Periodic orbits around the collinear libration points

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Abstract

The locations for the collinear libration points in the framework of the restricted three-body problem are determined when the bigger primary is a triaxial rigid body. The analysis of the periodic motion around these points is performed and given up to second order in the case that the initial state of the motion gives rise to periodic orbits. Moreover, some numerical results for the locations of collinear points are provided and the graphical investigations for the periodic motion are plotted, as well. It is worth mentioning that the collinear libration points and associated periodic orbits are considered the optimal placement to transfer a spacecraft to the nominal periodic orbits or to an associated stable manifold. ©2016 All rights reserved.

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1. Introduction

The restricted three-body problem is one of the most important branches in celestial mechanics. This is due to its valuable applications for orbit design and space navigation flights. The literature is rich in works about finding the positions of the libration pints, studying their linear stability, and obtaining the orbits of motion around them. Since the land mark work of Szebehely, see [24], which studied the original problem consisting of the restricted three-body problem with the primaries considered spherical, many researches have been made looking for a generalization of the problem by considering the elliptic problem rather than

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the circular problem, see [14]. While in the frame of the deviation of the shape of the primaries from the complete sphere, or their physical nature being radiating or having a magnetic field, were introduced in [20, 21].

In [18], they studied the periodic orbits generated by Lagrangian solutions of the restricted three-body problem whenever the bigger primary is radiating and the smaller is an oblate spheroid. The existence and stability of the libration points in the restricted problem, when the three bodies are oblate and the primaries are radiating, were studied in [10]. They also provided the periodic orbits around the triangular points. In addition, it was proved in [2] that the locations for the triangular points as well as their linear stability depend on the parameters regarding the first two even zonal harmonic of the more massive primary in the planar circular restricted three-body problem. Moreover, it was also showed therein that the triangular points become stable for $0 < \mu < \mu_c$, and unstable provided that $\mu_c \leq \mu \leq 1/2$, where μ_c is the critical mass parameter, which depends on the coefficients of zonal harmonic. Further, some numerical values regarding the positions of the triangular points, mass ratio and critical mass for the planets systems in our solar system, were provided.

The motion around the collinear equilibria when the bigger primary is an oblate spheroid was studied in [16]. Further, [25] deals with the motion around the collinear equilibrium points in the restricted three-body problem when the larger primary is radiating and the second primary is an oblate spheroid. For additional details, see [3, 4, 8, 13].

Furthermore, in [13], it was presented a comprehensive analytical study regarding the existence of the libration points and their linear stability in the restricted three-body problem under the effect of the first two even zonal harmonic parameters with respect to both primaries. They also found out the periodic orbits around the libration points, the expressions for semi-major and semi-minor axes, the eccentricities, and the periods of elliptical orbits, as well as the orientation of the principal axes. Moreover, there is an extensive and analytical study which has a great scientific sound in the framework of the restricted three-body problem. These studies on the existence of libration points, their linear stability, the periodic and secular solutions, as well as the solutions through numerical integration methods under the effects of the lack of sphericity and radiation pressure, has been introduced by E.I. Abouelmagd and his research group. See for example, but not limited to, [5, 6, 7, 9, 11, 12]. Furthermore, there are also some studies focus on this problem under the effect of the aforementioned perturbed forces, that must be taken into a consideration of the reader, see [1, 19, 23, 26].

However, the motion in the proximity of collinear libration points in the framework of the restricted three-body problem has attracted the interest of many other works, including [15, 17]. The periodic orbits around the collinear libration points represent an interesting case of motion in astrodynamics, since the moving body remains for infinite time in the proximity of an unstable equilibrium point. These orbits have a special importance, since they may be considered as the limiting case of asymptotic orbits to the Lyapunov periodic orbits (for further details, see [16]).

In this work, we present an analytical study regarding the periodic motion in the proximity of collinear libration points provided that the bigger primary is triaxial. This motion has been proved to be generally unstable in the case that the bigger primary is triaxial and radiating [22]. This also remains true whether the radiation effect is neglected. Nevertheless, for certain initial values for the velocity, it holds that periodic motions could exist around the three collinear points. The positions of the collinear points are obtained by considering the triaxiality of the bigger primary as a perturbation for the motion, and periodic orbits around them are obtained up to second order. Numerical results are provided for some chosen values for the parameter μ and the triaxiality terms of the bigger primary which are in turn functions of the moments of inertia of the bigger primary and the distance between the two primaries, respectively.

2. Equations of motion

In this paper, they are used the notations and terminology of Szebehely (1967). We will take the distance between the primaries equal to one; the sum of the masses of the primaries is also taken as the unit of mass.

The unit of time is also chosen as to make the gravitational constant is unity. Using dimensionless variables, when the bigger primary is a triaxial and the smaller is a spherical body as well as both of them are not radiating, the equations of motion of the infinitesimal mass in a synodic coordinate system (x, y) can be written in the following form (see [22])

$$\begin{cases} \Omega_x = \ddot{x} - 2n\dot{y}; \\ \\ \Omega_y = \ddot{y} + 2n\dot{x}, \end{cases} \tag{2.1}$$

where

$$\begin{cases} \Omega = \dfrac{n^2}{2}\left[(1-\mu)\, r_1^2 + \mu\, r_2^2\right] + \dfrac{1-\mu}{r_1} + \dfrac{\mu}{r_2} + \dfrac{1-\mu}{2r_1^3}\,(2\sigma_1 - \sigma_2) + \dfrac{3\,(1-\mu)\,y^2}{2r_1^5}\,(\sigma_2 - \sigma_1); \\ r_1^2 = (x-\mu)^2 + y^2; \\ r_2^2 = (x+1-\mu)^2 + y^2; \\ \mu = \dfrac{m_2}{m_1 + m_2}. \end{cases}$$

Note that $\mu \in (0, 1/2]$ is the parameter of mass ratio, m_1, m_2 are the masses of the primaries within $m_1 \geq m_2$,

$$\sigma_1 = \frac{A^2 - C^2}{5R^2}, \quad \sigma_2 = \frac{B^2 - C^2}{5R^2}, \quad \sigma_1,\, \sigma_2 \ll 1,$$

A, B, and C are the semi-axes of the triaxial rigid body, and R is the distance between the primaries, as well. Further, n is the mean motion of the primaries, which is governed by

$$n^2 = 1 + \frac{3}{2}\,(2\sigma_1 - \sigma_2).$$

It is worth mentioning that for an oblate body, $\sigma_1 = \sigma_2$.

3. Locations of the collinear points

If the collinear points are located in $y = 0$, then the derivatives of Ω with respect to x and y, resp., are given by

$$\begin{cases} \Omega_x = n^2 x - \dfrac{1-\mu}{r_1^2} - \dfrac{\mu}{r_2^2} - \dfrac{3(1-\mu)(2\sigma_1 - \sigma_2)}{2r_1^4}; \\ \\ \Omega_y = 0. \end{cases} \tag{3.1}$$

This leaves us a small parameter in the problem, namely, $\varepsilon = 2\sigma_1 - \sigma_2$, which takes the problem from the original case of a point mass to our case consisting of a triaxial rigid body. Thus, let us expand the locations of the collinear libration points as follows:

$$x_J = x_{J,0} + \sum_{i>0} \varepsilon^i\, x_{J,i}, \quad J = 1, 2, 3,$$

where $x_{J,0}$ are the coordinates of the collinear points whether $\sigma_1 = \sigma_2 = 0$. The above expansion is used to solve Eq. (3.1) in order to get the next locations for the collinear libration points:

$$x_J = x_{J,0} + \sum_{i=1}^{n} \varepsilon^i\, P_{J,i},$$

where $x_1 < -\mu < x_2 < 1 - \mu < x_3$, and $P_{J,i}$'s depend on μ and x_J's.

4. Motion around the collinear points

The three collinear points are proved to be generally unstable whether the bigger primary becomes triaxial (see [22]). However, for certain initial conditions, periodic and stable orbits may exist. We study the motion regarding the collinear points as follows. Let

$$x = x_J + \xi, \quad y = \eta,$$

where $|\xi|, |\eta| << 1$. Thus, substituting in Eq. (2.1) and expanding Ω_x, Ω_y in Taylor series up to second order around the collinear points, we obtain

$$\begin{cases} \ddot{\xi} - 2n\dot{\eta} = a_1\xi + a_2\xi^2 + a_3\eta^2; \\ \\ \ddot{\eta} + 2n\dot{\xi} = a_4\eta + a_3\xi\eta, \end{cases} \tag{4.1}$$

where a_1, a_2, a_3, and a_4 are the non-vanishing coefficients in that expansion. Their values are given below:

$$\begin{cases} a_1 = \dfrac{\partial^2\Omega}{\partial x^2}\bigg|_{x_j} = n^2 + \dfrac{2(1-\mu)}{r_1^3} + \dfrac{2\mu}{r_2^3} + \dfrac{6\varepsilon(1-\mu)}{r_1^5}; \\ \\ a_2 = \dfrac{\partial^3\Omega}{\partial x^3}\bigg|_{x_j} = -\dfrac{6(1-\mu)}{r_1^4} - \dfrac{6\mu}{r_2^4} - \dfrac{30\varepsilon(1-\mu)}{r_1^6}; \\ \\ a_3 = \dfrac{\partial^3\Omega}{\partial x\partial y^2}\bigg|_{x_j} = \dfrac{3(1-\mu)}{r_1^4} + \dfrac{3\mu}{r_2^4} + \dfrac{15\varepsilon(1-\mu)(1+2E)}{2r_1^6}; \\ \\ a_4 = \dfrac{\partial^2\Omega}{\partial y^2}\bigg|_{x_j} = n^2 - \dfrac{1-\mu}{r_1^3} - \dfrac{\mu}{r_1^3} - \dfrac{3\varepsilon(1-\mu)(1+2E)}{2r_1^5}, \end{cases}$$

and $E = 1 - \dfrac{1}{2 - \sigma_2/\sigma_1}$.

To solve Eq. (4.1), the involved coefficients are written in the next form:

$$\begin{cases} n^2 = n_0 + \varepsilon\, n_1; \\ \\ a_i = a_{i0} + \varepsilon\, a_{i1}, \end{cases}$$

where $i = 1, 2, 3, 4$, the values of a_{i0} and a_{i1} are the terms free of the small parameter ε, and the coefficients of it, respectively. Next, let us expand the variables ξ, η as

$$\begin{cases} \xi = \varepsilon\, \xi_1 + \varepsilon^2\, \xi_2; \\ \\ \eta = \varepsilon\, \eta_1 + \varepsilon^2\, \eta_2. \end{cases} \tag{4.2}$$

Substituting Eq. (4.2) into Eq. (4.1) and equating terms of the same order in ε with keeping the terms up to second order of ε, we obtain two system with second order, such that the first is linear (LS) while the second is non-linear system (NS).

$$\textbf{LS:} \quad \begin{cases} \ddot{\xi}_1 - 2\dot{\eta}_1 = a_{10}\,\xi_1; \\ \\ \ddot{\eta}_1 + 2\dot{\xi}_1 = a_{40}\,\eta_1. \end{cases} \tag{4.3}$$

NS:
$$\begin{cases} \ddot{\xi}_2 - 3\dot{\eta}_2 = a_{10}\,\xi_2 + a_{11}\,\xi_1 + a_{20}\,\xi_1^2 + a_{30}\,\eta_1^2; \\ \ddot{\eta}_2 + 3\dot{\xi}_2 = a_{40}\,\eta_2 + a_{41}\,\eta_1 + a_{30}\,\xi_1\,\eta_1. \end{cases} \tag{4.4}$$

The solution of the linear system (LS) is simply the linearized motion about the equilibrium collinear points in the classical sense. Seeking a periodic motion about the collinear points, the initial conditions are taken such that only the terms resulting in periodic motion remain. This is accomplished by choosing

$$\dot{\xi}_0 = \frac{2s^2}{s^2 + a_1}\,\eta_0, \quad \dot{\eta}_0 = -\frac{1}{2}\,(s^2 + a_1)\,\xi_0, \tag{4.5}$$

and hence,

$$\begin{cases} \xi_1 = \xi_0\,\cos st + \dfrac{\eta_0}{\Gamma}\,\sin st; \\ \eta_1 = \eta_0\,\cos st - \Gamma\,\xi_0\,\sin st, \end{cases} \tag{4.6}$$

where s is the angular frequency of motion, and ξ_0, η_0 are selected to match the initial conditions. In this context, the values of Γ and s will be controlled through the following expressions:

$$\begin{cases} \Gamma = \dfrac{s^2 + a_{10}}{2s}; \\ s = \sqrt{\dfrac{1}{2}\,(4 - a_{10} + a_{40}) + \sqrt{\dfrac{1}{4}\,(4 - a_{10} + a_{40})^2 - a_{10}\,a_{40}}}. \end{cases} \tag{4.7}$$

For a comprehensive and extended details about the method of constructions of Eqs. (4.5 - 4.7), see [7, 10].

A particular solution for Eq. (4.4) (NS), will be ruled by

$$\begin{cases} \xi_2 = \alpha_0 + \sum_{k=1}^{2}[\alpha_k^C\,\cos(ks)t + \alpha_k^S\,\sin(ks)t]; \\ \eta_2 = \beta_0 + \sum_{k=1}^{2}[\beta_k^C\,\cos(ks)t + \beta_k^S\,\sin(ks)t]. \end{cases} \tag{4.8}$$

Note that Eq. (4.8) describes the motion about the collinear points arising from the second order of the triaxiality. Substituting Eq. (4.6) into the system in Eq. (4.3), then we get the coefficients in Eq. (4.8), which are listed below:

$$\begin{cases} \alpha_0 = \dfrac{-1}{2\,\Gamma^2\,a_{10}}\,[(\Gamma^2\,a_{20} + \Gamma^4\,a_{30})\,\xi_0^2 + (a_{20} + \Gamma^2\,a_{30})\,\eta_0^2]; \\[2mm] \beta_0 = 0; \\[2mm] \alpha_1^C = -\xi_0\,\dfrac{a_{11}s^2 + 3\Gamma\,a_{41}s + a_{11}a_{14}}{s^4 + s^2\,(a_{10} + a_{40} - 9) + a_{10}a_{40}}; \\[3mm] \alpha_2^C = -\dfrac{a_{30}\,(\eta_0^2 - \Gamma^2\xi_0^2)}{12\Gamma s} - \dfrac{c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0}{2\Gamma^2\,(96s^5 + 24(a_{10} + a_{40} - 9)s^3 + 6a_{10}a_{40}s)}; \\[3mm] \alpha_1^S = \dfrac{-\eta_0}{\Gamma}\,\dfrac{a_{11}s^2 + 3\Gamma a_{41}s + a_{11}a_{14}}{s^4 + s^2(a_{10} + a_{40} - 9) + a_{10}a_{40}}; \\[3mm] \alpha_2^S = -\dfrac{\eta_0\xi_0}{\Gamma}\,\dfrac{4(a_{20} - \Gamma^2 a_{30})s^2 + 6\Gamma a_{30}s + a_{20}a_{40} - \Gamma^2 a_{30}a_{40}}{16s^4 + 4s^2(a_{10} + a_{40} - 9) + a_{10}a_{40}}; \\[3mm] \beta_1^C = -\dfrac{\eta_0}{\Gamma}\,\dfrac{\Gamma a_{41}s^2 + 3a_{11}s + \Gamma a_{10}a_{41}}{s^4 + (a_{10} + a_{40} - 9)s^2 + a_{10}a_{40}}; \\[3mm] \beta_2^C = -\dfrac{\eta_0\xi_0}{\Gamma}\,\dfrac{4\Gamma a_{30}\,s^2 + 6(a_{20} - \Gamma^2 a_{30})s + \Gamma a_{10}a_{30}}{16s^4 + 4(a_{10} + a_{40} - 9)s^2 + a_{10}a_{40}}; \\[3mm] \beta_1^S = \xi_0\,\dfrac{\Gamma a_{41}s^2 + 3a_{11}s + a_{10}\Gamma a_{41}}{s^4 + (a_{10} + a_{40} - 9)s^2 + a_{10}a_{40}}; \\[3mm] \beta_2^S = -\dfrac{4\Gamma a_{30}(\eta_0^2 - \Gamma^2\xi_0^2)s^2 + 6[(a_{20} - a_{30}\Gamma^2)\eta_0^2 + (a_{30}\Gamma^4 - a_{20}\Gamma^2)\xi_0^2]s + \Gamma a_{10}a_{30}(\eta_0^2 - \Gamma^2\xi_0^2)}{2\Gamma^2[16s^4 + 4(a_{10} + a_{40} - 9)s^2 + a_{10}a_{40}]}, \end{cases}$$

where the superscripts C, S of α and β refer to the coefficients of cosines and sines:

$$
\begin{cases}
c_0 = \dfrac{a_{10}a_{40}}{16}\, c_4; \\[2mm]
c_1 = \dfrac{a_{40}}{4}\, c_3; \\[2mm]
c_2 = 4\Gamma a_{30}(\eta_0^2 - \Gamma^2\xi_0^2)(a_{10} + a_{40}); \\[2mm]
c_3 = 24((a_{20} - \Gamma^2 a_{30})\,\eta_0^2 - (a_{20} - \Gamma^2 a_{30})\Gamma^2\xi_0^2); \\[2mm]
c_4 = 16\Gamma a_{30}(\eta_0^2 - \Gamma^2\xi_0^2).
\end{cases}
$$

After determining the quantities of variations $\xi_1, \xi_2, \eta_1, \eta_2$, we can state that Eq. (4.2) describes the possible periodic motion about the collinear equilibrium points up to second order.

5. Numerical results

In this section, some numerical results are provided for the problem in each point. In the first subsection, the locations of the collinear points within the existence of triaxial terms of order 0.01 and 0.1 are given in comparison with the case of a spherical body (corresponding to $\varepsilon = 0$). These locations are calculated for different values of the parameter, whose mass ratio is $\mu = 0.1, 0.2, 0.3, 0.4, 0.5$. In the second subsection, the orbits of the motion are plotted for $\varepsilon = 0.01$ at $\mu = 0.1, 0.2, 0.3, 0.4, 0.5$. The calculations are carried out at the initial position $(0.1, 0.1)$, when $\sigma_1 = 0.1, \sigma_2 = 0.19$. It is worth mentioning that when the calculations are made at other values of triaxial parameters also satisfying $\varepsilon = 0.01$, then the graphs showed no difference within an accuracy for the calculations of the order of 10^{-20}.

5.1. Comparison results for the locations of the collinear points

In this subsection, the locations of the collinear points x_1, x_2, and x_3, will be determined for two different values which represent the effect of triaxiality. To investigate the influence of the non-sphericity, such locations have been also calculated whether the bigger primary has spherical shape; see forthcoming Tables 1–3.

μ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.1$
0.1	-1.2596998329023315	-1.257244268416513	-1.2375104711119975
0.2	-1.2710486907398812	-1.267987369104578	-1.243242319805311
0.3	-1.2567346958119818	-1.2532537370836752	-1.2250162842749763
0.4	-1.230813769364947	-1.2270039093866836	-1.1960185960739376
0.5	-1.19840614455492	-1.194245806820715	-1.1610628825858291

Table 1: Locations of x_1.

μ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.1$
0.1	-0.6090351100232024	-0.6120723891238719	-0.6336397841054372
0.2	-0.438075958538366	-0.44215304301382785	-0.47002775453396095
0.3	-0.286129782050689	-0.29108338038823495	-0.3238099047695225
0.4	-0.14161752558401752	-0.1474171520016291	-0.18442328736484803
0.5	0.0000000000000000	-0.006685433830323481	-0.04775518486793163

Table 2: Locations of x_2.

μ	$\varepsilon = 0$	$\varepsilon = 0.01$	$\varepsilon = 0.1$
0.1	1.0416089085710600	1.051843583098057	1.1408674709495656
0.2	1.0828394642022434	1.0933782882169265	1.1835599768280145
0.3	1.123205958808682	1.134816029733530	1.2253756122557555
0.4	1.1620452673060393	1.1733068737679073	1.2656753426958351
0.5	1.198061445549200	1.2101263056681089	1.3035369762332791

Table 3: Locations of x_3.

5.2. Curves of motion around the collinear points

One of the key facts related to periodic trajectories around collinear libration points to be checked is if the moving body remains an infinite time in the vicinity of collinear points. Moreover, it holds that these orbits can be used to transfer the trajectory of the infinitesimal body (for instance, a spacecraft) to periodic orbits close to the nominal or to associate stable manifolds. The basic idea behind this approach depends on the fact that making a spacecraft to follow a reference periodic orbit is often not critical, while it is much more important to save enough fuel for station-keeping during the planned mission term. Thus, a special attention has been paid in this subsection on the graphs of periodic orbits in the proximity of collinear points having several values for mass ratio μ and period T, where T is the dimensionless periodic time given by $T = 2\pi/s$, see Figs. (1)-(15).
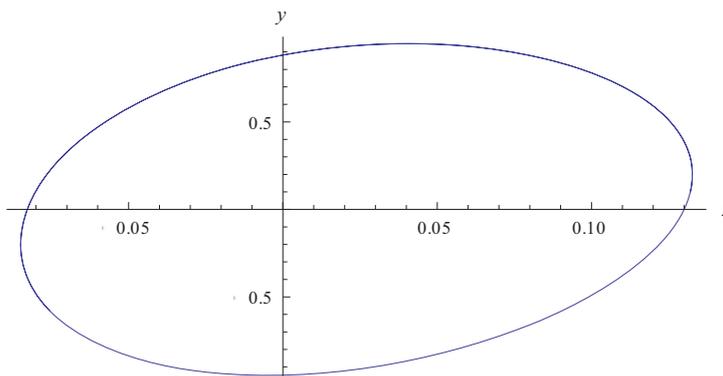
Figure 1: Periodic orbits around x_1, when $\mu = 0.1$ and $T = 1.9793$.
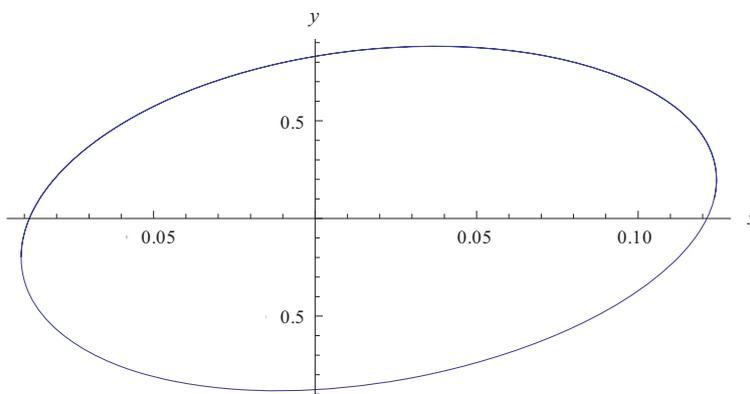
Figure 2: Periodic orbits around x_1, when $\mu = 0.2$ and $T = 2.07021$.
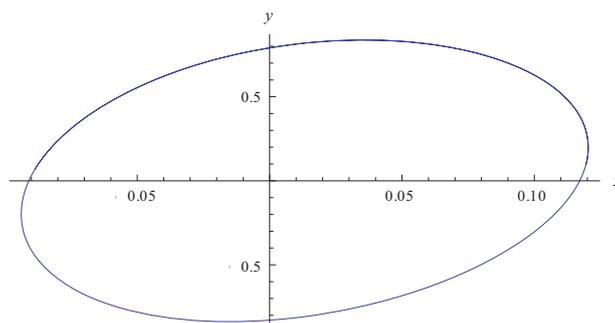
Figure 3: Periodic orbits around x_1, when $\mu = 0.3$ and $T = 2.14099$.
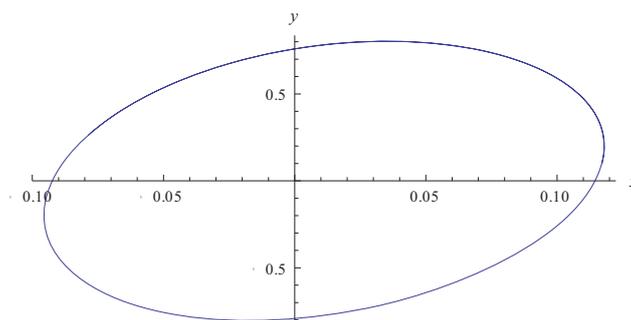
Figure 4: Periodic orbits around x_1, when $\mu = 0.4$ and $T = 2.20247$.
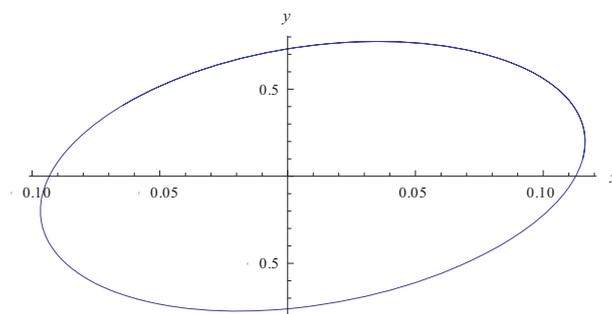
Figure 5: Periodic orbits around x_1, when $\mu = 0.5$ and $T = 2.2584$.
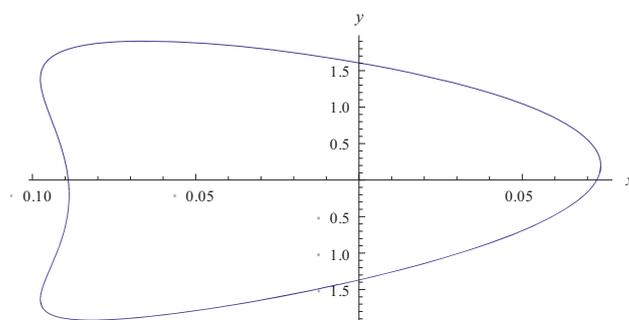
Figure 6: Periodic orbits around x_2, when $\mu = 0.1$ and $T = 3.1185$.
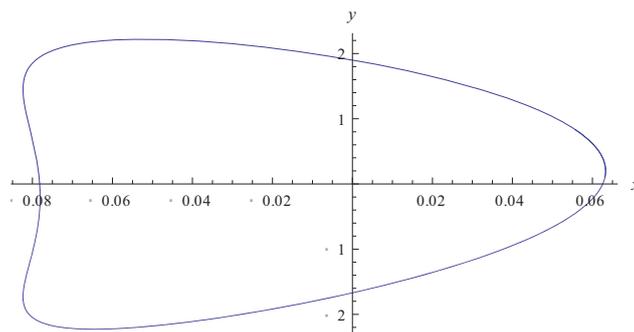
Figure 7: Periodic orbits around x_2, when $\mu = 0.2$ and $T = 2.97723$.
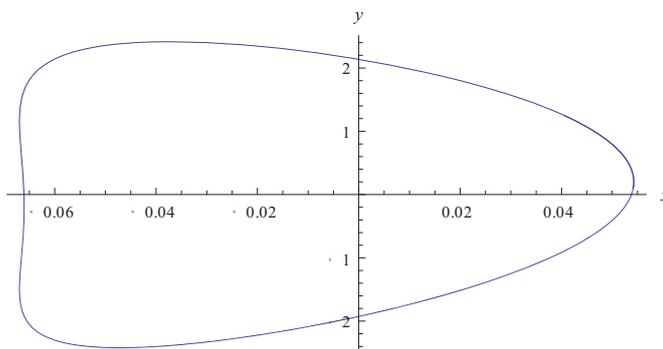
Figure 8: Periodic orbits around x_2, when $\mu = 0.3$ and $T = 2.90639$.
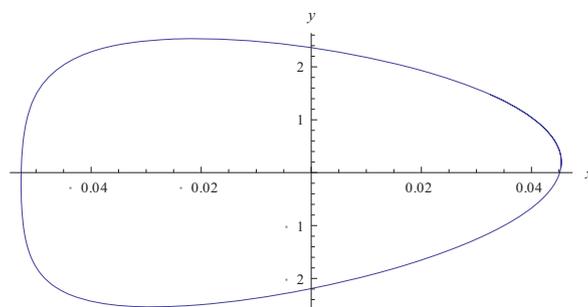
Figure 9: Periodic orbits around x_2, when $\mu = 0.4$ and $T = 2.8729$.
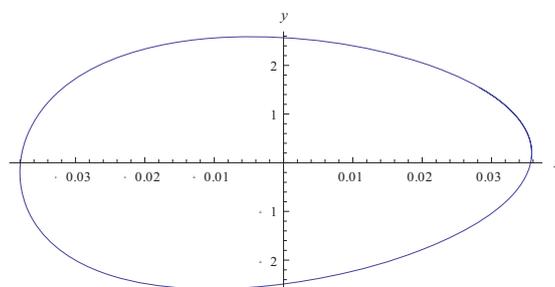
Figure 10: Periodic orbits around x_2, when $\mu = 0.5$ and $T = 2.86683$.
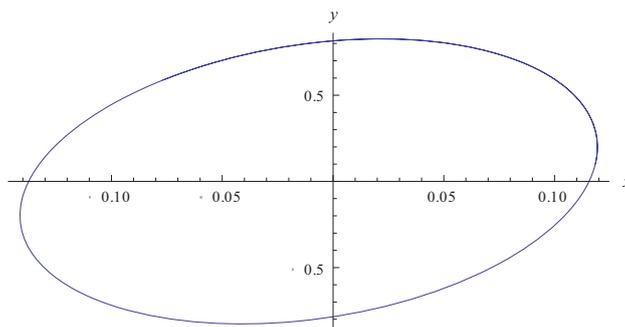
Figure 11: Periodic orbits around x_3, when $\mu = 0.1$ and $T = 2.32001$.
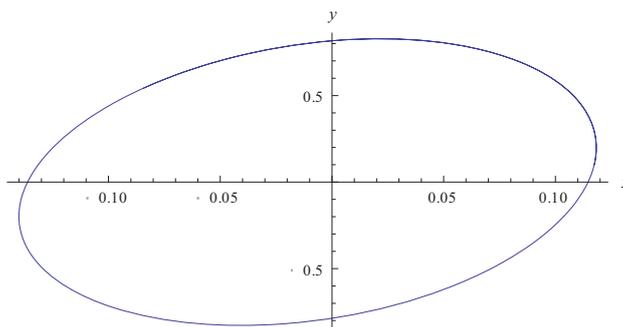
Figure 12: Periodic orbits around x_3, when $\mu = 0.2$ and $T = 2.29955$.
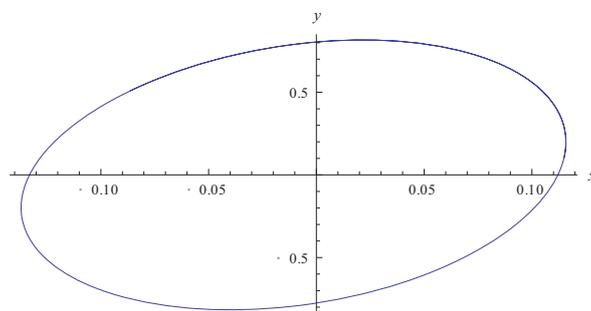
Figure 13: Periodic orbits around x_3, when $\mu = 0.3$ and $T = 2.28855$.
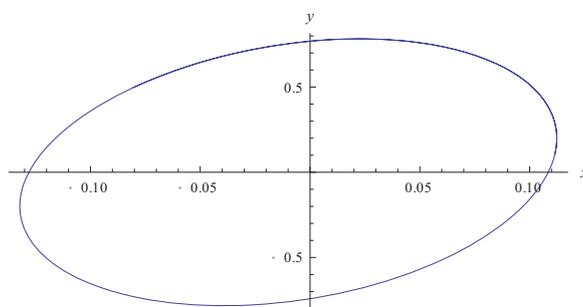
Figure 14: Periodic orbits around x_3, when $\mu = 0.4$ and $T = 2.29387$.
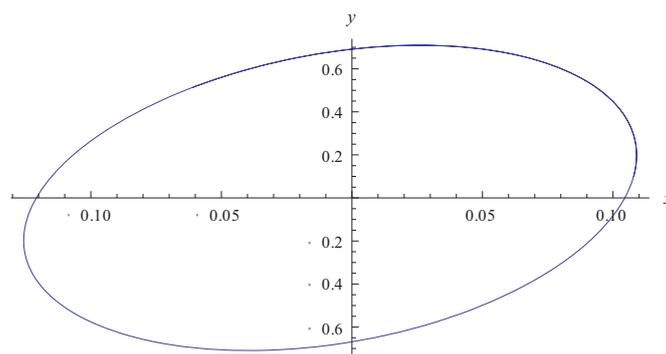
Figure 15: Periodic orbits around x_3, when $\mu = 0.5$ and $T = 2.33084$.

It should be noted that, since the motion in the vicinity of the collinear points are unstable in general, there is a special interest to find their locations and associated periodic orbits, as well. These points provide a useful platform to investigate both the solar system and the universe. This importance comes from a growing interest in missions to collinear libration points and associated periodic orbits.

6. Conclusion

In this paper, the equations of motion in the framework of the restricted three-body problem are found where the bigger primary is a triaxial rigid body. The locations of the collinear libration points have been also evaluated. The analytical study on the periodic solution around these points is constructed and given up to second order in the case that the initial state of the motion gives rise to periodic orbits. Furthermore, some numerical results are established for the locations of each collinear libration point with the existence of triaxial terms. These results are compared with the case consisting of the bigger primary has a spherical shape for different values regarding the mass ratio. Moreover, the curves of motion in the proximity of these points have been also plotted. Finally, we would like to point out that the model of the restricted three-body problem results interesting in space dynamics, and in particular, if the model includes the effect of the triaxiality. In that case, significant improvements for theories regarding the motion for certain satellites in our solar system will follow.

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