A Kochen-Specker system has at least 21 vertices

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A Kochen-Specker system S is a finite set of points on the (open) northern hemisphere, such that there is no 010-coloring; that is: there is no $\{0,1\}$-valued coloring with

1. three pairwise orthogonal points are assigned $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$ and
2. two orthogonal points are not assigned $(1,1)$.

point	$\sim$ direction of magnetic field in measurement of SPIN-1
coloring	$\sim$ non-contextual deterministic theory

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Theorem (Kochen-Specker)

There is a Kochen-Specker system. Thus: there is no non-contextual deterministic theory predicting the measurement of a SPIN-1 particle.

The smallest Kochen-Specker system?

Kochen-Specker	1975	≤ 117
Penrose, Peres (indep.)	1991	
Conway	$\sim$ 1995	
Arends, Wampler, Ouaknine	2009	

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U&W	july?	≥ 22 or $= 21$
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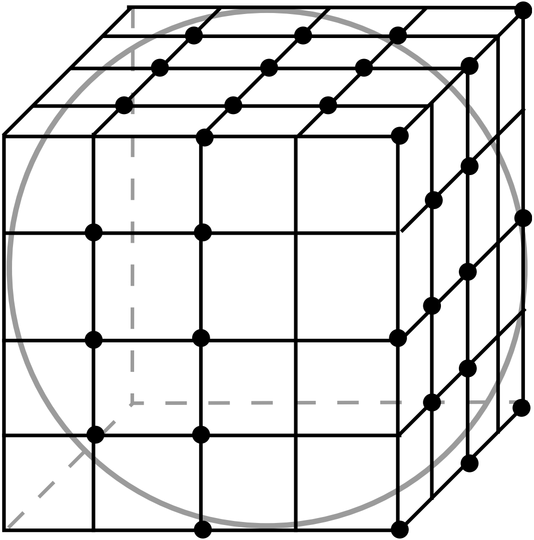
Conway's record



It is a problem about graphs

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A 010-coloring of a graph, is a $\{0, 1\}$-vertex coloring, such that

1. every triangle is colored $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$ and
2. no adjacent vertices are colored both 1.

It is a problem about graphs

There is a Kochen-Specker system with n points
if and only if
there is a embeddable and non-010-colorable graph on n vertices.

Restrict the search

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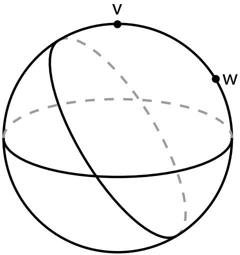

Restrict the search

(The orthogonality graph of) a minimal Kochen-Specker system is

connected;	$\sim 10^{26.4}$
squarefree and	$\sim 10^{10.2}$
has minimal vertex degree 3;	$\sim 10^{7.5}$

The candidates

Our computation found the following number of non 010-colorable squarefree graphs with minimal vertex degree 3

$\#V$	# candidates
≤ 16	0
17	1
18	2
19	19
20	441

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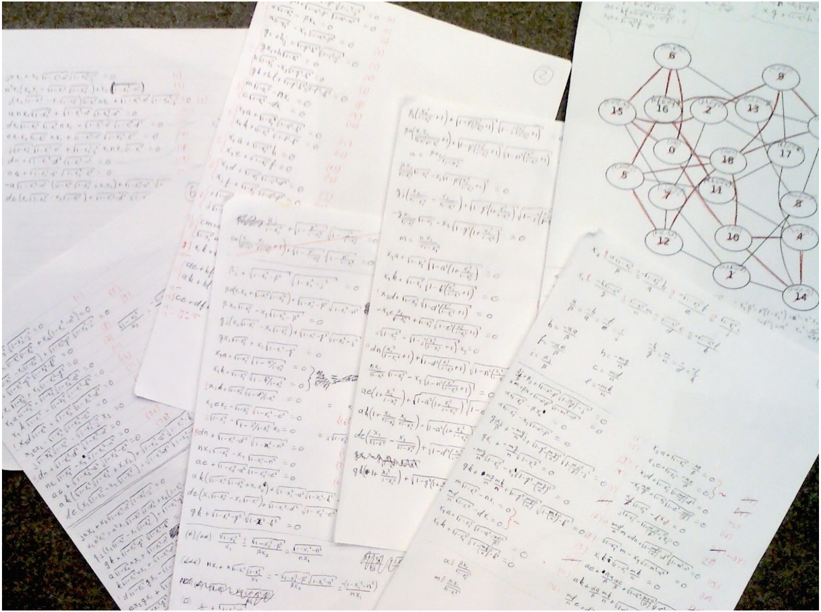
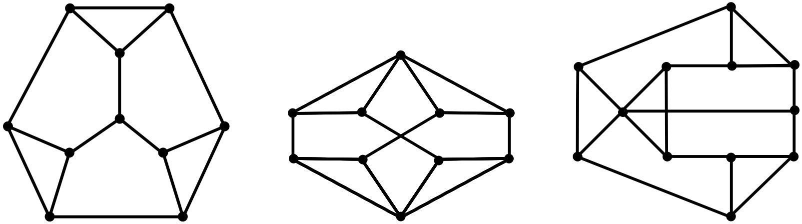
21	≥ 7616

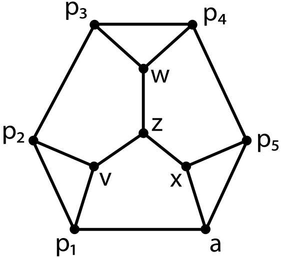


Unembeddable subgraphs

All these candidates contain as a subgraph one of these unembeddable graphs.



Pen and paper proof of unembeddability



Suppose this graph is embeddable.

Note that v and a are distinct points orthogonal to p_1. Thus p_1 is fixed. Observe: p_1 is collinear to $v \times a$.

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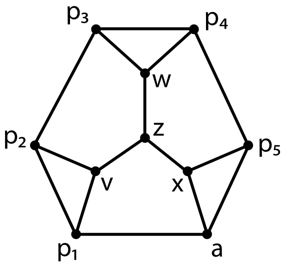

Suppose this graph is embeddable.

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Similarly: p_2 is collinear to $v \times (v \times a)$. And so on. We see a is collinear to $x \times (x \times (w \times (w \times (v \times (v \times a)))))$.

Pen and paper proof of unembeddability

We may assume $z = (0,0,1)$, $x = (1,0,0)$, $v = (v_1, v_2, 0)$, $w = (w_1, w_2, 0)$ and $a = (0, a_2, a_3)$. We have:

$$\begin{pmatrix} 0 \\ a_2 \\ a_3 \end{pmatrix} \text{ is collinear to } \begin{pmatrix} 0 \\ -a_2 v_1 w_2 (v_1 w_1 + v_2 w_2) \\ -a_3 (v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2) \end{pmatrix}$$

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That is:

$$\begin{aligned} v_1 w_2 \langle v, w \rangle &= v_1 w_2 (v_1 w_1 + v_2 w_2) \\ &= v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2 \\ &= (v_1^2 + v_2^2) w_1^2 + (v_1^2 + v_2^2) w_2^2 \\ &= w_1^2 + w_2^2 = 1. \end{aligned}$$

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Since v and w are not collinear, we have by Cauchy-Schwarz $|\langle v, w \rangle| < 1$. Note $|v_1|, |w_2| \leq 1$. Thus: $|v_1 w_2 \langle v, w \rangle| < 1$. Contradiction.

Example of automized cross product chasing

```
load_package redlog;
rlset R;
procedure d(x,y);
    (first x) * (first y) +
    (second x) * (second y) +
    (third x) * (third y);
procedure k(x,y);
    {(second x)*(third y) - (third x)*(second y),
     (third x)*(first y) - (first x)*(third y),
     (first x)*(second y) - (second x)*(first y)};
v0c1 := 1; v0c2 := 0; v0c3 := 0;
v1c1 := 0; v1c2 := 1; v1c3 := 0;
v0 := {v0c1, v0c2, v0c3};
v1 := {v1c1, v1c2, v1c3};
v2 := {v2c1, v2c2, v2c3};
v3 := {v3c1, v3c2, v3c3};
v2c1 := 0;
neq0 := k(v0,k(v3,v1));
```

(snip)

```
neq29 := k(k(k(k(v3,v1),v1),v2),k(k(v3,v0),v3));
phi :=
       (first neq0 neq 0 or
        second neq0 neq 0 or
        third neq0 neq 0) and
```

(snip)

```
       (first neq29 neq 0 or
        second neq29 neq 0 or
        third neq29 neq 0) and
       d(v2,v0) = 0 and
       d(k(k(v3,v0),v3),k(k(k(k(v3,v1),v1),v2),v2)) = 0 and
        true;
rlqe ex(v3c3,
     ex(v3c2,
     ex(v3c1,
     ex(v2c3,
     ex(v2c2,phi)))));
```

Source code, paper and experimental results can be found at

`kochen-specker.info`

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Some open problems:

- If G is embeddable, is there a S such that $G = \mathcal{G}(S)$.
- Is every embeddable graph, grid embeddable? That is: using points of the form $(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}}, \frac{z}{\sqrt{n}})$ for $x, y, z, n \in \mathbb{Z}$.