Variance reduction for Russian-roulette [1]

László Szécsi, László Szirmay-Kalos, Csaba Kelemen

Department of Control Engineering and Information Technology
Technical University of Budapest,
Budapest, Magyar Tudósok krt. 2., H-1117, HUNGARY
Email: szirmay@iit.bme.hu

ABSTRACT

Russian-roulette is one of the most important techniques to compute infinite dimensional integrals in an unbiased way. However, Russian roulette is also responsible for adding large amount of noise. This paper examines Russian roulette and a related problem, the sampling of combined BRDFs and proposes two improvements that can reduce the additional noise of Russian-roulette and random elementary BRDF selection, keeping also the unbiasedness of the method. The first improvement takes advantage that the light transfer is computed on several wavelengths simultaneously, thus the distribution of the energy on the wavelengths should be more precisely taken into account when Russian-roulette is made to terminate the walk or to select randomly from the elementary BRDFs. The second improvement gets rid of the fundamental assumption of Russian roulette that the contribution is zero when the walk is terminated. If we have a better estimation for the incoming radiance at this case, this estimation can be used instead, which can significantly reduce the additional noise.

Keywords: Global illumination, random walks, Russian-roulette, spectral rendering, photon map.

1 Introduction

Random walk global illumination algorithms have to evaluate an infinite sequence of the integrals of the following form:

$$L^r(\vec{x}, \omega) = \int_{\Omega} w(\omega, \vec{x}, \omega') \cdot L^{\text{in}}(\vec{x}, \omega') \, d\omega'$$

where $L^r(\vec{x}, \omega)$ is the radiance reflected at point $\vec{x}$ in direction ω,

$$w(\omega, \vec{x}, \omega') = f_r(\omega', \vec{x}, \omega) \cdot \cos\theta'$$

is the probability density of the light scattering, which equals to the product of the BRDF function f_r and the cosine of angle between direction ω' and the surface normal at the given point. Finally, $L^{\text{in}}(\vec{x}, \omega')$ is the incoming radiance, which equals to the emission and the reflected radiance of that point which is visible from $\vec{x}$ at direction ω'.

When Monte-Carlo integration is used, the integrand is divided and simultaneously multiplied by a probability density $p(\omega')$, converting the integral to an expected value:

$$\int_{\Omega} \frac{w(\omega')}{p(\omega')} \cdot L^{\text{in}}(\omega') \cdot p(\omega') \, d\omega' = E\left[\frac{w(\omega')}{p(\omega')} \cdot L^{\text{in}}(\omega')\right].$$

A Monte-Carlo quadrature would generate random directions in the domain with probability density p and estimate the expected value as an average of the integrand values for these direc-

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tions:

$$E\left[\frac{w(\omega')}{p(\omega')}\cdot L^{\rm in}\right]\approx\frac{1}{M}\sum_{j=1}^{M}\frac{w(\omega'_j)}{p(\omega'_j)}\cdot L^{\rm in}(\omega'_j).$$

The incoming radiance $L^{\rm in}$ is the sum of the emission and the reflected radiance at the visible point. Since the reflected radiance of the visible point should be obtained by a similar integral, the sample process has to be repeated, giving rise to an infinite recursion. One way of avoiding this is to limit the levels and assuming that after d reflections the light transfer can be neglected. This approach distorts the result and makes it biased. Russian roulette [1], on the other hand, solves the problem of infinite recursion providing still asymptotically correct, unbiased results. The solution is based on randomization, that is, the error of considering only finite length samples is converted to a random noise with zero mean. In this way, as the number of samples increases, this error vanishes. Similar randomization happens when we have combined BRDFs, defined as the sum of elementary BRDFs, because we can sample the direction only according to the elementary BRDFs.

This paper examines these randomization problems. In section 1 we review and analyze Russian-roulette and the different possibilities of combined BRDF sampling. Then two improvements are presented. Section 2 proposes to take into account the spectral properties to reduce the noise of the randomization. Section 3, on the other hand, includes a rough estimate of the reflected radiance when the walk is terminated, instead of using zero as happens in the classical method. We show that this simple technique can significantly reduce the variance.

1.1 Russian roulette

Russian roulette further randomizes the integral quadrature corresponding to a given reflection and before taking a sample it decides randomly with probability s whether it really evaluates the integrand at the sample point or simply assumes that the integrand is zero without any calculations. In order to compensate the not computed terms, when the integrand is really computed, it is divided by probability s. This randomization introduces a new random variable, called *randomized reflected radiance* $L^{\rm rr}$, which is equal to $w/p\cdot L^{\rm in}/s$ if the integrand is evaluated and zero otherwise. The Monte-Carlo quadrature, which provides the estimate as an expected value will still be correct:

$$E[L^{\rm rr}]=s\cdot E\left[L^{\rm rr}\mid \text{evaluated}\right]+$$

$$(1-s)\cdot E\left[L^{\rm rr}\mid \text{not evaluated}\right]=$$

$$s\cdot E\left[\frac{w}{p}\cdot L^{\rm in}\cdot\frac{1}{s}\right]+(1-s)\cdot 0=E\left[\frac{w}{p}\cdot L^{\rm in}\right]=L^r. \tag{1}$$

The variance of the new estimator, on the other hand, is increased:

$$D^2[L^{\rm rr}]=E[(L^{\rm rr})^2]-E^2[L^{\rm rr}]=$$

$$s\cdot E\left[\left(\frac{L^{\rm rr}}{s}\right)^2\right]+(1-s)\cdot 0-E^2[L^{\rm rr}]=$$

$$\left(\frac{1}{s}-1\right)\cdot E\left[\left(\frac{w}{p}\cdot L^{\rm in}\right)^2\right]+D^2\left[\frac{w}{p}\cdot L^{\rm in}\right]. \tag{2}$$

Note that $D^2\left[w/p\cdot L^{\rm in}\right]$ is the variance of the original estimator not using Russian roulette, thus the additional variance of Russian roulette is the first term of equation 2.

Random walk algorithms set the continuation probability s to minimize the fluctuation of the new estimator $w/p\cdot L^{\rm in}/s$. Since there is usually no information about the incoming radiance $L^{\rm in}$, s is set to approximate w/p. If ideal BRDF sampling is used, then p is proportional to w and thus s is the ratio of proportionality. The proportionality ratio can be derived from the fact that p integrates to 1 since it is a real probability density, while w integrates to the albedo of the surface defined by $a(\vec{x},\omega)=\int_\Omega w(\omega,\vec{x},\omega')\,d\omega'$. Thus s is usually set to approximate the local albedo.

1.2 BRDF sampling for materials of multiple reflection type

Practical reflection models incorporate different simple BRDFs. For example, a lot of materials can be well modeled by a sum of diffuse and specular reflections. Methods are available to sample directions according to either the diffuse or the specular BRDF but not for the sum of them.

Fortunately, Russian-roulette can also be extended to handle these cases. Suppose that the scattering probability density is available in the form of a sum of the weights corresponding to elementary BRDFs:

$$w = w_1 + w_2 + \ldots + w_n.$$

Thus the radiance of a single reflection is:

$$L^r = \int_\Omega (w_1 + w_2 + \ldots + w_n) \cdot L^{\text{in}}\ d\omega'.$$

Assume that probability density p_i can be found to mimic an elementary scattering density w_i and these densities should be used to sample the composed integrand. We have two options to attack this problem. Either the integral is decomposed to a sum corresponding to different scattering densities and the terms are sampled separately, or we use the elementary sampling densities to sample the integrand as a whole and combine the results of the estimators. In the following, we review and compare these methods.

1.2.1 Decomposing the integrand

Method 1 decomposes the reflected radiance according to the elementary scattering densities [7]:

$$L^r = \int_\Omega w \cdot L^{\text{in}}\ d\omega' = \sum_{i=1}^{n} \int_\Omega w_i \cdot L^{\text{in}}\ d\omega'.$$

These integrals are then estimated using the probability density that mimics the elementary scattering probability:

$$L^r = \sum_{i=1}^{n} \int_\Omega \frac{w_i}{p_i} \cdot L^{\text{in}} \cdot p_i\ d\omega' = \sum_{i=1}^{n} E\left[\frac{w_i}{p_i} \cdot L^{\text{in}}\right].$$

This sum can also be computed by Monte-Carlo techniques. Let us select the ith term of this sum, i.e. the ith BRDF, with probability s_i and weight the resulting radiance by $1/s_i$ or stop the walk with probability $1 - \sum_i s_i$. Thus the new random variable representing the reflected radiance is

$$L_1^{\text{rr}} = \frac{w_i}{s_i p_i} \cdot L^{\text{in}}$$

if the ith model is used, and 0 if no model is selected.

This is a Monte-Carlo estimation of the sum. According to importance sampling, the variance will be small if $(w_i/p_i) \cdot L^{\text{in}}/s_i$ can be made nearly constant. Since we usually do not have a-priori information about L^{in}, $w_i/(p_i s_i)$ can be made a constant number. Thus to obtain a low-variance estimator, an elementary BRDF should be selected with the probability of its transformed weight w_i/p_i. Unfortunately, this is still not optimal when Russian roulette is also used, since the termination of the walk has zero contribution. Note that the weight w_i/p_i may either be equal or approximate the albedo, thus a low-variance estimator selects an elementary BRDF with the probability of its albedo.

1.2.2 Combination of sampling schemes

We can also suppose that we have different sampling schemes for the incoming direction, thus an estimator can be obtained by using all the techniques and combining their results. Let us use a weighted sum for such combination:

$$\sum_{i=1}^{n} \alpha_i \cdot \int_\Omega \frac{w}{p_i} \cdot L^{\text{in}} \cdot p_i\ d\omega' = \sum_{i=1}^{n} \alpha_i \cdot E\left[\frac{w}{p_i} \cdot L^{\text{in}}\right].$$

where α_i is the weight of the expected value corresponding to density p_i. The expected value is correct if $\sum_{i=1}^{n} \alpha_i$. Again, we can use Monte-Carlo estimation for this sum. Let us select the ith term of this sum, i.e. the ith BRDF, with probability s_i and weight the resulting radiance by $1/s_i$ or stop the walk with probability $1 - \sum_i s_i$. Thus the random reflected radiance variable L^{rr} is

$$\alpha_i \cdot \frac{w}{s_i p_i} \cdot L^{\text{in}}$$

if the ith model is used, and 0 if no model is selected. In this Monte-Carlo estimation of the sum the variance will be small if $\alpha_i \cdot w/p_i \cdot L^{\text{in}}/s_i$ can be made nearly constant. Taking into account the requirement that the sum of weights is 1, we can obtain:

$$\alpha_i = \frac{s_i}{\sum_{j=1}^{n} s_j}.$$

Substituting this into the formula of the reflected radiance, we can obtain the following estimate of method 2:

$$L_2^{\text{rr}} = \frac{w}{p_i \sum_j s_j} \cdot L^{\text{in}}.$$

The method discussed so far applied a static weighting of different techniques. However, it seems worth using a weighting scheme that depends on the generated direction as well [2]. The formal basis of such combination is given by the theory of multiple importance sampling [9]. Multiple importance sampling combines different sampling techniques in a way that their advantages are preserved, i.e. the variance of the combined estimator is smaller than the individual estimators and not far from the optimum. One of such weighting schemes, called the balance heuristic, sets the weights proportional to the probability density of the individual methods. Formally, this means that we always divide with the average density of the combined techniques, no matter which method generated the sample. Thus the random reflected radiance of this method 3 is:

$$L_3^{\mathrm{rr}} = \frac{w}{\sum_{j=1}^{n} s_j p_j} \cdot L^{\mathrm{in}}$$

if the ith model is used, and 0 if no model is selected.

1.2.3 Comparison of the BRDF selection methods

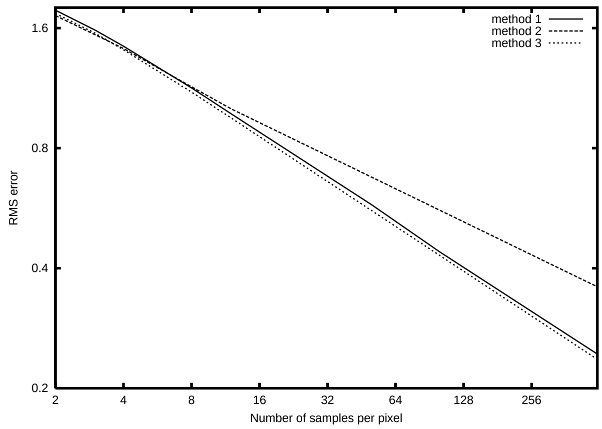

Figure 1: *Comparison of the three random BRDF selection mechanism.*

In figure 1 we compared the error curves of the three discussed method using a path tracing algorithm to render the Cornell Box scene. We can notice that the performances of method 1 and method 3 are similar but method 2 is poorer. The reason of the worse performance of method 2 comes from the fact that we assumed that the probability density of the individual methods are good to sample the integrand in the whole domain. However, the specular sampling cannot sample the whole domain of the diffuse reflection. This problem is solved by method 1 separating the integrand and by method 3 using a mixed probability density.

In the following section, we introduce a spectral optimization that can be used with all mentioned methods. In order to have unified notations, if the ith elementary BRDF is used, then we shall denote the random reflected radiance estimation as

$$L^{\mathrm{rr}} = \frac{W_i}{s_i} \cdot L^{\mathrm{in}}.$$

where factor W_i represents the following weights for three discussed methods, respectively:

$$\frac{w_i}{p_i}, \qquad \frac{ws_i}{p_i \sum_j s_j}, \qquad \frac{ws_i}{\sum_j p_j s_j}.$$

2 Spectral optimization of Russian roulette

In this section we consider the spectral properties of the carried radiance and the scattering density in order to optimize Russian roulette and BRDF selection. The BRDF f_r, the scattering density w and the albedo are not scalar values, since they depend on the wavelength and the light is usually transferred on several (3, 8, 16, etc.) wavelengths simultaneously in global illumination algorithms. On the other hand, the continuation probability s must be scalar, thus the “proportionality” with the albedo should be given a special interpretation. The usual technique is to make the continuation probability proportional to the average (or weighted average) of the albedo values at different wavelengths, and working with the “luminance” of the albedo instead of the wavelength dependent albedo. Unfortunately, this can be very inefficient if the scene has highly saturated colors. Assume, for example, that the simulation is carried out on three wavelengths corresponding to the red, green and blue colors and a blue light source illuminates an ideally reflecting, yellow (red + green) wall. In this case, the reflected radiance on all wavelengths will be zero at the wall, thus it is no use continuing the walk. However, the average of the albedos on the three wavelengths is 2/3, thus Russian roulette will continue the completely useless computation with probability 2/3.

The same problem can occur when the elementary BRDFs are selected randomly. Assume that our surface has diffuse reflection in red and green (the surface is yellow) and the specular reflection in blue. When the surface is lit by blue color, the diffuse BRDF cannot contribute to the reflected light, thus it is not worth selecting. However, if the selection mechanism is based on the average of the reflectances of different wavelengths, it can happen that the irrelevant diffuse BRDF is selected with 2/3 while the relevant specular BRDF only with 1/3 probability.

In this section we propose solutions for the mentioned problems occurring when the light is transferred on multiple wavelengths. We have to take into account the spectral properties of the radiance or importance accumulated up to the given reflection point. In path tracing we start at the eye, walk in the scene generating the continuation direction by BRDF sampling and Russian roulette, then gather the emission (or the reflection of the direct light sources) of the visited point. The found illumination value is multiplied by the product of the scattering densities divided by the sampling probability densities and the continuation probabilities. Thus the weight of an illumination value at the mth reflection is:

$$F = \frac{W[1]}{s[1]} \circ \frac{W[2]}{s[2]} \circ \ldots \circ \frac{W[m]}{s[m]},$$

where $W[k]$ is the weight at the kth reflection and $\circ$ is the diadic product:

$$[a^{(1)}, \ldots, a^{(l)}] \circ [b^{(1)}, \ldots, b^{(l)}] =$$

$$[a^{(1)} \cdot b^{(1)}, \ldots, a^{(l)} \circ b^{(1)}].$$

The result F is also a vector:

$$F = [F^{(1)}, \ldots, F^{(l)}]$$

On the other hand, if we use light tracing the initial point and direction is sampled with probability density p_e and the radiance estimator after the mth reflection is:

$$F = \frac{L^e}{p_e} \circ \frac{W[1]}{s[1]} \circ \frac{W[2]}{s[2]} \circ \ldots \circ \frac{W[m]}{s[m]}.$$

In both gathering and shooting algorithms, at the current reflection we can select from n elementary BRDFs or terminate the walk, with probabilities $s_1, s_2, \ldots, s_n$ and $1 - \sum_i s_i$, respectively. If elementary BRDF i is selected, then the new weight after the reflection is:

$$\left[F^{(1)} \cdot \frac{W_i^{(1)}}{s_i}, \ldots, F^{(l)} \cdot \frac{W_i^{(l)}}{s_i}\right].$$

Using the intuition of importance sampling, we have to make the selection aiming at making the power associated with a ray constant. Since now we have a vector of weights, an appropriate average of the elements can be kept close to constant. Let us use a wavelength dependent additional weighting function h to provide the weighted average of the weights, which can be either constant or follow the visual sensitivity curve. Thus we intend to keep

$$\sum_l F^{(l)} \cdot \frac{W_i^{(l)}}{s_i} \cdot h_l = \sum_l F^{(l)} \cdot h_l.$$

Thus the selection probabilities s_i are

$$s_i = \frac{\sum_l F^{(l)} \cdot W_i^{(l)} \cdot h_l}{\sum_l F^{(l)} \cdot h_l}.$$

Note that this formula incorporates the spectral properties of the radiance or importance carried to the given reflection point in variable F. Thus when we arrive with green light at a red surface, the selection probability will be automatically zero. Either other elementary BRDFs take the role or the walk is terminated.

Figure 2 shows the error curves of a path tracing program implemented with the classical and the spectral Russian roulette. The test scene of the left figure was a room where the sum of the albedo and the emission was 1 at each point and on all wavelengths. In this case the solution can be obtained analytically [8]. The diffuse albedo of the walls for the three wavelengths was a different permutation of the values $(1.0, 0.5, 0.2)$, to simulate a scene with saturated colors. The test scene of the right image was the standard Cornell Box. For both test scenes, significantly less rays were needed to achieve the same level of error. We observed a speedup of 30 to 50 percent.

3 Russian-roulette with incoming radiance estimation

The other main problem of Russian roulette is that when the walk is terminated, it assumes that the incoming radiance is zero. The additional variance increase introduced by this assumption is:

$$\left(\frac{1}{s} - 1\right) \cdot E\left[\left(\frac{w}{p} \cdot L^{\text{in}}\right)^2\right].$$

Note that this variance can be significant if the expected radiance is far from being zero and the continuation probability is small. This can happen if the incoming illumination is large.

In order to attack this problem, assume that we have some rough estimation $\tilde{L}$ for the incoming radiance L^{in} at the given point. When the walk is decided to be terminated, we use this rough estimate instead of assuming that the incoming

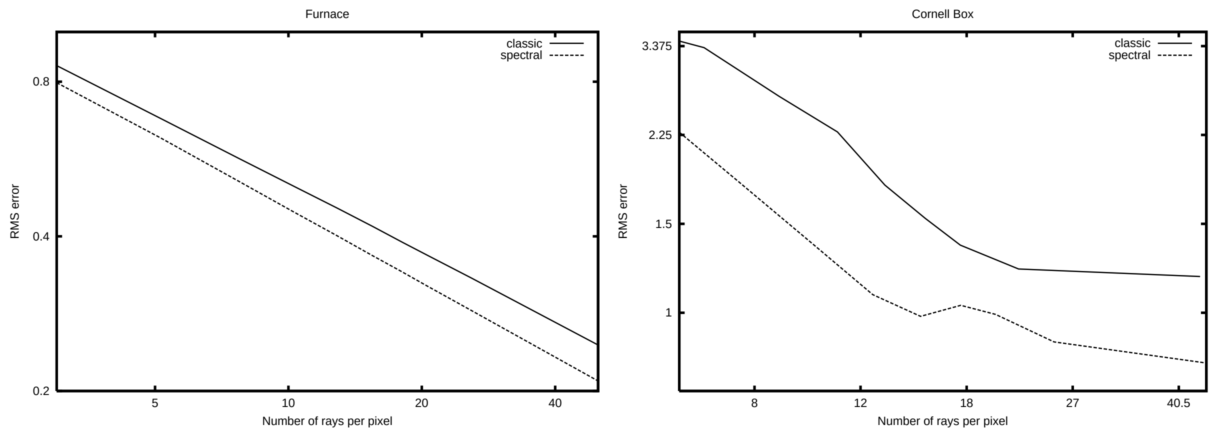

Figure 2: Comparison conventional and the spectral Russian roulette for a "furnace" and for the Cornell Box.

radiance is zero. If the walk is continued, then a linear combination of the actually computed radiance $w/p \cdot L^{\text{in}}$ and the reflection of this estimate $a \cdot \tilde{L}$ is inserted in the estimator, that is, we use

$$\alpha \cdot \frac{w}{p} \cdot L^{\text{in}} + \beta \cdot a \cdot \tilde{L},$$

where a is the local albedo, which also represents the response for directionally uniform illumination. The α and β values of this linear combination can be determined from the requirement that the expected value of this estimator should be correct:

$$L^r = s \cdot \left(\alpha \cdot E\left[\frac{w}{p} \cdot L^{\text{in}}\right] + \beta \cdot a \cdot \tilde{L}\right) + (1-s) \cdot a \cdot \tilde{L}.$$

From this requirement we obtain that

$$\alpha = \frac{1}{s}, \quad \beta = -\frac{1-s}{s},$$

thus the estimator for continuing the walk is

$$L^{\text{rr}} = \frac{w/p \cdot L^{\text{in}}}{s} - \frac{(1-s) \cdot a \cdot \tilde{L}}{s}.$$

On the other hand, when the walk is terminated the estimator is $L^{\text{rr}} = a \cdot \tilde{L}$.

Let us compute the variance of this estimator:

$$D^2[L^{\text{rr}}] = s \cdot E\left[\left(\frac{w/p \cdot L^{\text{in}}}{s} - \frac{(1-s) \cdot a \cdot \tilde{L}}{s}\right)^2\right] +$$

$$(1-s) \cdot (a\tilde{L})^2 - E^2[L^{\text{rr}}] =$$

$$\left(\frac{1}{s} - 1\right) \cdot E\left[\left(\frac{w}{p} \cdot L^{\text{in}} - a \cdot \tilde{L}\right)^2\right] + D^2\left[\frac{w}{p} \cdot L^{\text{in}}\right].$$

Comparing this result to equation 2, we can conclude that the added variance of the random termination is reduced to

$$\left(\frac{1}{s} - 1\right) \cdot E\left[\left(\frac{w}{p} \cdot L^{\text{in}} - a \cdot \tilde{L}\right)^2\right]$$

.

This improvement can, for example, be used to reduce the continuation probability and thus increase the speed of the method. In order to evaluate the potential speed up, let us suppose that the original Russian roulette uses s_{old} while the new method s_{new} continuation probability, which are set to provide the same error. Solving the

$$\left(\frac{1}{s_{old}} - 1\right) \cdot E\left[(\frac{w}{p} \cdot L^{\text{in}})^2\right] =$$

$$\left(\frac{1}{s_{new}} - 1\right) \cdot E\left[(\frac{w}{p} \cdot L^{\text{in}} - a \cdot \tilde{L})^2\right]$$

equation, we can obtain:

$$s_{new} = \frac{s_{old}}{(1 - s_{old})/d + 1}$$

where

$$d = \frac{E\left[(w/p \cdot L^{\text{in}} - \tilde{L})^2\right]}{E\left[(w/p \cdot L^{\text{in}})^2\right]}$$

is the goodness of estimation $\tilde{L}$.

In order to find the speed up ratio, we rely on the fact that the average length of the random walk is

$1/(1-s)$, where s is the continuation probability. The speed up is then the average number of the rays to be shot by the old method divided by the number of rays shot by the new method:

$$\text{speedup} = \frac{1/(1-s_{old})}{1/(1-s_{new})} = \frac{1}{1-(1-d)\cdot s_{old}}.$$

Note that this can be high if s_{old} is large and d is small. In the optimal case, when $d=0$, the speed up is $1/(1-s_{old})$, which means that the walk can be stopped after the first hit.

The different techniques that can provide estimations $\tilde{L}$ are discussed in the next section. At this point we should just emphasize that this estimation can be very rough, the modified Russian roulette will compensate its error. However, the speed up factor depends on the accuracy of this estimation.

3.1 Estimating the incoming radiance

The incoming radiance $\tilde{L}$ can be estimated either from the analysis of the scene or from the data gathered during a preprocessing phase.

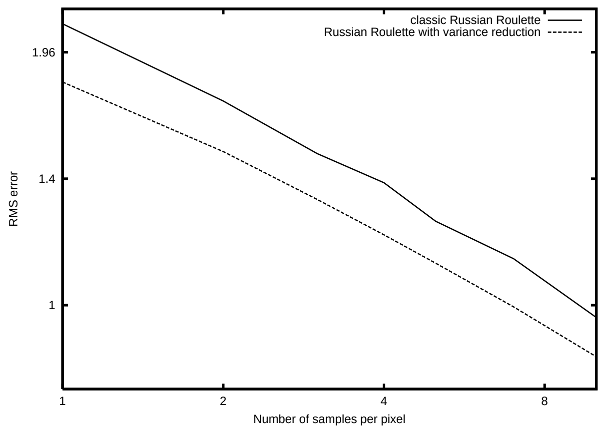

Figure 3: *The effect of the global radiance estimation.*

Suppose that the scene is closed. In this case, we can approximate the average radiance in the scene, which can be regarded as an estimate for $\tilde{L}$. The total emitted power of the light sources is

$$\Phi^e = \int_S \int_\Omega L^e(\vec{x},\omega)\cdot\cos\theta \; d\vec{x}d\omega$$

where S is the set of all surface points, L^e is the emitted radiance and θ is the angle between the direction of the emission and the surface normal. This emitted power will be multiplied by the albedo at each reflection. Suppose that the average albedo in the scene is $\tilde{a}$. The reflected power in the scene is the sum of the single reflection, double reflection, etc., that is:

$$\Phi^r \approx \Phi^e \cdot (\tilde{a} + \tilde{a}^2 + \ldots) = \frac{\tilde{a}\Phi^e}{1-\tilde{a}}.$$

From the average power, we can obtain the average radiance:

$$\tilde{L}(\vec{x},\omega) \approx \frac{1}{\pi S}\cdot\frac{\tilde{a}\Phi^e}{1-\tilde{a}}.$$

We used a conservative estimate and did not take into account the direct illumination in the estimate of $\tilde{L}$. The reason is that the direct illumination can have high variation and can be estimated poorly without computing it, which can result in an overestimation of $\tilde{L}$. Examining the variance formulae, we had better underestimate $\tilde{L}$.

Figure 3 shows the effects of the proposed estimate. We can notice that the new method required about 30% less samples to achive the same level of error.

4 Conclusions

In this paper we analyzed Russian roulette and random elementary BRDF sampling. We concluded that when combined BRDFs are sampled with the probability densities of the elementary BRDFs, then either the integrand decomposition method or multiple importance sampling should be used. We also proposed a spectral version of Russian roulette and the random BRDF selection and showed that this improvement resulted in 30-50 percent speed up. Finally, we examined the application of a simple estimate for the incoming radiance for the case when Russian roulette terminates the walk. Even if this estimate is obtained from a single value, the speedup is an additional 30 percent. Our future goal is to use more accurate estimates that are available in photon map [3], or from virtual light sources [4], which can result in more significant speedups.

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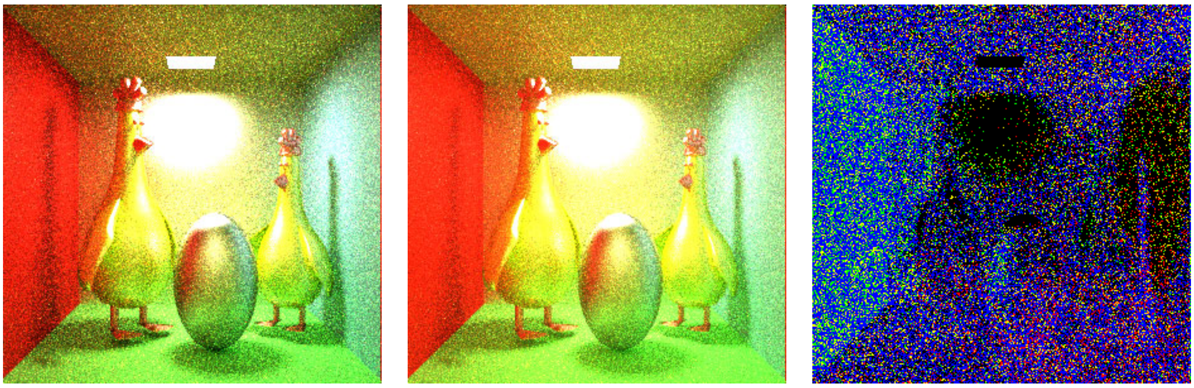

Figure 4: Cornell Chickens rendered by path tracing with classical Russian roulette (left), by estimated reflected radiance and the image of pixels where the estimation is negative. We used 100 samples per pixel and the rendering time was 245 sec.

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