States of Convex Sets

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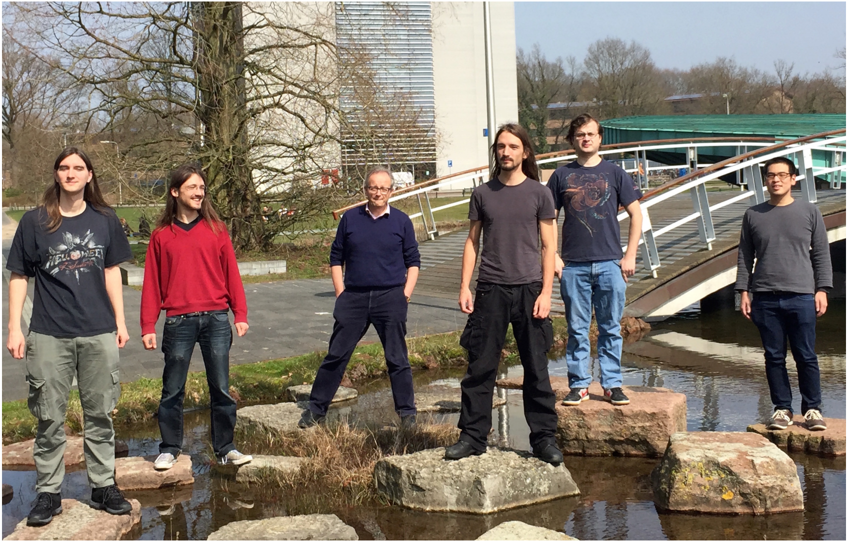
The categorical quantum logic group in Nijmegen



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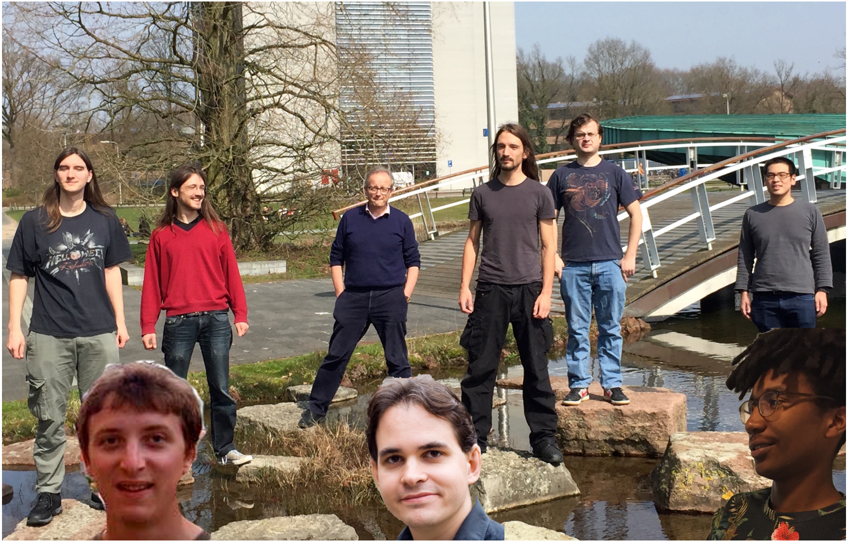

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In contrast to the friendly competition at Oxford: they emphasize to axiomatize what is unique and non-classical about quantum mechanics.

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   but we'll come to that!

Oxford & Nijmegen

Setting

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Sets : $\mathcal{K}\ell(\mathcal{D})$: $\mathbf{vN}^{\mathsf{op}}$

Setting

Classical	:	Probabilistic	:	Quantum
Sets	:	$\mathcal{K}\ell(\mathcal{D})$	:	$\mathbf{vN}^{\mathrm{op}}$
sets with maps		sets with probabilistic maps		von Neumann algebras with c.p. unital normal linear maps

Logic?

	Sets	$\mathcal{K}\ell(\mathcal{D})$	$\mathbf{vN}^{\mathrm{op}}$
	classical	probabilistic	quantum
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effectus*	✓	✓	✓

* see next page

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- these diagrams are pullbacks:

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(Rather weak assumptions!)

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$1 \xrightarrow{\lambda} 1+1$	scalar

Examples of states and predicates

State	Predicate	Validity	Scalars
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Examples of operatorions on states and predicates

- Negation of predicate: $$\begin{tikzcd} X \arrow[r, "p"] \arrow[rr, bend right, "\neg p"'] & 1+1 \arrow[r, "{[\kappa_2,\kappa_1]}"] & 1+1 \end{tikzcd}$$

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- Predicates p, q are summable whenever there is a b such that

$$\begin{tikzcd} & X \arrow[dl, "p"'] \arrow[d, "b"'] \arrow[dr, "q"] & \\ 1+1 & 1+1+1 \arrow[l, "{[\kappa_1,\kappa_2,\kappa_2]}"] \arrow[r, "{[\kappa_2,\kappa_1,\kappa_2]}"'] & 1+1 \end{tikzcd}$$

and then their sum is given by $p \ovee q = [\kappa_1, \kappa_1, \kappa_2] \circ b$.

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So what? They block treating conditional probability in an effectus.

Cancellative Convex Sets

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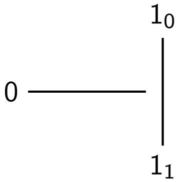

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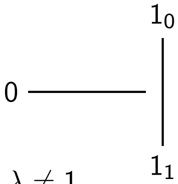

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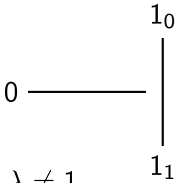

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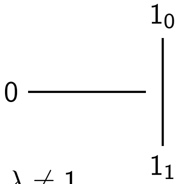

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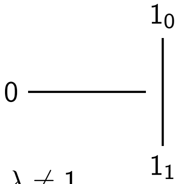

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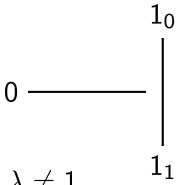

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4. The full subcategory $\mathbf{CConv}_{[0,1]}$ of $\mathbf{Conv}_{[0,1]}$ of cancellative convex sets over $[0,1]$ is an effectus!

Normalisation

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C has normalisation:
For every $1 \overset{\sigma}{\to} X + 1$ with $\sigma \neq \kappa_2$ there is a unique $1 \overset{\omega}{\to} X$ such that the following diagram commutes.

$$\begin{tikzcd}
1 \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & X + 1 \\
X + 1 \arrow[r, "!+\mathrm{id}"'] & 1 + 1 \arrow[u, "\omega+\mathrm{id}"']
\end{tikzcd}$$

Conclusion and references

$$\begin{tikzcd}
\mathbf{EMod}_{[0,1]}^{\mathsf{op}} \arrow[rr, bend left, "\mathrm{Stat}"] \arrow[rr, phantom, "\top"] & & \mathbf{CConv}_{[0,1]} \arrow[ll, bend left, "\mathrm{Pred}"] \\
& \mathbf{C} \arrow[ul, "\mathrm{Pred}"] \arrow[ur, "\mathrm{Stat}"'] &
\end{tikzcd}$$

1. Every category above is an effectus;
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Conclusion and references

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1. Every category above is an effectus;
   every functor above preserves coproducts.
2. For the relation with conditional probability,
   see Section 6 of the paper.
3. For more about effectuses:
   Bart Jacobs, *New Directions in Categorical Logic, [...]*,
   arXiv:1205.3940v3.