{I}_\mathcal{V}} \times \Delta^{\mathcal{I}_\mathcal{V}}\right) \to [0,1]\right\}$ for all $R \in \mathbf{R}$
- $C^{\mathcal{I}_\mathcal{V}} \in \left\{\left(\Delta^{\mathcal{I}_\mathcal{V}}\right) \to [0,1]\right\}$ for all $C \in \mathbf{C}$
- $a^{\mathcal{I}_\mathcal{V}} \in \Delta^{\mathcal{I}_\mathcal{V}}$ for all $a \in N_I$

and the following conditions hold for every $R \in \mathbf{R}$, $C, D \in \mathbf{C}$ and $A = \{a_1, \ldots, a_n\} \subseteq N_I$ with any $x, y \in \Delta^{\mathcal{I}_\mathcal{V}}$ and $a \in N_I$.

(universal role)	$U^{\mathcal{I}_\mathcal{V}}(x,y) = 1$
(top concept)	$\top^{\mathcal{I}_\mathcal{V}}(x) = 1$
(bottom concept)	$\bot^{\mathcal{I}_\mathcal{V}} = 0$
(set of individuals)	$A^{\mathcal{I}_\mathcal{V}}(a) = \begin{cases} 1, & a \in A \\ 0, & a \notin A \end{cases}$
(conjunction)	$(C \sqcap D)^{\mathcal{I}_\mathcal{V}}(x) = C^{\mathcal{I}_\mathcal{V}}(x) \otimes D^{\mathcal{I}_\mathcal{V}}(x)$
(disjunction)	$(C \sqcup D)^{\mathcal{I}_\mathcal{V}}(x) = C^{\mathcal{I}_\mathcal{V}}(x) \oplus D^{\mathcal{I}_\mathcal{V}}(x)$
(negation)	$(\neg C)^{\mathcal{I}_\mathcal{V}}(x) = \ominus C^{\mathcal{I}_\mathcal{V}}(x)$
(universal restriction)	$(\forall R.C)^{\mathcal{I}_\mathcal{V}}(x) = \min_{y \in \Delta^{\mathcal{I}_\mathcal{V}}}\left(r^{\mathcal{I}_\mathcal{V}}(x,y) \rhd C^{\mathcal{I}_\mathcal{V}}(y)\right)$
(existential restriction)	$(\exists R.C)^{\mathcal{I}_\mathcal{V}}(x) = \max_{y \in \Delta^{\mathcal{I}_\mathcal{V}}}\left(r^{\mathcal{I}_\mathcal{V}}(x,y) \otimes C^{\mathcal{I}_\mathcal{V}}(y)\right)$

A tuple $\mathcal{I}_\mathcal{V} = \left(\Delta^{\mathcal{I}_\mathcal{V}}, \cdot^{\mathcal{I}_\mathcal{V}}\right)$ is called a vague interpretation over (N_R, N_C, N_I) with domain $\Delta^{\mathcal{I}_\mathcal{V}}$.

We will now construct a vague interpretation that is equivalent to the interpretation of example 2. Again we will assume that $\Delta^{\mathcal{I}_\mathcal{V}} = N_I$ and will not take the universe of discourse into account, but only the named individuals instead.

Example 11

Given the following elementary description.

$$N_R = \{\mathtt{r}\}$$
$$N_C = \{\mathtt{A}, \mathtt{B}\}$$
$$N_I = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$$

A vague interpretation $\mathcal{I}_\mathcal{V} = \left(\Delta^{\mathcal{I}_\mathcal{V}}, \cdot^{\mathcal{I}_\mathcal{V}}\right)$ over (N_R, N_C, N_I) could be defined by the following mappings.

$$\begin{array}{rrr} \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{a},\mathtt{a}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{a},\mathtt{b}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{a},\mathtt{c}) = 1 \\ \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{b},\mathtt{a}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{b},\mathtt{b}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{b},\mathtt{c}) = 0 \\ \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{c},\mathtt{a}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{c},\mathtt{b}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{c},\mathtt{c}) = 1 \\ \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{a}) = 1 & \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 0 & \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{c}) = 1 \\ \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{a}) = 1 & \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 0 & \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{c}) = 0 \end{array}$$

And again, the values for composite concepts are already determined by the definition of the vague interpretation on the elementary description. For the composite concepts of example 3, we get the following results.

$$\begin{aligned} (\mathtt{A} \sqcap \mathtt{B})^{\mathcal{I}_\mathcal{V}}(x) &= \mathtt{A}^{\mathcal{I}_\mathcal{V}}(x) \otimes \mathtt{B}^{\mathcal{I}_\mathcal{V}}(x) \\ &= \begin{cases} 1 \otimes 1 = \min(1,1) = 1 & x = \mathtt{a} \\ 0 \otimes 0 = \min(0,0) = 0 & x = \mathtt{b} \\ 1 \otimes 0 = \min(1,0) = 0 & x = \mathtt{c} \end{cases} \end{aligned}$$

$$\begin{aligned} (\mathtt{B} \sqcup \neg\mathtt{A})^{\mathcal{I}_\mathcal{V}}(x) &= \mathtt{B}^{\mathcal{I}_\mathcal{V}}(x) \oplus \left(\ominus \mathtt{A}^{\mathcal{I}_\mathcal{V}}(x)\right) \\ &= \begin{cases} 1 \oplus (\ominus 1) = \max(1,0) = 1 & x = \mathtt{a} \\ 0 \oplus (\ominus 0) = \max(0,1) = 1 & x = \mathtt{b} \\ 0 \oplus (\ominus 1) = \max(0,0) = 0 & x = \mathtt{c} \end{cases} \end{aligned}$$

$$\begin{aligned} (\exists \mathtt{r}.\mathtt{A})^{\mathcal{I}_\mathcal{V}}(x) &= \max_{y \in N_I} \left(r^{\mathcal{I}_\mathcal{V}}(x,y) \otimes \mathtt{A}^{\mathcal{I}_\mathcal{V}}(y)\right) \\ &= \begin{cases} \max(0 \otimes 1, 0 \otimes 0, 1 \otimes 1) = \max(0,0,1) = 1 & x = \mathtt{a} \\ \max(0 \otimes 1, 0 \otimes 0, 0 \otimes 1) = \max(0,0,0) = 0 & x = \mathtt{b} \\ \max(0 \otimes 1, 0 \otimes 0, 1 \otimes 1) = \max(0,0,1) = 1 & x = \mathtt{c} \end{cases} \end{aligned}$$

$$\begin{aligned} (\forall \mathtt{r}.(\{\mathtt{a}\} \sqcap \mathtt{B}))^{\mathcal{I}_\mathcal{V}}(x) &= \min_{y \in N_I} \left(r^{\mathcal{I}_\mathcal{V}}(x,y) \triangleright \left(\{\mathtt{a}\}^{\mathcal{I}_\mathcal{V}}(y) \oplus \mathtt{B}^{\mathcal{I}_\mathcal{V}}(y)\right)\right) \\ &= \begin{cases} \min(0 \triangleright (1 \oplus 1), 0 \triangleright (0 \oplus 0), 1 \triangleright (0 \oplus 0)) = 0 & x = \mathtt{a} \\ \min(0 \triangleright (1 \oplus 1), 0 \triangleright (0 \oplus 0), 0 \triangleright (0 \oplus 0)) = 1 & x = \mathtt{b} \\ \min(0 \triangleright (1 \oplus 1), 0 \triangleright (0 \oplus 0), 1 \triangleright (0 \oplus 0)) = 0 & x = \mathtt{c} \end{cases} \end{aligned}$$

We will now continue with the definition of a vague knowledge base. A vague knowledge base consists of axioms as before. The axioms are extended by a value, that states the degree of truth of an axiom. Notice that it is not the probability of an axiom to be true, but the intensity or precision of the axioms statement.

Definition 15: vague axiom

Let $\mathbf{C}$ be a set of concepts. For every two concepts $C, D \in \mathbf{C}$ and every $p \in [0,1]$, the expression $C \sqsubseteq D = p$ is called a vague concept inclusion (VCI). A vague TBox, denoted by $\mathcal{T}_{\mathcal{V}}$, is a finite set of VCIs.

$$\mathcal{T}_{\mathcal{V}} \subset \{C \sqsubseteq D = p \mid C, D \in \mathbf{C}, p \in [0,1]\}$$

Each axiom $\Phi \in \mathcal{T}_{\mathcal{V}}$ is called a vague terminological axiom.

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles and $\mathbf{C}$ be the set of concepts over that elementary description. For every individual $a \in N_I$, every concept $C \in \mathbf{C}$ and every $p \in [0,1]$, the expression $C(a) = p$ is called a vague concept assertion. For every two individuals $a, b \in N_I$, every role $R \in \mathbf{R}$ and every $p \in [0,1]$, the expression $R(a,b) = p$ is called a vague role assertion. A vague ABox, denoted by $\mathcal{A}_{\mathcal{V}}$, is a finite set of vague concept and role assertions.

$$\begin{aligned}\mathcal{A}_{\mathcal{V}} \subset &\{C(a) = p \mid a \in N_I, C \in \mathbf{C}, p \in [0,1]\} \cup \\ &\{R(a,b) = p \mid a, b \in N_I, R \in \mathbf{R}, p \in [0,1]\}\end{aligned}$$

Each axiom $\phi \in \mathcal{A}$ is called a vague assertional axiom.

A tuple $\mathcal{K}_{\mathcal{V}} = (\mathcal{T}_{\mathcal{V}}, \mathcal{A}_{\mathcal{V}})$ is called a vague knowledge base (KB), if and only if it consists of a set of vague terminological axioms $\mathcal{T}_{\mathcal{U}}$ and a set of vague assertional axioms $\mathcal{A}_{\mathcal{U}}$.

Since the vague interpretation highly depends on the choice of the operators, it is a rather difficult task for an author of a knowledge base to state a precise value for the degree of precision of an axiom. Therefore most vague logics implement a fuzzy model that allows to formulate imprecise statements with a fuzzy degree of that imprecision. But in this paper we will only allow statements with a single value between 0 and 1 for the vagueness.

Example 12

Let (N_R, N_C, N_I) be the following elementary description.

$$\begin{aligned}N_R &= \{\mathtt{r}\} \\ N_C &= \{\mathtt{A}, \mathtt{B}\} \\ N_I &= \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}\end{aligned}$$

Then this is a vague knowledge base.

$$
\begin{aligned}
\mathcal{T}_\mathcal{V} = \{ & \\
& \neg\mathtt{B} \sqsubseteq \exists\mathtt{r}.\top = 1, \\
& \exists\mathtt{r}.\{\mathtt{a},\mathtt{b}\} \sqsubseteq \bot = 1, \\
& \mathtt{A} \sqsubseteq \mathtt{B} = 0.2 \\
& \} \\
\mathcal{A}_\mathcal{V} = \{ & \\
& (\mathtt{A} \sqcap \mathtt{B})(\mathtt{a}) = 1, \\
& (\mathtt{A} \sqcup \mathtt{B})(\mathtt{b}) = 1, \\
& (\forall\mathtt{r}.\mathtt{A})(\mathtt{c}) = 1, \\
& \mathtt{B}(\mathtt{c}) = 0.3, \\
& \mathtt{r}(\mathtt{a},\mathtt{c}) = 1, \\
& \mathtt{r}(\mathtt{b},\mathtt{c}) = 0.1, \\
& \mathtt{r}(\mathtt{c},\mathtt{c}) = 1 \\
& \} \\
\mathcal{K}_\mathcal{V} = (\, & \mathcal{T}_\mathcal{V}, \mathcal{A}_\mathcal{V} \,)
\end{aligned}
$$

We will now define what consistency means for a vague knowledge base. The definition is very similar to classical description logic. It is extended so that not only the boolean value must fit to the model, but also the correct value for the vagueness.

Definition 16: vague model

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description. Let $\mathcal{I}_\mathcal{V} = (\Delta^{\mathcal{I}_\mathcal{V}}, \cdot^{\mathcal{I}_\mathcal{V}})$ be a vague interpretation over (N_R, N_C, N_I).

- Let $C, D \in \mathbf{C}$ be two concepts and $p \in [0,1]$.
  In $\mathcal{I}_\mathcal{V}$ holds a vague concept inclusion (denoted $\mathcal{I}_\mathcal{V} \models C \sqsubseteq D = p$)
  if and only if $\min_{x \in \Delta^{\mathcal{I}_\mathcal{V}}} \left(C^{\mathcal{I}_\mathcal{V}}(x) \triangleright D^{\mathcal{I}_\mathcal{V}}(x) \right) = p$.
- Let $\mathcal{T}_\mathcal{V}$ be a vague TBox.
  In $\mathcal{I}_\mathcal{V}$ holds $\mathcal{T}_\mathcal{V}$ (denoted $\mathcal{I}_\mathcal{V} \models \mathcal{T}_\mathcal{V}$)
  if and only if $\mathcal{I}_\mathcal{V} \models \Phi$ for every vague concept inclusion $\Phi \in \mathcal{T}_\mathcal{V}$.
- Let $a \in N_I$ be an individual, $C \in \mathbf{C}$ be a concept and $p \in [0,1]$.
  In $\mathcal{I}_\mathcal{V}$ holds a vague concept assertion (denoted $\mathcal{I}_\mathcal{V} \models C(a)$)
  if and only if $C^{\mathcal{I}_\mathcal{V}}(a^{\mathcal{I}_\mathcal{V}}) = C(a)$.
- Let $a, b \in N_I$ be two individuals, $R \in \mathbf{R}$ be a role and $p \in [0,1]$.

In $\mathcal{I}_\mathcal{V}$ holds a vague role assertion (denoted $\mathcal{I}_\mathcal{V} \models R(a,b)$)
if and only if $R^{\mathcal{I}_\mathcal{V}}(a^{\mathcal{I}_\mathcal{V}}, b^{\mathcal{I}_\mathcal{V}}) = R(a,b)$.

- Let $\mathcal{A}_\mathcal{V}$ be a vague ABox.
  In $\mathcal{I}_\mathcal{V}$ holds $\mathcal{A}_\mathcal{V}$ (denoted $\mathcal{I}_\mathcal{V} \models \mathcal{A}_\mathcal{V}$)
  if and only if $\mathcal{I}_\mathcal{V} \models \phi$ for every vague concept and role assertion $\phi \in \mathcal{A}$.

- Let $\mathcal{K}_\mathcal{V} = (\mathcal{T}_\mathcal{V}, \mathcal{A}_\mathcal{V})$ be a knowledge base.
  In $\mathcal{I}_\mathcal{V}$ holds the knowledge base (denoted $\mathcal{I}_\mathcal{V} \models \mathcal{K}_\mathcal{V}$)
  if and only if $\mathcal{I}_\mathcal{V} \models \mathcal{T}_\mathcal{V}$ and $\mathcal{I}_\mathcal{V} \models \mathcal{A}_\mathcal{V}$.

If $\mathcal{I}_\mathcal{V} \models \mathcal{K}_\mathcal{V}$, then $\mathcal{I}_\mathcal{V}$ is called a model of $\mathcal{K}_\mathcal{V}$. If a vague knowledge base $\mathcal{K}_\mathcal{V}$ is given and there exists at least one vague interpretation $\mathcal{I}_\mathcal{V}$ such that $\mathcal{I}_\mathcal{V} \models \mathcal{K}_\mathcal{V}$, then $\mathcal{K}_\mathcal{V}$ is called consistent.

The vague interpretation $\mathcal{I}_\mathcal{V}$ from example 11 is not a model of the knowledge base $\mathcal{K}$ from example 12. One can easily see this by looking at the axiom $(\mathtt{A} \sqcup \mathtt{B})(\mathtt{b}) = 1$. Since $\mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 0$ and $\mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 0$, it is $(\mathtt{A} \sqcup \mathtt{B})^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \oplus \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 0 \oplus 0 = 0 \neq 1$.

We will now give an example of another vague interpretation that is a model of the knowledge base from example 12.

Example 13

Let (N_R, N_C, N_I) be the elementary description and $\mathcal{K}_\mathcal{V} = (\mathcal{T}_\mathcal{V}, \mathcal{A}_\mathcal{V})$ be the knowledge base from example 12. Let now $\mathcal{I}_\mathcal{V}$ be a vague interpretation over (N_R, N_C, N_I) with the following mappings.

$$\begin{array}{rrr}
\mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{a},\mathtt{a}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{a},\mathtt{b}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{a},\mathtt{c}) = 1 \\
\mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{b},\mathtt{a}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{b},\mathtt{b}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{b},\mathtt{c}) = 0.1 \\
\mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{c},\mathtt{a}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{c},\mathtt{b}) = 0 & \mathtt{r}^{\mathcal{I}_\mathcal{V}}(\mathtt{c},\mathtt{c}) = 1 \\
\mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{a}) = 1 & \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 1 & \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{c}) = 1 \\
\mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{a}) = 1 & \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) = 0.2 & \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{c}) = 0.3
\end{array}$$

We will now show, that $\mathcal{I}_\mathcal{V} \models \mathcal{K}_\mathcal{V}$ by showing that $\mathcal{I}_\mathcal{V} \models \Phi$ for each vague terminological axiom $\Phi \in \mathcal{T}_\mathcal{V}$ and then showing that $\mathcal{I}_\mathcal{V} \models \phi$ for each vague assertional axiom $\phi \in \mathcal{A}_\mathcal{V}$.

$$\begin{aligned}
&\min_{x \in N_I} \left((\neg \mathtt{B})^{\mathcal{I}_\mathcal{V}}(x) \rhd (\exists \mathtt{r}.\top)^{\mathcal{I}_\mathcal{V}}(x) \right) \\
&\quad = \min_{x \in N_I} \left(\ominus \mathtt{B}^{\mathcal{I}_\mathcal{V}}(x) \rhd \max_{y \in N_I} \left(\mathtt{r}^{\mathcal{I}_\mathcal{V}}(x,y) \otimes \top^{\mathcal{I}_\mathcal{V}}(y) \right) \right) \\
&\quad = \min_{x \in N_I} \left(\ominus \mathtt{B}^{\mathcal{I}_\mathcal{V}}(x) \rhd \mathtt{r}^{\mathcal{I}_\mathcal{V}}(x, \mathtt{c}) \right) \\
&\quad = \min \left(\ominus 1 \rhd 1, \ominus 0.2 \rhd 0.1, \ominus 0.3 \rhd 1 \right)
\end{aligned}$$

$$
\begin{aligned}
&= \min(1,1,1)\\
&= 1\\
&\min_{x\in N_I}\left(\left(\exists \texttt{r}.\{\texttt{a},\texttt{b}\}\right)^{\mathcal{I}_\mathcal{V}}(x) \rhd \bot^{\mathcal{I}_\mathcal{V}}(x)\right)\\
&= \min_{x\in N_I}\left(\max_{y\in N_I}\left(\texttt{r}^{\mathcal{I}_\mathcal{V}}(x,y)\otimes\{\texttt{a},\texttt{b}\}^{\mathcal{I}_\mathcal{V}}(y)\right)\rhd \bot^{\mathcal{I}_\mathcal{V}}(x)\right)\\
&= \min(0\rhd 0, 0\rhd 0, 0\rhd 0)\\
&= 0\\
&\min_{x\in N_I}\left(\texttt{A}^{\mathcal{I}_\mathcal{V}}(x)\rhd \texttt{B}^{\mathcal{I}_\mathcal{V}}(x)\right)\\
&= \min(1\rhd 1, 1\rhd 0.2, 1\rhd 0.3)\\
&= 0.2
\end{aligned}
$$

$$
\begin{aligned}
(\texttt{A}\sqcap\texttt{B})^{\mathcal{I}_\mathcal{V}}(\texttt{a}) &= \texttt{A}^{\mathcal{I}_\mathcal{V}}(\texttt{a})\otimes\texttt{B}^{\mathcal{I}_\mathcal{V}}(\texttt{a}) = 1\otimes 1 = 1\\
(\texttt{A}\sqcup\texttt{B})^{\mathcal{I}_\mathcal{V}}(\texttt{b}) &= \texttt{A}^{\mathcal{I}_\mathcal{V}}(\texttt{b})\oplus\texttt{B}^{\mathcal{I}_\mathcal{V}}(\texttt{b}) = 1\oplus 0.2 = 1\\
(\forall\texttt{r}.\texttt{A})^{\mathcal{I}_\mathcal{V}}(\texttt{c}) &= \min_{y\in\Delta^{\mathcal{I}_\mathcal{V}}}\left(r^{\mathcal{I}_\mathcal{V}}(\texttt{c},y)\rhd\texttt{A}^{\mathcal{I}_\mathcal{V}}(y)\right) = \min(0\rhd 1, 0\rhd 1, 1\rhd 1) = 1\\
\texttt{B}^{\mathcal{I}_\mathcal{V}}(\texttt{c}) &= 0.3\\
\texttt{r}^{\mathcal{I}_\mathcal{V}}(\texttt{a},\texttt{c}) &= 1\\
\texttt{r}^{\mathcal{I}_\mathcal{V}}(\texttt{b},\texttt{c}) &= 0.1\\
\texttt{r}^{\mathcal{I}_\mathcal{V}}(\texttt{c},\texttt{c}) &= 1
\end{aligned}
$$

This is also the only vague interpretation that models the given vague knowledge base in Goedel logic. For some other logics, such as Lukasiewicz logic, the knowledge base is not even consistent. We don't show it here, but want to strengthen again that vague logic highly depends on the chosen logic and this has to be taken into account when modeling a knowledge base.

Nevertheless, the other task for knowledge bases, the inference of new knowledge, exists in vagueness theory, too. It works almost like in classical description logic [4].

Definition 17: inference for vague knowledge bases

Let (N_R, N_C, N_I) be an elementary description. Let $\mathbf{R}$ be the set of roles over that elementary description and let $\mathbf{C}$ be the set of concepts over that elementary description. Let $\mathcal{K}_\mathcal{V} = (\mathcal{T}_\mathcal{V}, \mathcal{A}_\mathcal{V})$ be a consistent vague knowledge base.

A vague terminological axiom $\Phi \in \{C \sqsubseteq D = p \mid C, D \in \mathbf{C}, p \in [0,1]\}$ is a logical consequence of $\mathcal{K}_\mathcal{V}$, denoted by $\mathcal{K}_\mathcal{V} \models \Phi$ if and only if for all $\mathcal{I}_\mathcal{V} \models \mathcal{K}_\mathcal{V}$ also $\mathcal{I}_\mathcal{V} \models (\mathcal{T}_\mathcal{V} \cup \{\Phi\}, \mathcal{A}_\mathcal{V})$.

A vague assertional axiom $\phi \in \{C(a) = p \mid a \in N_I, C \in \mathbf{C}, p \in [0,1]\} \cup \{R(a,b) = p \mid a, b \in N_I, C \in \mathbf{C}, p \in [0,1]\}$ is a logical consequence of $\mathcal{K}_\mathcal{V}$, denoted by $\mathcal{K}_\mathcal{V} \models \phi$, if and only if for all $\mathcal{I}_\mathcal{V} \models \mathcal{K}_\mathcal{V}$ also $\mathcal{I}_\mathcal{V} \models (\mathcal{T}_\mathcal{V}, \mathcal{A}_\mathcal{V} \cup \{\phi\})$.

In the last example we present axioms that are inferable from example 12.

Example 14

Let (N_R, N_C, N_I) be the elementary description and $\mathcal{K}_\mathcal{V} = (\mathcal{T}_\mathcal{V}, \mathcal{A}_\mathcal{V})$ be the knowledge base from example 12. Then the following axiom is inferable.

$$\mathtt{B}(\mathtt{b}) = 0.2$$

This logical consequence is obvious when looking at the axiom $\mathtt{A} \sqsubseteq \mathtt{B} = 0.2$.

$$\begin{aligned}
0.2 &= \min_{x \in N_I} \left(\mathtt{A}^{\mathcal{I}_\mathcal{V}}(x) \rhd \mathtt{B}^{\mathcal{I}_\mathcal{V}}(x)\right) \\
&= \min\left(\mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{a}) \rhd \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{a}), \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \rhd \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}), \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{c}) \rhd \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{c})\right) \\
&= \min\left(1 \rhd 1, \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \rhd \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}), 1 \rhd 0.3\right) \\
&= \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \rhd \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \\
&= \begin{cases} 1 & \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \leq \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \\ \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) & \mathtt{A}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) > \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b}) \end{cases} \\
&= \mathtt{B}^{\mathcal{I}_\mathcal{V}}(\mathtt{b})
\end{aligned}$$

5 Conclusion

We demonstrated three types of logic. The classical description logic $\mathcal{ALC}$ can manage knowledge up to a certain degree, the amount of composite concepts and roles is very limited, but it is easy to handle. We extended $\mathcal{ALC}$ by the management of uncertainty and vagueness respectively. For both, uncertainty and vagueness theory, we chose the easiest possible way to extend the knowledge base by assigning only a single value between 0 and 1 to each axiom. We showed what consistency means in each logic and how new knowledge could be inferred. We hinted to further extensions of those logics that allow to store and manage knowledge with higher complexity. There are also several algorithms to solve or ease certain problems.

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