# Naïve Bayes and Logistic Regression

Required reading:

- Mitchell draft chapter (see course website)

Recommended reading:

- Mitchell, 6.10 (text learning example)
- Bishop, Chapter 3.1.3, 3.1.4
- Ng and Jordan paper

Machine Learning 10-701

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# Naïve Bayes and Logistic Regression

- Design learning algorithms based on our understanding of probability
- Two of the most widely used
- Interesting relationship between these two
- Generative and Discriminative classifiers

# Bayes Rule

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Which is shorthand for:

$$(\forall i, j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i) P(Y = y_i)}{P(X = x_j)}$$

Random Variable

It's ith possible value

# Bayes Rule

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Equivalently:

$$(\forall i, j) P(Y = y_i | X = x_j) = \frac{P(X = x_j | Y = y_i) P(Y = y_i)}{\sum_k P(X = x_j | Y = y_k) P(Y = y_k)}$$

# Bayes Rule

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Common abbreviation:

$$(\forall i, j) \;\; P(y_i | x_j) = \frac{P(x_j | y_i) P(y_i)}{P(x_j)}$$

# Bayes Classifier

Training data:

| | | | | | | |
|---|---|---|---|---|---|---|
| $X$ | | | | | | $Y$ |
| Sky | Temp | Humid | Wind | Water | Forecst | EnjoySpt |
| Sunny | Warm | Normal | Strong | Warm | Same | Yes |
| Sunny | Warm | High | Strong | Warm | Same | Yes |
| Rainy | Cold | High | Strong | Warm | Change | No |
| Sunny | Warm | High | Strong | Cool | Change | Yes |

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Learning = estimating P(X|Y), P(Y)

Classification = using Bayes rule to calculate P(Y | $X^{new}$)

# Bayes Classifier

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|---|---|---|---|---|---|---|
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$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How shall we represent P(X|Y), P(Y)?

How many parameters must we estimate?

# Bayes Classifier

Training data:

| $X$ | | | | | | $Y$ |
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$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

How shall … esent P(X|Y), P(Y)?

How … arameters must we estimate?

Full joint $P(X_1 \ldots X_n \mid Y)$
usually impractical!

# Naïve Bayes

Naïve Bayes assumes

$X = \langle X_1, \ldots, X_n \rangle$, Y discrete-valued

$$P(X_1 \ldots X_n | Y) = \prod_i P(X_i | Y)$$

i.e., that $X_i$ and $X_j$ are conditionally independent given Y, for all $i \neq j$

# Conditional Independence

Definition: X is <u>conditionally independent</u> of Y given Z, if the probability distribution governing X is independent of the value of Y, given the value of Z

$$(\forall i, j, k) P(X = x_i | Y = y_j, Z = z_k) = P(X = x_i | Z = z_k)$$

Which we often write

$$P(X|Y,Z) = P(X|Z)$$

E.g.,

$$P(Thunder|Rain, Lightning) = P(Thunder|Lightning)$$

Naïve Bayes uses assumption that the $X_i$ are conditionally independent, given Y

then:

$$P(X_1, X_2|Y) = P(X_1|X_2, Y)P(X_2|Y)$$
$$= P(X_1|Y)P(X_2|Y)$$

$$P(X_1 ... X_n|Y) = \prod_i P(X_i|Y)$$

How many parameters needed now for P(X|Y)? P(Y)?

$$\theta_{ij} \equiv P(X = x_i|Y = y_j) \qquad \pi_j \equiv P(Y = y_j)$$

# Naïve Bayes classification

Bayes rule:

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) P(X_1 \ldots X_n | Y = y_k)}{\sum_j P(Y = y_j) P(X_1 \ldots X_n | Y = y_j)}$$

Assuming conditional independence:

$$P(Y = y_k | X_1 \ldots X_n) = \frac{P(Y = y_k) \prod_i P(X_i | Y = y_k)}{\sum_j P(Y = y_j) \prod_i P(X_i | Y = y_j)}$$

So, classification rule for $X^{new} = \langle X_1, \ldots, X_n \rangle$ is:

$$Y^{new} \leftarrow \arg\max_{y_k} \; P(Y = y_k) \prod_i P(X_i | Y = y_k)$$

# Naïve Bayes Algorithm

- Train Naïve Bayes (examples)

  for each* value $y_k$

  estimate $\pi_k \equiv P(Y = y_k)$

  for each* value $x_{ij}$ of each attribute $X_i$

  estimate $\theta_{ijk} \equiv P(X_i = x_{ij} | Y = y_k)$

- Classify ($X^{new}$)

$$Y^{new} \leftarrow \arg\max_{y_k} \quad P(Y = y_k) \prod_i P(X_i | Y = y_k)$$

* parameters must sum to 1

# Estimating Parameters: $Y$, $X_i$ discrete-valued

## Maximum likelihood estimates:

$$\widehat{\pi}_k = \widehat{P}(Y = y_k) = \frac{\#D\{Y = y_k\}}{|D|}$$

$$\widehat{\theta}_{ijk} = \widehat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D\{X_i = x_{ij} \wedge Y = y_k\}}{\#D\{Y = y_k\}}$$

## MAP estimates (uniform Dirichlet priors):

$$\widehat{\pi}_k = \widehat{P}(Y = y_k) = \frac{\#D\{Y = y_k\} + l}{|D| + lR}$$

$$\widehat{\theta}_{ijk} = \widehat{P}(X_i = x_{ij} | Y = y_k) = \frac{\#D\{X_i = x_{ij} \wedge Y = y_k\} + l}{\#D\{Y = y_k\} + lM}$$

# Naive Bayes: Example

Consider *PlayTennis* again, and new instance

$\langle Outlk = sun, Temp = cool, Humid = high, Wind = strc$

Want to compute:

$$Y^{new} \leftarrow \arg\max_{y_k} P(Y = y_k) \prod_i P(X_i | Y = y_k)$$

$P(y)\ P(sun|y)\ P(cool|y)\ P(high|y)\ P(strong|y) = .005$

$P(n)\ P(sun|n)\ P(cool|n)\ P(high|n)\ P(strong|n) = .021$

$$\rightarrow Y^{new} = n$$

# Learning to classify text documents

- Classify which emails are spam
- Classify which emails are meeting invites
- Classify which web pages are student home pages

How shall we represent text documents for Naïve Bayes?

# Article from rec.sport.hockey

```
Path: cantaloupe.srv.cs.cmu.edu!das-news.harvard.e
From: xxx@yyy.zzz.edu (John Doe)
Subject: Re: This year's biggest and worst (opinic
Date: 5 Apr 93 09:53:39 GMT

I can only comment on the Kings, but the most
obvious candidate for pleasant surprise is Alex
Zhitnik. He came highly touted as a defensive
defenseman, but he's clearly much more than that.
Great skater and hard shot (though wish he were
more accurate). In fact, he pretty much allowed
the Kings to trade away that huge defensive
liability Paul Coffey. Kelly Hrudey is only the
biggest disappointment if you thought he was any
good to begin with. But, at best, he's only a
mediocre goaltender. A better choice would be
Tomas Sandstrom, though not through any fault of
his own, but because some thugs in Toronto decidec
```

# Learning to Classify Text

Target concept $Interesting?: Document \rightarrow \{+, -\}$

1. Represent each document by vector of words
   - one attribute per word position in document
2. Learning: Use training examples to estimate
   - $P(+)$
   - $P(-)$
   - $P(doc|+)$
   - $P(doc|-)$

Naive Bayes conditional independence assumption

$$P(doc|v_j) = \prod_{i=1}^{length(doc)} P(a_i = w_k|v_j)$$

where $P(a_i = w_k|v_j)$ is probability that word in position $i$ is $w_k$, given $v_j$

one more assumption:
$P(a_i = w_k|v_j) = P(a_m = w_k|v_j), \forall i, m$

# Baseline: Bag of Words Approach


| | |
|---|---|
| aardvark | 0 |
| about | 2 |
| all | 2 |
| Africa | 1 |
| apple | 0 |
| anxious | 0 |
| ... | |
| gas | 1 |
| ... | |
| oil | 1 |
| … | |
| Zaire | 0 |

## Twenty NewsGroups

Given 1000 training documents from each group
Learn to classify new documents according to which newsgroup it came from

| | |
|---|---|
| comp.graphics | misc.forsale |
| comp.os.ms-windows.misc | rec.autos |
| comp.sys.ibm.pc.hardware | rec.motorcycles |
| comp.sys.mac.hardware | rec.sport.baseball |
| comp.windows.x | rec.sport.hockey |
| | |
| alt.atheism | sci.space |
| soc.religion.christian | sci.crypt |
| talk.religion.misc | sci.electronics |
| talk.politics.mideast | sci.med |
| talk.politics.misc | |
| talk.politics.guns | |

Naive Bayes: 89% classification accuracy

LEARN_NAIVE_BAYES_TEXT($Examples, V$)

1. *collect all words and other tokens that occur in Examples*

- $Vocabulary \leftarrow$ all distinct words and other tokens in $Examples$

2. *calculate the required $P(v_j)$ and $P(w_k|v_j)$ probability terms*

- For each target value $v_j$ in $V$ do
  - $docs_j \leftarrow$ subset of $Examples$ for which the target value is $v_j$
  - $P(v_j) \leftarrow \frac{|docs_j|}{|Examples|}$
  - $Text_j \leftarrow$ a single document created by concatenating all members of $docs_j$
  - $n \leftarrow$ total number of words in $Text_j$ (counting duplicate words multiple times)
  - for each word $w_k$ in $Vocabulary$
    * $n_k \leftarrow$ number of times word $w_k$ occurs in $Text_j$
    * $P(w_k|v_j) \leftarrow \frac{n_k+1}{n+|Vocabulary|}$

For code, see

www.cs.cmu.edu/~tom/mlbook.html

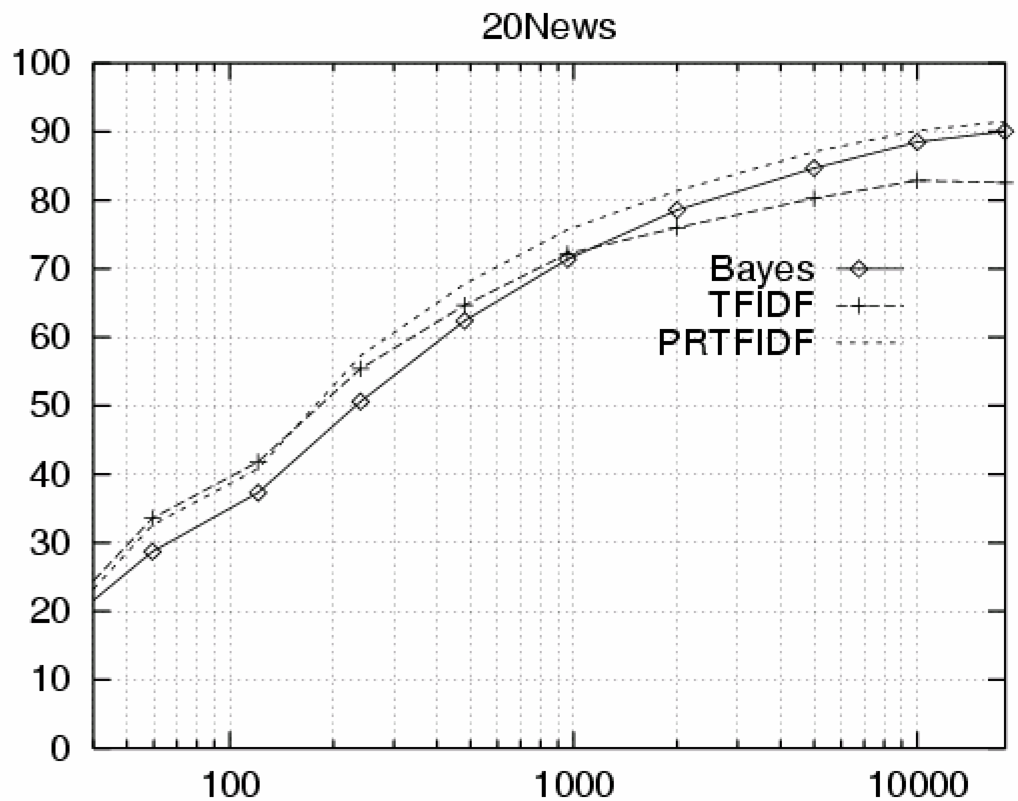
click on "Software and Data"

CLASSIFY_NAIVE_BAYES_TEXT($Doc$)

- $positions \leftarrow$ all word positions in $Doc$ that contain tokens found in $Vocabulary$
- Return $v_{NB}$, where

$$v_{NB} = \operatorname*{argmax}_{v_j \in V} P(v_j) \prod_{i \in positions} P(a_i|v_j)$$

# Learning Curve for 20 Newsgroups




Accuracy vs. Training set size (1/3 withheld for test)

# What if we have continuous $X_i$ ?

Eg., image classification: $X_i$ is i[th] pixel

Gaussian Naïve Bayes (GNB): assume

$$P(X_i = x \mid Y = y_k) = \frac{1}{\sigma_{ik}\sqrt{2\pi}} e^{\frac{-(x-\mu_{ik})^2}{2\sigma_{ik}^2}}$$

Sometimes assume variance

- is independent of Y (i.e., $\sigma_i$),
- or independent of $X_i$ (i.e., $\sigma_k$)
- or both (i.e., $\sigma$)

# Estimating Parameters: $Y$ discrete, $X_i$ continuous

Maximum likelihood estimates:

jth training example

$$\widehat{\mu}_{ik} = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j X_i^j \delta(Y^j = y_k)$$

δ(x)=1 if x true, else 0

$$\widehat{\sigma}_{ik}^2 = \frac{1}{\sum_j \delta(Y^j = y_k)} \sum_j (X_i^j - \widehat{\mu}_{ik})^2 \delta(Y^j = y_k)$$

# Example: GNB for classifying mental states

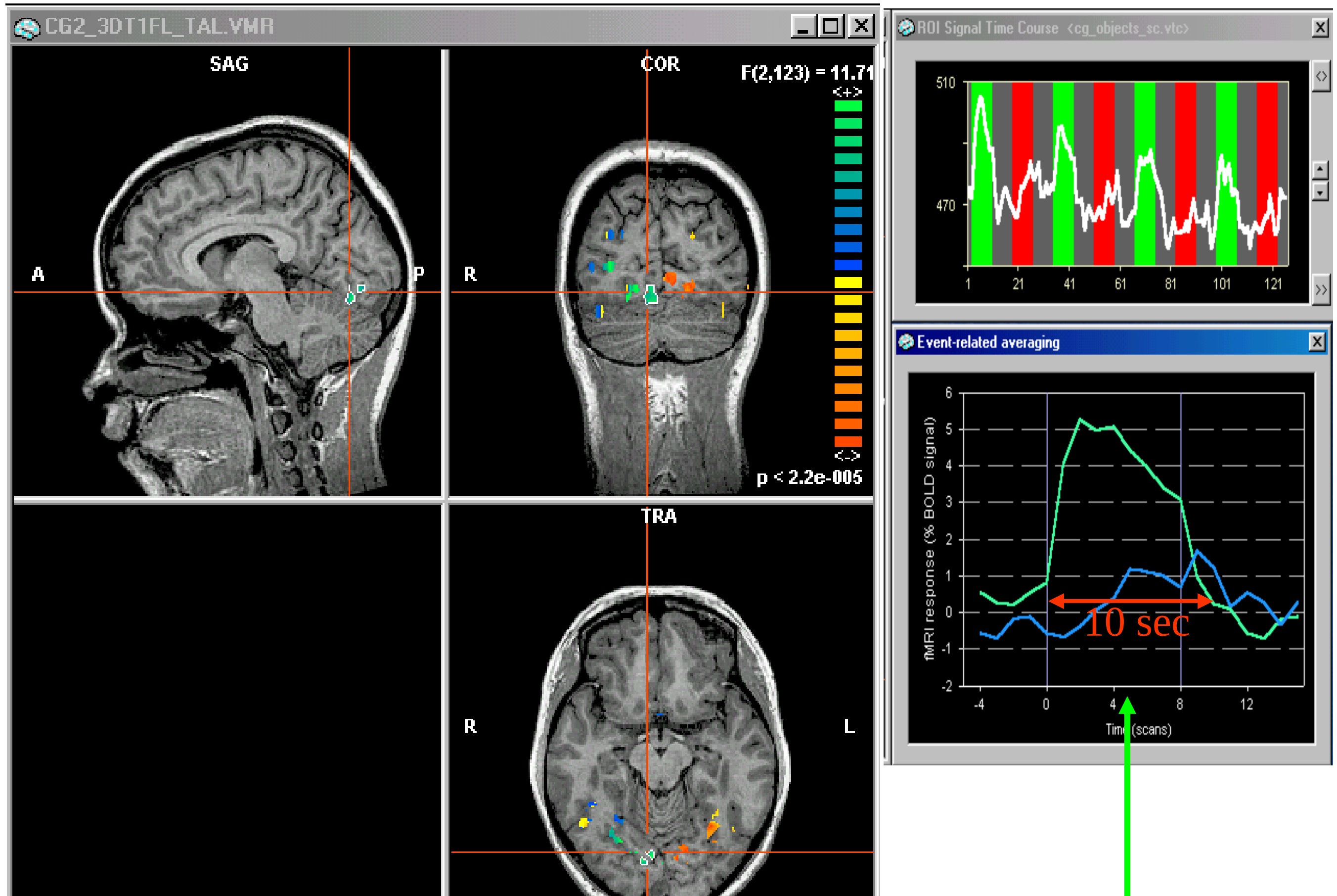
~1 mm resolution

~2 images per sec.

15,000 voxels/image

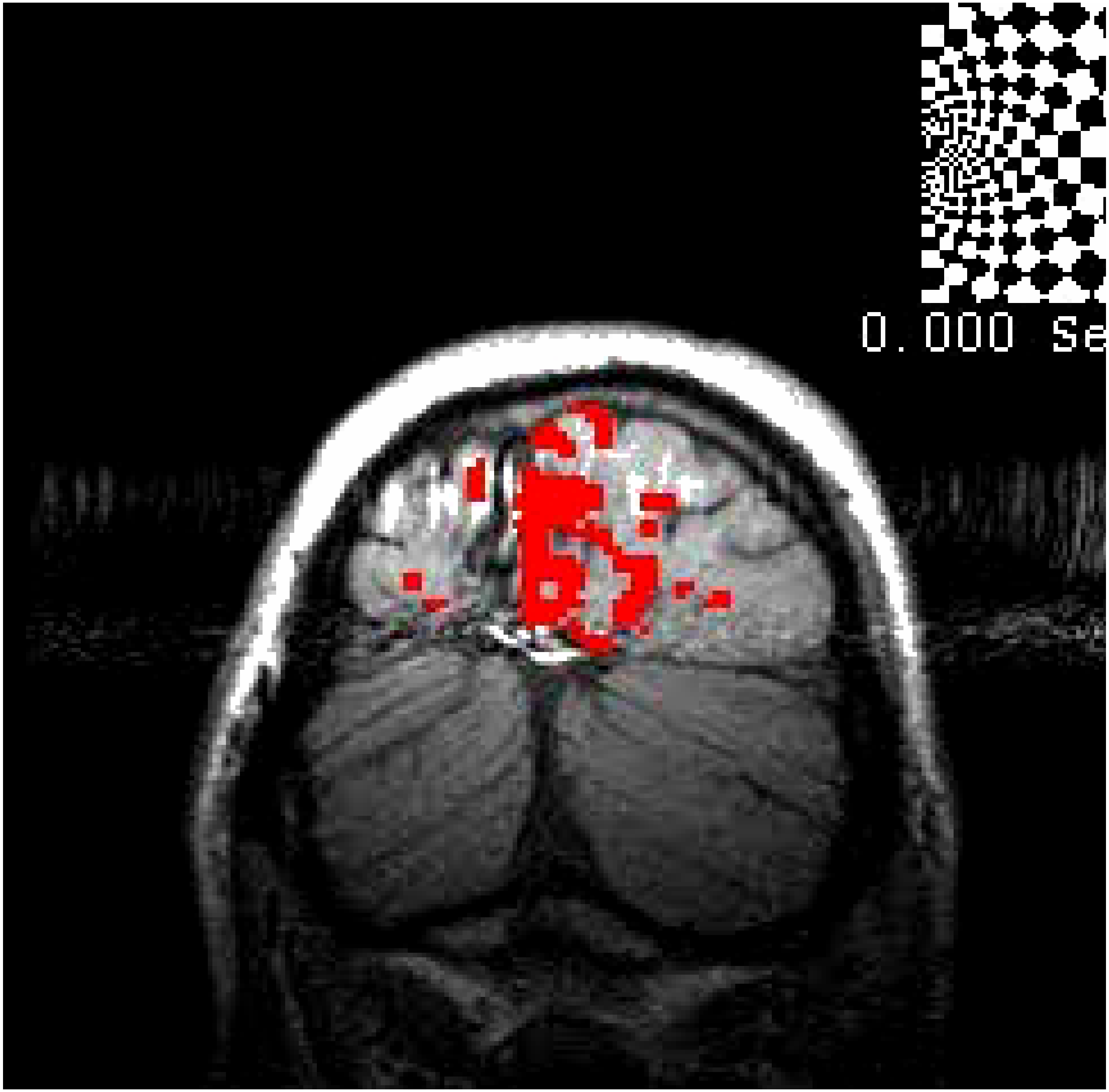
non-invasive, safe

measures Blood Oxygen Level Dependent (BOLD) response



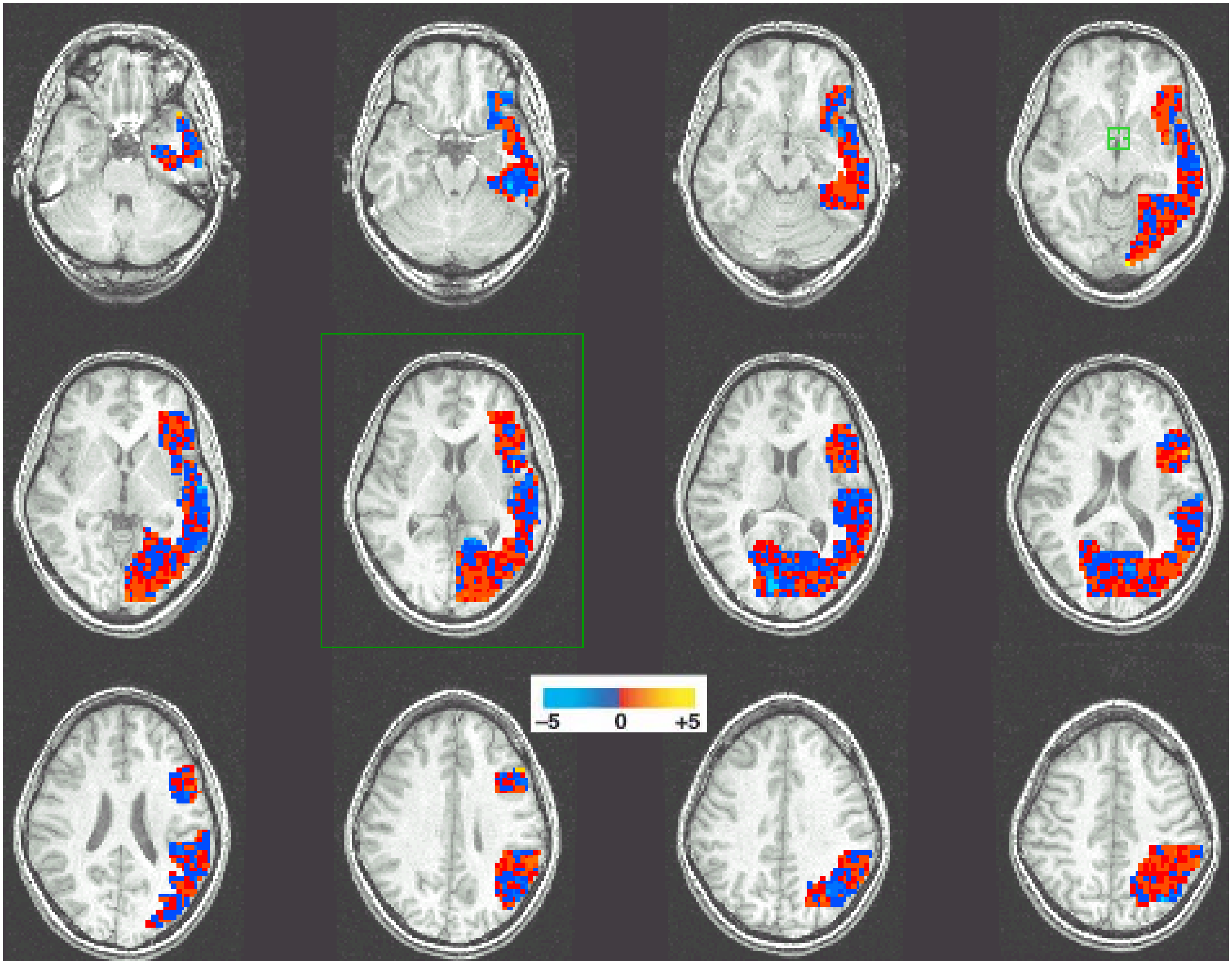

Typical impulse response

Brain scans can track activation with precision and sensitivity



# Gaussian Naïve Bayes: Learned $\mu_{voxel,word}$

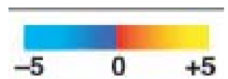
P(BrainActivity | WordCategory = People)



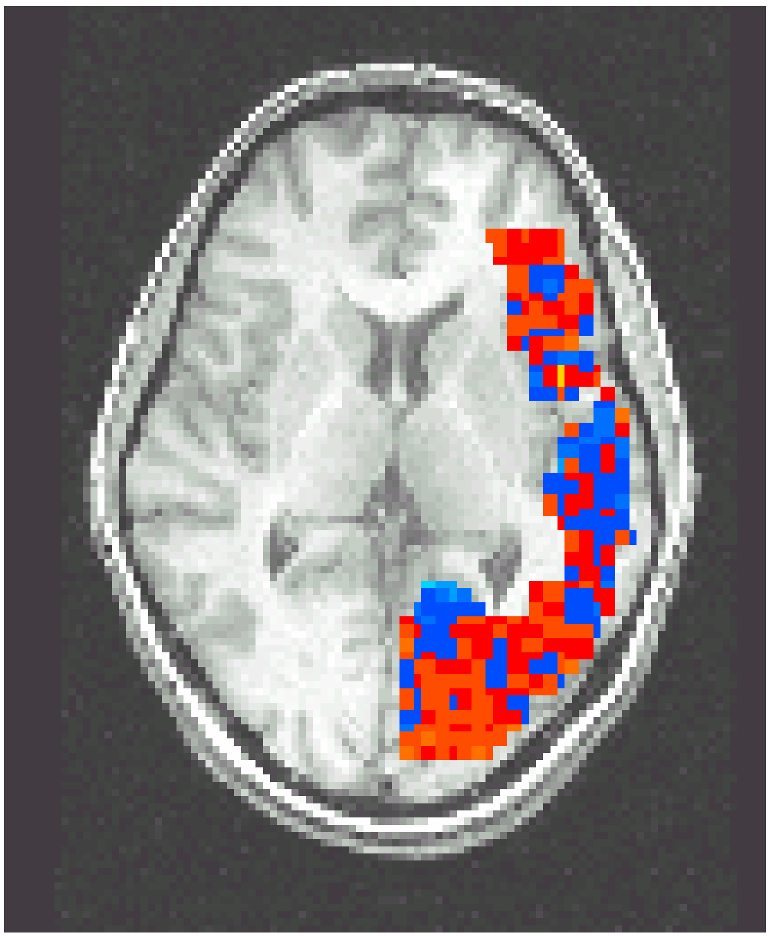
# Learned Bayes Models – Means for P(BrainActivity | WordCategory)

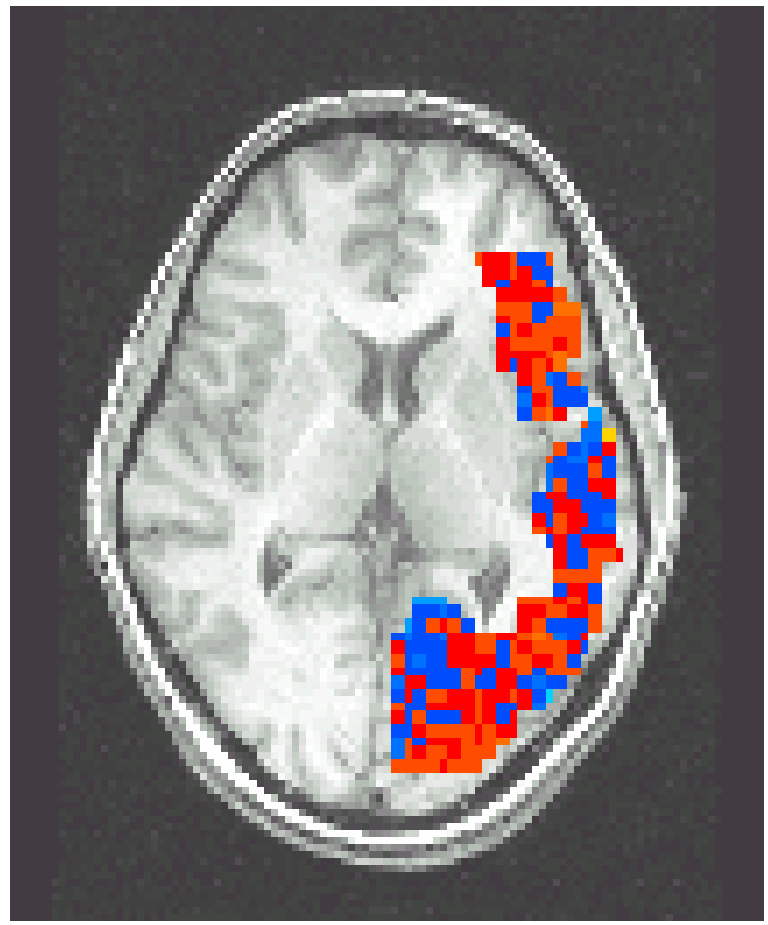
Pairwise classification accuracy: 85%

People words




Animal words

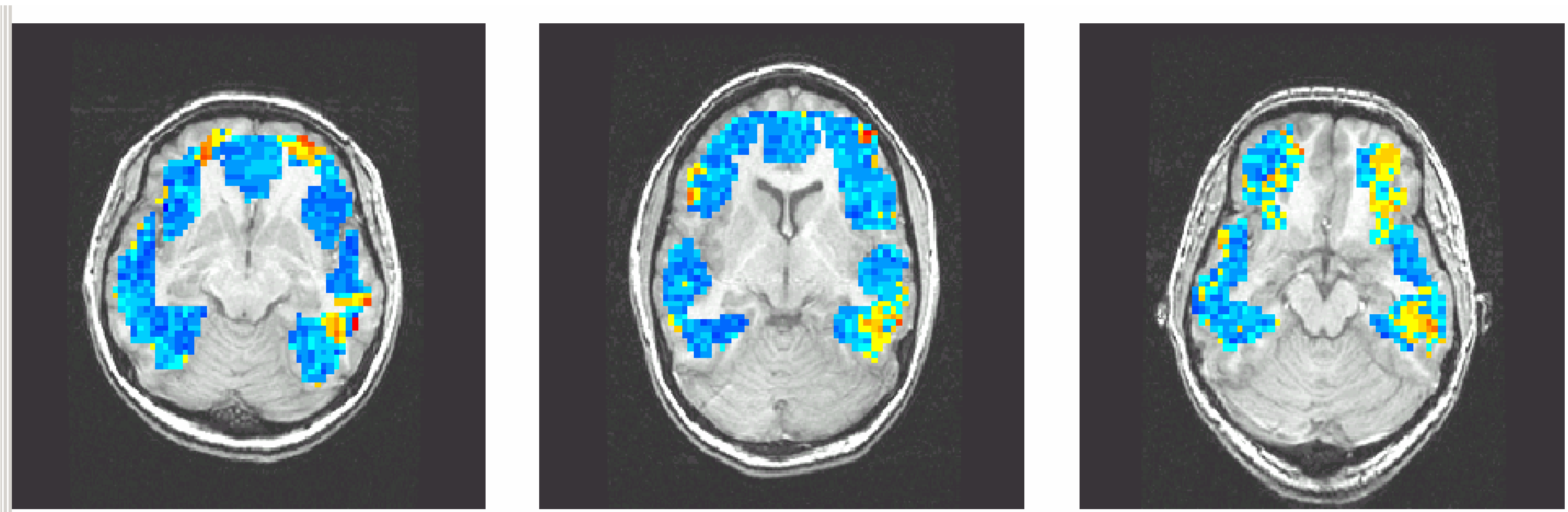




Plot of single-voxel classification accuracies.

Gaussian naïve Bayes classifier

(yellow and red are most predictive).



Subject 1 Subject 2 Subject 3

# What you should know:

- Learning (generative) classifiers based on Bayes rule
- Conditional independence
  - What it is
  - Why it's important
- Naïve Bayes assumption and its consequences
- Naïve Bayes with discrete inputs, continuous inputs