1 Internal rate of return, bonds, yields

1.1 Internal rate of return

Given a deterministic cash flow steam, $(x_0, x_1, \ldots, x_n)$, where x_i (allowed to be positive, 0 or negative) denotes the flow at time period i (years say), we already studied the *net present value*,

$$NPV = \sum_{i=0}^{n} \frac{x_i}{(1+r)^i}.$$

Here r is the known (annual say) interest rate available to us all. For comparison purposes, if the investment stream actually came from only withdrawing and depositing money in a bank account at interest rate r, then $NPV = 0$ (for example if you place \$100 in a savings account for 1 year at fixed rate r, then the cash flow stream is $(-100, 100(1+r))$ and NPV$=$ $100 + 100(1+r)(1+r)^{-1} = 0$). But an arbitrary cash flow does not come from such a simple bank account scheme, hence NPV is typically quite different from 0. (Hopefully positive!) But this motivates hunting for a value of r that would result in $NPV = 0$:

Definition 1.1 *The internal rate of return (IRR) of the stream is a number $r > 0$ such that*

$$\sum_{i=0}^{n} \frac{x_i}{(1+r)^i} = 0.$$

It is the interest rate implied by the cash flow stream (not the current real interest rate, whatever it may be). Changing variables $a = (1+r)^{-1}$, equivalently we must solve for a positive root $0 < a < 1$ of a degree n polynomial:

$$\sum_{i=0}^{n} x_i a^i = x_0 + a x_1 + \cdots + a^n x_n = 0; \tag{1}$$

then $r = 1/a - 1$.

In general, there is no such solution but there always is under suitable sufficient conditions such as

IRR Sufficient Conditions:

If $x_0 < 0$ and $x_i \geq 0,\ 1 \leq i \leq n$ and $\sum_{i=0}^{n} x_i > 0$, then a solution $r > 0$ exists. This condition means that we put money in only at time 0 (initial investment), and then get back money at all other times, and that the sum over all those other times exceeds our initial investment. (Easy example: buy a bond).

A proof of the above: Let $f(a) = x_0 + a x_1 + \cdots + a^n x_n$, as from (1). Then under the sufficient conditions we have $f(0) < 0$ and $f(1) > 0$; thus by the continuity of the function $f = f(a)$ (e.g., the intermediate value theorem of calculus) there must exist a value of $a \in (0,1)$ for which $f(a) = 0$.

IRR can be used as an alternative to NPV for purposes of comparing two different streams to decide which is better. The idea is that of the two, you would choose the one having the largest IRR. (Of course, just as any stream with $NPV < 0$ would be avoided, any stream with IRR $< r =$ current interest rate would be avoided too.) Unfortunately it is possible that the two methods yield different conclusions; that is, IRR might rank your first steam higher than the second, while NPV might rank your second steam higher than your first!

Finally we point out that a solution to IRR must be solved for numerically in general, using *Newton's method*, for example.

1.2 Bonds, fixed income securities

A Bond is an example of a *fixed income* security, meaning that the payoff is essentially predetermined, deterministic, fixed: You invest a fixed amount of money now and are guaranteed fixed, known payoffs in the future. (Stock on the other hand is a so-called *risky security*, different since the payoff is random and potentially highly volatile.) Here we introduce some basics of bonds, their payoff structure, yields, and prices.

Bonds with no coupons

Buying a bond means you are lending money now (time $t = 0$) to some institution (government, business, etc.) that needs to raise capital, and are promised back the money back at a prespecified future time with a profit. Along the way, there may be so-called *coupon* payments, meaning, for example, that every 6 months until maturity, you receive a fixed amount too. The simplest case, however, is when there are no coupons, a *zero coupon bond*. For example, suppose you buy a 5-year \$1000 bond, which means that 5 years from now you will receive a *face-value* of \$1000, but nothing in between. The price P that you pay now is the present value, $P = 1000/(1 + r)^5$, where r is the interest rate. For example if $r = 0.04$, then $P = 822$. The interest rates used for such bonds depend on the length of maturity; you would receive a higher rate for a longer time period. For example, if instead of 5 years, you bought the bond for 10 years then you might receive a rate of 0.06 instead of 0.04, and then $P = 1000/(1.06)^{10} = 558$. These various interest rates are referred to as *spot rates*.

During the lifetime of your bond before maturity, interest rates might change causing the price of new bonds to be different than what you paid. If the rates go up then the bond price goes down, whereas if the rates go down, then the bond price goes up. Bonds are typically traded on the open market, so you could, for example, sell your 5 year \$1000 face-value bond (bought at time $t = 0$) one year later (at time $t = 1$) as a 4 year \$1000 face-value bond at a price of the form $P = 1000/(1 + r)^4$, where r is the appropriate spot rate for 4-year bonds at time $t = 1$.

Bonds with coupons

Most bonds give you fixed amount payments at regular intervals up to the face value termination (maturity) date; these payments are referred to as *coupon* payments since some time ago when buying a bond you would be given paper coupons to turn in for your payments. For example, *USA Treasury Notes and Bonds* are bonds that pay you a fixed amount every 6 months up to termination at which time you get both the last fixed amount plus the face value. The price of such a bond can be computed by using present values with current spot rates (e.g., the current zero coupon rates).

2-year \$1000 bond example

For example, consider a 2-year \$1000 bond, that has coupons every 6 months in the amount of \$25, for a total of four times until $t = 2$ years at which time you receive \$1025. To price this bond we need to know the current rates $r_{0.5},\ r_1,\ r_{1.5}$ and r_2; the spot rates for bonds having maturity from 6 months to 2 years. The first payment of 25 has PV of $25(1+r_{0.5}/2)^{-1}$, the second has PV of $25(1+r_1)^{-1}$, the third has PV of $25(1+r_{1.5}(1.5))^{-1}$ and the last has PV of $1025(1+r_{1.5}(2))^{-1}$; thus the price P is given by

$$P = 25(1+(0.50)r_{0.5})^{-1} + 25(1+(1)r_1)^{-1} + 25(1+(1.5)r_{1.5})^{-1} + 1025(1+(2)r_2)^{-1}.$$

In the above formula we used simple linear scaling for the interest rate computations (since we did not specify the type of compounding). Let's instead assume continuous compounding in which case the price becomes

$$P = 25e^{-(0.50)r_{0.5}} + 25e^{-(1)r_1} + 25e^{-(1.50)r_{1.5}} + 1025e^{-(2)r_2}. \tag{2}$$

For example, suppose that $r_{0.5} = 0.03,\ r_1 = 0.040,\ r_{1.5} = 0.045$ and $r_2 = 0.50$. Then $P \approx 999$.

Bond yields

Given a bond, we can solve for the implied interest rate, that is, the IRR of the cash flow stream induced by the bond (IRR is defined in Section 1.1); $(x_0, x_1, \ldots, x_n)$ where $x_0 = -P < 0$, and $x_i =$ the payment at the i^{th} period. This rate, denoted by λ, is called the *yield* of the bond, and it always exists because the IRR sufficient conditions given in Section 1.1 hold.

Let us suppose for example, that a 2-year \$1000 bond is issued with price P as in (2). Then the yield is the solution λ to

$$25e^{-(0.50)\lambda} + 25e^{-(1)\lambda} + 25e^{-(1.50)\lambda} + 1025e^{-(2)\lambda} = P = 999. \tag{3}$$

Letting $a = e^{-(0.50)\lambda}$ we equivalently must solve for a zero of a 4th degree polynomial,

$$25a + 25a^2 + 25a^3 + 1025a^4 - 999 = 0.$$

The solution (solved numerically) is $a \approx .976$; $\lambda = -2\ln(a) = 0.0486$.

Bond yield formula

Here we offer a general formula for finding the yield λ of a given bond that has price P. Let us assume that the face value is denoted by F, the coupon payments are given $m \geq 2$ times per year (every $1/m$ years). Let us assume further that K denotes the coupon amount per period, and that there are $1 \leq n \leq m$ periods remaining. We wish to compute the IRR, that is, the implied annual interest rate λ that, when compounded every $1/m$ units of time for computing the PV of the bond payoffs, would produce the price P: Solve for $\lambda > 0$ such that

$$P = \frac{F}{[1+(\lambda/m)]^n} + \sum_{i=1}^{n} \frac{K}{[1+(\lambda/m)]^i} \tag{4}$$

$$= \frac{F}{[1+(\lambda/m)]^n} + \frac{K}{(\lambda/m)}\Big\{1 - \frac{1}{[1+(\lambda/m)]^n}\Big\}. \tag{5}$$

The formula is derived as follows: There are $n \geq 1$ periods remaining, so the PV of the face value F is given by $\frac{F}{[1+(\lambda/m)]^n}$. Meanwhile, the i^{th} remaining coupon payment has a PV of $\frac{K}{[1+(\lambda/m)]^i}$, and these payments are summed over all n remaining periods. The final simplified form of the answer is due to the formulas for geometric series discussed in Lecture Notes 1 (in the Appendix, for example). In our 2-year \$1000 bond example, $m = 2$, $n = 4$, $F = 1000$ and $K = 25$.

If compounding is continuous, then the formula is given by

$$P = Fe^{-\lambda(\frac{n}{m})} + \sum_{i=1}^{n} Ke^{-\lambda(\frac{i}{m})} \tag{6}$$

$$= Fe^{-(\frac{\lambda}{m})^n} + Ke^{-(\frac{\lambda}{m})}\Big\{\frac{1 - e^{-(\frac{\lambda}{m})^n}}{1 - e^{-(\frac{\lambda}{m})}}\Big\}. \tag{7}$$

By changing variables, $a = \lambda/m$, $e^{-\frac{\lambda}{m}}$ respectively, in either case, finding λ reduces to the problem of finding the zero of a polynomial (in a) of degree $n+1$.

1.3 *Certificate of Deposit (CD)*

A *Certificate of Deposit (CD)* is a fixed income security offered by your bank. It is just like a bond except there are no coupons and it can't be re-sold on the open market: You must wait until maturity to get the agreed upon face value F. Typically maturity for a CD is from 6 months to several years, unlike bonds which might have maturities as long as 30 years. In essence, a CD is like a savings account that offers a higher interest rate since you are agreeing not to take any withdraws until a fixed date in the future (at which time you withdraw F). Just as for a zero coupon bond, the price of such a CD with maturity (say) T is given by Fe^{-rT} where r is the spot rate for such a CD.