BINOMIAL COEFFICIENTS, THE BRACKET FUNCTION, AND COMPOSITIONS WITH RELATIVELY PRIME SUMMANDS

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To Professor Alfred T. Brauer on the occasion of his 70th birthday.

It is known [5] that a necessary and sufficient condition for p to be prime is that for every natural number n

$$(1) \qquad \binom{n}{p} \equiv \left[\frac{n}{p}\right] \pmod{p},$$

where $[x]$ denotes the greatest integer less than or equal to x.

Indeed this result is equivalent to the congruence

$$(1 - x)^k \equiv 1 - x^k \pmod{p}$$

as is evident from the generating functions

$$(2) \qquad \sum_{n=k}^{\infty} \binom{n}{k} x^{n-k} = (1 - x)^{-k-1}, \quad |x| < 1$$

and

$$(3) \qquad \sum_{n=k}^{\infty} \left[\frac{n}{k}\right] x^{n-k} = (1 - x)^{-1} (1 - x^k)^{-1}, \quad |x| < 1 .$$

These results and some extensions of (1) in a recent paper [2] suggest that there is more than a casual relation between the binomial coefficients and the bracket function. In the present paper this relation is made evident by exhibiting an expansion of the binomial coefficients in terms of the bracket function, and conversely. These expansions give congruences equivalent to (1), and the expansions are a special case of a general inversion theorem. In the course of the analysis we obtain novel results concerning the compositions (ordered partitions) of a natural number into relatively prime summands. Expansions involving unordered partitions are also developed.

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The compositions (Zergliederungen) of n into positive summands are given as the solution of the Diophantine equation

(4) $$a_1 + a_2 + \ldots + a_k = n, \quad (a_i \geq 1)$$

whereas the partitions (Zerfällungen) of n into positive summands are given by the same equation together with the restriction that

$$1 \leq a_1 \leq a_2 \leq \ldots \leq a_k .$$

Thus the compositions of 4 into positive summands in all are:

4; 3+1; 1+3; 2+2; 2+1+1; 1+2+1; 1+1+2; 1+1+1+1 .

The partitions are: 4; 1+3; 2+2; 1+1+2; 1+1+1+1 .

Catalan [3], [4], [6, Vol. 2, 114, 126] proved in 1838 that the equation

(5) $$a_1 + a_2 + \ldots + a_k = n, \quad (a_i \geq 0)$$

has $\binom{n+k-1}{k-1}$ solutions. He then observed in 1868 that equation (4) has $\binom{n-1}{k-1}$ solutions. In fact this follows by adding 1 to each summand in (5). A direct proof of the enumeration is not difficult. Indeed (Cf. Bachmann [1, Vol. 2, 105-7]; MacMahon [8, Vol. 1, 150-1]; Riordan [10, 124]) if $C_k(n)$ be the number of compositions of n into k positive summands, then

$$(x + x^2 + x^3 + \ldots)^k = \sum_{n=k}^{\infty} C_k(n)\, x^n$$

$$= \left(\frac{x}{1-x}\right)^k = \sum_{n=k}^{\infty} \binom{n-1}{k-1} x^n ,$$

from which the result is evident. P. Paoli [6, Vol. 2, 107] anticipated Catalan in 1780.

We may state this basic result in the enumerative form

(6) $$C_k(n) = \binom{n-1}{k-1} = \sum_{\substack{a_1 + \ldots + a_k = n \\ a_i \geq 1}} 1$$

The simple identity

$$\binom{n}{k} = \sum_{j=k}^{n} \binom{j-1}{k-1}$$

now allow us to infer that

$$(7) \qquad \binom{n}{k} = \sum_{j=k}^{n} C_k(j) = \sum_{j=k}^{n} \sum_{\substack{a_1 + \dots + a_k = j \\ a_i \geq 1}} 1 .$$

With this expansion we are now in a position to assert

Theorem 1.

$$(8) \qquad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] \sum_{\substack{a_1 + \dots + a_k = j \\ (a_1, \dots, a_k) = 1}} 1 .$$

Proof. The expansion is evident from (7). When we restrict the solutions of the equation $a_1 + a_2 + \dots + a_k = j$ to those which are relatively prime, it is evident that we may restore the equality by counting how many multiples of j there are, less than or equal to n, and this is precisely the meaning of $[n/j]$.

A simple example will illustrate. On the one hand, by (7)

$$\binom{10}{3} = \sum_{j=3}^{10} \sum_{\substack{a_1+a_2+a_3 = j \\ a_i \geq 1}} 1 = 1+3+6+10+15+21+28+36 = 120 .$$

However, not all the partitions of j are formed by relatively prime integers. These cases are 6 = 2 + 2 + 2; 8 = 2 + 2 + 4 = 2 + 4 + 2 = 4 + 2 +2; 9 = 3 + 3 + 3; 10 = 2 + 2 + 6 = 2 + 6 + 2 = 6 + 2 + 2 = 2 + 4 + 4 = 4 + 2 + 4 = 4 + 4 + 2. Removing the common factors, we could just as well have written such solutions in the forms 3 = 1 + 1 + 1; 4 = 1 + 1 + 2 = 1 + 2 + 1 = 2 + 1 + 1; 3 = 1 + 1 + 1; 5 = 1 + 1 + 3 = 1 + 3 + 1 = 3 + 1

$+ 1 = 1 + 2 + 2 = 2 + 1 + 2 = 2 + 2 + 1$, provided that we regroup and count multiplicities. This gives

$$\binom{10}{3} = 3(1) + 2(3) + 2(6) + (10 - 1) + 15 + (21 - 3) + (28 - 1) + (36 - 6),$$

or

$$\binom{10}{3} = \sum_{j=3}^{10} \left[\frac{10}{j}\right] \sum_{\substack{a_1+a_2+a_3 = j \\ (a_1, a_2, a_3) = 1}} 1$$

We shall obtain expansion (8) by an entirely different approach later in this paper.

For the sake of completeness we wish to show that Theorem 1 is equivalent to the following result due to J. Schröder [11]. Schröder proved the following Theorem 2.

$$\binom{n}{k} = \sum_{\substack{(a_1, a_2, \ldots, a_k) = 1 \\ 1 \le a_i \le n-k+1}} \left[\frac{n}{a_1 + a_2 + \ldots + a_k}\right]. \tag{9}$$

As far as the writer has been able to determine, this is one of the very few expansions in the literature of the sort under discussion. Schröder proved the formula by an enumeration in k-dimensional space and an induction from k to $k + 1$. As for the equivalence of (9) and (8), we have

$$\sum_{\substack{(a_1, \ldots, a_k) = 1 \\ 1 \le a_i \le n-k+1}} \left[\frac{n}{a_1 + \ldots + a_k}\right] = \sum_{1 \le j \le n-k+1} \left[\frac{n}{k+j-1}\right] \sum_{\substack{a_1+\ldots+a_k=k+j-1 \\ (a_1, \ldots, a_k)=1}} 1$$

$$= \sum_{1 \le j-k+1 \le n-k+1} \left[\frac{n}{j}\right] \sum_{\substack{a_1+\ldots+a_k = j \\ (a_1, \ldots, a_k)=1}} 1 ,$$

which is our relation (8) and the steps are reversible.

In view of Schröder's approach, it is of interest to make some remarks here about lattice points. By a lattice point in k-space is meant a point $(a_1, a_2, \ldots, a_k)$ where the coordinates a_i are integers. If we view space from the origin $(0, \ldots, 0)$ and assume that the presence of a point may block our view of points further out along the same ray, then we may speak of visible lattice points. In order for a point to be a visible lattice point it is necessary and sufficient that $(a_1, a_2, \ldots, a_k) = 1$. Thus we may state the theorem of Schröder in the form of

Theorem 3. Let $V_j(k)$ = the number of visible lattice points in k-space, seen from the origin, and lying on the hyperplane $a_1 + a_2 + \ldots + a_k = j + k - 1$. Then

$$(10) \qquad \binom{n}{k} = \sum_{j=1}^{n-k+1} V_j(k) \left[\frac{n}{k+j-1} \right] .$$

Thus, in 2-space,

$$\binom{n}{2} = \sum_{j=1}^{n-1} V_j(2) \left[\frac{n}{j+1} \right] ,$$

where $V_j(2)$ is the number of visible lattice points lying entirely within the first quadrant and on the line $x + y = j + 1$. The successive values of $V_j(2)$ $(j = 1, 2, \ldots)$ here are 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, ... and we always have in this case $V_j(2) \le j$, since the line segment in question has just this many lattice points in all.

In general we evidently have the estimate

$$(11) \qquad V_j(k) \le \binom{j+k-2}{k-1} .$$

As other examples of Theorem 1 we have

$$\binom{n}{3} = \left[\frac{n}{3}\right] + 3\left[\frac{n}{4}\right] + 6\left[\frac{n}{5}\right] + 9\left[\frac{n}{6}\right] + 15\left[\frac{n}{7}\right] + 18\left[\frac{n}{8}\right] + \ldots$$

$$\binom{n}{4} = \left[\frac{n}{4}\right] + 4\left[\frac{n}{5}\right] + 10\left[\frac{n}{6}\right] + 20\left[\frac{n}{7}\right] + 34\left[\frac{n}{8}\right] + 56\left[\frac{n}{9}\right] + \ldots$$

Since the equation $a_1 + a_2 + \ldots + a_k = k$, $(a_i \geq 1)$, has the sole solution $1 + 1 + \ldots + 1 = k$, we have from Theorem 1 the (equivalent) Corollary 1.

(12)
$$\binom{n}{k} - \left[\frac{n}{k}\right] = \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j)$$

where the number-theoretic function $R_k(j)$ is defined by

(13)
$$R_k(j) = \sum_{\substack{a_1 + \ldots + a_k = j \\ (a_1, \ldots, a_k) = 1}} 1$$

and is the number of compositions of j into k relatively prime positive summands.

In order to relate our expansion to congruence (1) we shall now study the arithmetic nature of the function $R_k(j)$.

First of all, it is easy to use (2), (8), and (3) in order to develop a generating function for $R_k(j)$. Indeed we have

$$\frac{x^k}{(1-x)^{k+1}} = \sum_{n=k}^{\infty} \binom{n}{k} x^n = \sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j)$$

$$= \sum_{j=k}^{\infty} R_k(j) \sum_{n=j}^{\infty} \left[\frac{n}{j}\right] x^n$$

$$= \sum_{j=k}^{\infty} R_k(j) \frac{x^j}{(1-x)(1-x^j)} ,$$

and the lower summation index may be changed to $j = 1$ since $R_k(j) = 0$ if $j < k$. Thus we have established

Theorem 4. The number-theoretic function $R_k(j)$ is the coefficient in the Lambert series

(14)
$$\sum_{j=1}^{\infty} R_k(j) \frac{x^j}{1-x^j} = \frac{x^k}{(1-x)^k} ,$$

or, equivalently,

$$(15) \qquad \sum_{j=1}^{\infty} R_k(j) \frac{x^j}{1-x^j} = \sum_{j=k}^{\infty} C_k(j), x^j .$$

It may be of interest to compare this result with the Lambert series for the Euler totient function (Cf. Knopp [7, 466-7]):

$$(16) \qquad \sum_{j=1}^{\infty} \phi(j) \frac{x^j}{1-x^j} = \frac{x}{(1-x)^2} .$$

Now [7, 466-7] it is known that the Lambert series

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} A_n x^n$$

is equivalent to the relation

$$A_n = \sum_{d|n} a_d ,$$

and so we have from (15) that

$$(17) \qquad C_k(n) = \sum_{d|n} R_k(d) .$$

We invert this expansion by the Möbius inversion theorem and so find

<u>Theorem 5</u>. <u>The number of compositions (ordered partitions) of the integer</u> n <u>into</u> k <u>relatively prime positive summands is given by</u>

$$(18) \qquad R_k(n) = \sum_{d|n} C_k(d)\,\mu(n/d) = \sum_{d|n} \binom{d-1}{k-1} \mu(n/d) .$$

Therefore we also have Theorem 1 in the equivalent form:

<u>Theorem 6</u>.

$$(19) \qquad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] \sum_{d|j} \binom{d-1}{k-1} \mu(j/d) .$$

We have presented what seems a natural way to arrive at relation

(19), but we now give a very short derivation on the basis of a famous formula of E. Meissel. First of all we note the general lemma

(20) $$\sum_{j \le x} \sum_{d | j} f(d, j) = \sum_{d \le x} \sum_{\substack{j \le x \\ d | j}} f(d, j)$$

$$= \sum_{d \le x} \sum_{m \le x/d} f(d, md)$$

valid for any number-theoretic function $f(d, j)$.

Meissel (1850 [6, Vol. 1, 441] proved that for all real $x \ge 1$

(21) $$\sum_{m \le x} \left[\frac{x}{m}\right] \mu(m) = 1 \quad .$$

Thus we have

$$\sum_{1 \le j \le x} \left[\frac{x}{j}\right] \sum_{d | j} \binom{d-1}{k-1} \mu(j/d)$$

$$= \sum_{d \le x} \binom{d-1}{k-1} \sum_{m \le x/d} \left[\frac{x/d}{m}\right] \mu(m)$$

$$= \sum_{d \le x} \binom{d-1}{k-1} = \binom{[x]}{k} \quad ,$$

and this gives us (more generally than Theorem 6)

<u>Theorem 7.</u> <u>For any real</u> $x \ge 1$, <u>and natural numbers</u> $k \ge 1$,

(22) $$\binom{[x]}{k} = \sum_{1 \le j \le x} \left[\frac{x}{j}\right] \sum_{d | j} \binom{d-1}{k-1} \mu(j/d) \quad .$$

The arithmetical nature of $R_k(n)$ is of interest and in view of (12) the congruence (1) is evidently equivalent to

<u>Theorem 8.</u> <u>The congruence</u>

(23) $$R_k(n) \equiv 0 \pmod{k}$$

<u>is true for all natural numbers $n \ge k + 1$ if and only if k is prime.</u>

Our proof will depend on some elementary results about the binomial coefficients and the Möbius function.

Now

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \quad . \tag{24}$$

Therefore, if p is any prime which divides each divisor d of n, then

$$\sum_{p \mid d,\, d \mid n} \mu(n/d) = \sum_{pd' \mid pm} \mu(pm/pd') = \sum_{d' \mid m} \mu(m/d') = \begin{cases} 1, & m = 1, \\ 0, & m > 1, \end{cases}$$

or therefore

$$\sum_{p \mid d,\, d \mid n} \mu(n/d) = \begin{cases} 1, & n = p \\ 0, & n > p \end{cases} \quad . \tag{25}$$

Now it is familiar that

$$\binom{d-1}{p-1} \equiv \begin{cases} 0 \pmod p, & p \nmid d \;, \\ 1 \pmod p, & p \mid d \;, \end{cases} \tag{26}$$

and so we have

$$\sum_{d \mid n} \binom{d-1}{p-1} \mu(n/d) \equiv \begin{cases} 0 \pmod p, & \text{for } p \nmid d,\ d \mid n, \text{ i.e. } p \nmid n \;, \\ \sum\limits_{d \mid n} \mu(n/d), & \text{for } p \mid d,\ d \mid n, \text{ i.e. } p \mid n \;. \end{cases}$$

Thus in any case $(p \mid n$ or $p \nmid n)$ we have by this and (25) that

$$\sum_{d \mid n} \binom{d-1}{p-1} \mu(n/d) \equiv 0 \pmod p \tag{27}$$

for all integers $n \geq p + 1$ if p is a prime.

As for the converse, suppose that $R_k(n) \equiv 0 \pmod k$ for all $n \geq k + 1$. Then, in virtue of (17) we should have

$$C_k(n) \equiv \begin{cases} 1 \pmod{k}, & n \geq k+1, \text{ if } k \mid n, \\ 0 \pmod{k}, & n \geq k+1, \text{ if } k \nmid n, \end{cases}$$

or, equivalently,

$$\binom{n-1}{k-1} \equiv \left[\frac{n}{k}\right] - \left[\frac{n-1}{k}\right] \pmod{k} \quad \text{for all } n \geq k+1 \quad .$$

Summing both sides from $n = k + 1$ to $n = m$ we should then have

$$\binom{m}{k} - 1 \equiv \left[\frac{m}{k}\right] - 1 \pmod{k} \quad \text{for all } m \geq k+1 \quad ,$$

or that is

$$\binom{m}{k} \equiv \left[\frac{m}{k}\right] \pmod{k} \quad \text{for all } m \geq k+1 \quad .$$

But this can happen only when $k =$ prime as we know from our original congruence (1) for which a separate proof is known.

As a matter of fact then, congruence (23) is a simple consequence of (1).

Table of Values of $R_k(n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13.	n
1	1	0	0	0	0	0	0	0	0	0	0	0	0	
2		1	2	2	4	2	6	4	6	4	10	4	12	
3			1	3	6	9	15	18	27	30	45	42	66	
4				1	4	10	20	34	56	80	120	154	220	
5					1	5	15	35	70	125	210	325	495	
6						1	6	21	56	126	252	461	792	
7							1	7	28	84	210	462	924	
8								1	8	36	120	330	792	
9									1	9	45	165	495	
10										1	10	55	220	
11											1	11	66	
12												1	12	
13													1	
⋮														
k														

The numbers $R_k(n)$ form an interesting modification of the familiar Pascal array. We have from (18) the modified binomial theorem relation

$$\sum_{k=1}^{n} R_k(n)\, x^{k-1} = \sum_{d|n} \mu(n/d)(x+1)^{d-1} \quad . \tag{28}$$

In particular, when $x = 1$ this sum represents the total number of compositions of n into relatively prime summands. These values, 1, 1, 3, 6, 15, 27, 63, 120, 252, 495, 1023, 2010, 4095, ... afford a check of the table.

We note a few special values of $R_k(n)$:

$$R_k(p^s) = \binom{p^s - 1}{k-1} - \binom{p^{s-1} - 1}{k-1}, \quad s \geq 1,\ p = \text{prime}, \tag{29}$$

$$R_k(pq) = \binom{pq-1}{k-1} - \binom{p-1}{k-1} - \binom{q-1}{k-1} + \binom{0}{k-1}, \quad p, q \text{ primes}, \tag{30}$$

$$R_k(p^2q) = \binom{p^2q-1}{k-1} - \binom{pq-1}{k-1} - \binom{p^2-1}{k-1} + \binom{p-1}{k-1} \quad , \tag{31}$$

with similar formulas for other cases. The expansion always contains as even number of binomial coefficients when $n \geq 2$ since $R_1(n) = 0$ for $n \geq 2$.

It is of interest to translate (18) into terms of Dirichlet series. It is easily shown that the formal relation involves Riemann's Zeta function and is

$$\sum_{n=1}^{\infty} C_k(n)\, n^{-s} = \zeta(s) \sum_{n=1}^{\infty} R_k(n)\, n^{-s} \quad , \tag{32}$$

and this also follows from (17).

Having found the expansion (19) of a binomial coefficient in terms of the bracket function, it is natural to look for an inverse expansion.

Put

$$\left[\frac{n}{k}\right] = \sum_{j=k}^{n} \binom{n}{j} A_k(j) .$$

Then

$$\sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \left[\frac{j}{k}\right] = \sum_{r=k}^{n} A_k(r) \sum_{j=r}^{n} (-1)^{n-j} \binom{n}{j} \binom{j}{r}$$

$$= A_k(n) ,$$

since the inner summation is merely a well-known Kronecker delta.

Thus an expansion inverse to (19) is given by

Theorem 9.

$$(33) \qquad \left[\frac{n}{k}\right] = \sum_{j=k}^{n} \binom{n}{j} \sum_{d=k}^{j} (-1)^{j-d} \binom{j}{d} \left[\frac{d}{k}\right] .$$

Since $A_k(k) = 1$ we have an analogy to (12)

Corollary 2.

$$(34) \qquad \left[\frac{n}{k}\right] - \binom{n}{k} = \sum_{j=k+1}^{n} \binom{n}{j} A_k(j) ,$$

where

$$(35) \qquad A_k(j) = \sum_{d=k}^{j} (-1)^{j-d} \binom{j}{d} \left[\frac{d}{k}\right] .$$

For $A_k(j)$ we next develop an expansion inverse to (14). Indeed, we have from (35), (2), and (3)

$$\sum_{j=k}^{\infty} A_k(j) \left(\frac{x}{1-x}\right)^j = \sum_{d=k}^{\infty} (-1)^d \left[\frac{d}{k}\right] \sum_{j=d}^{\infty} \binom{j}{d} \left(\frac{x}{x-1}\right)^j$$

$$= (1-x) \sum_{d=k}^{\infty} \left[\frac{d}{k}\right] x^d = \frac{x^k}{1-x^k} .$$

It is evident from (35) that $A_k(j) = 0$ for $j < k$, so we have

Theorem 10. The expansion inverse to (14) is

$$\sum_{j=1}^{\infty} A_k(j) \left(\frac{x}{1-x}\right)^j = \frac{x^k}{1-x^k} \quad . \tag{36}$$

Now it is evident that (14) and (36) imply a pair of orthogonal relations involving the functions $R_k(j)$ and $A_k(j)$. By a routine calculation we find upon substitution of the one expansion into the other that we have

Theorem 11. The numbers $R_k(j)$ and $A_k(j)$ satisfy the orthogonality relations

$$\sum_{j=k}^{n} R_k(j)A_j(n) = \delta_k^n \tag{37}$$

and

$$\sum_{j=k}^{n} A_k(j)R_j(n) = \delta_k^n \quad . \tag{38}$$

Thus we have also established a general inversion theorem, of which (19) and (33) are special cases. We have

Theorem 12. For any two sequences $f(n, k)$, $g(n, k)$

$$f(n, k) = \sum_{j=k}^{n} g(n, j)R_k(j) \tag{39}$$

if and only if

$$g(n, k) = \sum_{j=k}^{n} f(n, j)A_k(j) \quad , \tag{40}$$

where $R_k(j)$ is given by (18) and $A_k(j)$ by (35).

An alternative form of (35) is easily gotten by way of the recurrence

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1} \quad .$$

Indeed we find that

$$A_k(n) = \sum_{j=k}^{n} (-1)^{n-j} \binom{n-1}{j-1} \left\{ \left[\frac{j}{k}\right] - \left[\frac{j-1}{k}\right] \right\},$$

but $[j/k] - [(j-1)/k] = 1$ or 0 accordingly as $k | j$ or $k \nmid j$, whence Theorem 13.

(41) $$A_k(n) = \sum_{\substack{k \le j \le n \\ k|j}} (-1)^{n-j} \binom{n-1}{j-1} = \sum_{1 \le m \le n/k} (-1)^{n-mk} \binom{n-1}{mk-1}$$

Table of Values of $A_k(n)$

	1	2	3	4	5	6	7	8	9	10	11	12	13..n
1	1	0	0	0	0	0	0	0	0	0	0	0	0
2		1	-2	4	-8	16	-32	64	-128	256	-512	1024	-2048
3			1	-3	6	-9	9	0	-27	81	-162	243	-243
4				1	-4	10	-20	36	-64	120	-240	496	-952
5					1	-5	15	-35	70	-125	200	-255	275
6						1	-6	21	-56	126	-252	463	-804
7							1	-7	28	-84	210	-462	924
8								1	-8	36	-120	330	-792
9									1	-9	45	-165	495
10										1	-10	55	-220
11											1	-11	66
12												1	-12
13													1
⋮													
k													

The numbers $A_k(n)$ also form an interesting modification of the Pascal array, and the companion to (28) is

$$(42)\qquad \sum_{k=1}^{n} A_k(n)\, x^{k-1} = \sum_{j=1}^{n} (-1)^{n-j}\binom{n}{j} \sum_{k=1}^{j} \left[\frac{j}{k}\right] x^{k-1} .$$

When $x = 1$ we recall that

$$\sum_{k=1}^{j} \left[\frac{j}{k}\right] = \sum_{k=1}^{j} \tau(k) , \quad \text{where } \tau(k) = \sum_{d\mid k} 1 ,$$

and so we have

$$\begin{aligned}
\sum_{k=1}^{n} A_k(n) &= \sum_{j=1}^{n} (-1)^{n-j}\binom{n}{j} \sum_{k=1}^{j} \tau(k) \\
&= \sum_{k=1}^{n} \tau(k) \sum_{j=k}^{n} (-1)^{n-j}\binom{n}{j} \\
&= \sum_{k=1}^{n} (-1)^{n-k}\binom{n-1}{k-1} \tau(k) \\
&= \sum_{k=0}^{n-1} (-1)^{n-1-k}\binom{n-1}{k} \tau(k+1) .
\end{aligned}$$

This result is easily inverted, and we may state these formulas as

Theorem 14. For all integers $n \geq 0$, and $\tau(k)$ = number of divisors of k,

$$(43)\qquad \sum_{j=1}^{n+1} A_j(n+1) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} \tau(k+1) = \Delta_{x,1}^{n}\, \tau(x) \Big|_{x=1}$$

and inversely

$$(44)\qquad \tau(n+1) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=1}^{k+1} A_j(k+1) .$$

The first few values of the sum (43) are 1, 1, -1, 2, -5, 13, -33, 80, -184, 402, -840, ... For example, we have the following difference table:

$$
\begin{array}{ccccccccccccc}
1 & & 2 & & 2 & & 3 & & 2 & & 4 & \dots & \tau(n) \\
 & 1 & & 0 & & 1 & & -1 & & 2 & & & \\
 & & -1 & & 1 & & -2 & & 3 & & & & \\
 & & & 2 & & -3 & & 5 & & & & & \\
 & & & & -5 & & 8 & & & & & & \\
 & & & & & 13 & & & & & & &
\end{array}
$$

The arithmetical nature of $A_k(n)$ is of interest. In view of (34) and (1) we evidently have a result analogous to (23). In fact we have

Theorem 15. *The congruence*

$$A_k(n) \equiv 0 \pmod{k} \tag{45}$$

is true for all natural numbers $n \geq k + 1$ *if and only if* k *is prime.*

Indeed this congruence follows easily from (1) since we have

$$
\begin{aligned}
A_k(n) &= \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \left[\frac{j}{k}\right] \\
&\equiv \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \binom{j}{k} \pmod{k} \text{ for all } n \geq k \text{ if and only if } k = \text{prime}, \\
&= \delta_k^n \equiv 0 \pmod{k} \text{ for all } n \geq k+1.
\end{aligned}
$$

We should like next to return to relation (28) and give another congruence involving $R_k(n)$. It is known [6, Vol. 1, 84-86] that

$$\sum_{d \mid n} \mu(d)\, a^{n/d} \equiv 0 \pmod{n} \tag{46}$$

for all integers $a \geq 1$. In fact Gegenbauer showed that

$$\sum_{d \mid n} f(d)\, a^{n/d} \equiv 0 \pmod{n} \quad \text{whenever} \quad \sum_{d \mid n} f(d) \equiv 0 \pmod{n}.$$

Gauss proved (46) when a = prime. Thus we have from (28) that

$$a \sum_{k=1}^{n} R_k(n)\, (a-1)^{k-1} \equiv 0 \pmod{n}, \quad (a \geq 1,\ n \geq 1) \tag{47}$$

and in particular this holds for $a = 2$. Thus the numbers 2, 2, 6, 12, 30, 54, 126, 240, 504, 990, 2046, 4020, 8190, ... are, respectively, divisible by 1, 2, 3, 4, 5, ... thereby affording a check of the column sums in the table of values of $R_k(n)$ given previously.

It should be remarked that any formula, such as (18), which gives the number of compositions of n into k relatively prime positive summands also solves the problem of counting how many compositions are possible when the summands have greatest common divisor g; for clearly if $R_k(n, g)$ is this number, then

$$(48)\quad R_k(n, g) = \sum_{\substack{a_1+\dots+a_k = n \\ (a_1, \dots, a_k) = g}} 1 = \sum_{\substack{b_1+\dots+b_k = n/g \\ (b_1, \dots, b_k) = 1}} 1 = \begin{cases} 0, & g \nmid n, \\ R_k(n/g), & g \mid n. \end{cases}$$

Thus far we have restricted out attention to compositions. It may therefore be of some interest to consider the possibility of expansion of a binomial coefficient in terms of bracket functions and partitions. Let

$$(49)\qquad p(n, k) = \sum_{\substack{1 \le b_1 \le b_2 \le \dots \le b_k \le n \\ b_1 + b_2 + \dots + b_k = n}} 1$$

so that $p(n, k)$ is the number of partitions of n into k positive summands. Consider a typical partition $n = b_1 + \dots + b_k$. If 1 occurs a_1 times, 2 occurs a_2 times, etc., then it is well known (e.g. Cf. [1, Vol. 2, 102]) that we may restate (49) in the form

$$(50)\qquad p(n, k) = \sum_{\substack{a_1 + 2a_2 + 3a_3 + \dots + na_n = n \\ a_1 + a_2 + \dots + a_n = k,\ a_i \ge 0}} 1 .$$

We recall that if we form an arrangement of k marks $(c_1 c_2 c_1 c_4 c_3 \dots)$, where c_1 occurs a_1 times, c_2 occurs a_2 times, etc., with $k = a_1 + \ldots + a_n$, $a_i \geq 0$, then the total number of distinct such arrangements (permutations) which may be formed is enumerated by the expression

$$\frac{k!}{a_1!\,a_2!\ldots a_n!} .$$

This expression then enumerates the <u>compositions</u> of n into k positive summands corresponding to a given <u>partition</u> $n = b_1 + b_2 + \ldots + b_k$. It follows from this that we may change relation (50) into an enumeration of compositions by introducing the above ratio of factorials (instead of just counting 1 for each partition). Thus we evidently have proved

<u>Theorem 16.</u> <u>For all natural numbers</u> n <u>and</u> k

$$(51)\quad \binom{n-1}{k-1} = \sum_{\substack{a_1 + 2a_2 + 3a_3 + \ldots + na_n = n \\ a_1 + a_2 + \ldots + a_n = k,\ a_i \geq 0}} \frac{k!}{a_1!\,a_2!\ldots a_n!} .$$

Again we may argue as we did in going from (6) to (7), whence we have established

<u>Theorem 17.</u>

$$(52)\quad \binom{n}{k} = \sum_{j=k}^{n} \sum_{\substack{a_1 + 2a_2 + 3a_3 + \ldots + ja_j = j \\ a_1 + a_2 + \ldots + a_j = k,\ a_i \geq 0}} \frac{k!}{a_1!\,a_2!\ldots a_j!} .$$

We may next apply the same argument here which we used to obtain Theorem 1, which is to say that we may restrict our attention to relatively prime summands, but have the same total enumeration of compositions, by introducing the bracket function. We evidently have

<u>Theorem 18.</u>

$$(53)\quad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] \sum_{\substack{a_1 + 2a_2 + 3a_3 + \ldots + ja_j = j \\ a_1 + a_2 + \ldots + a_j = k \\ (a_1,\, a_2,\, \ldots,\, a_j) = 1}} \frac{k!}{a_1!\,a_2!\ldots a_j!}$$

It follows that the inner sum gives another way of expressing $R_k(j)$, that is, we conclude that

$$(54) \qquad R_k(n) = \sum_{\substack{a_1 + 2a_2 + 3a_3 + \ldots + na_n = n \\ a_1 + a_2 + \ldots + a_n = k \\ (a_1, a_2, \ldots, a_n) = 1}} \frac{k!}{a_1! a_2! \ldots a_n!} ,$$

and of course the arithmetical properties we found for $R_k(n)$ then apply to this summation also. Thus, also, in Theorem 12, our main inversion theorem, we have several ways of expressing the coefficients $R_k(n)$ and $A_k(n)$.

Some further consequences of Theorem 12 and the other expansions in this paper will be presented later.

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LETTER TO THE EDITOR

B.G. BAUMGART
Glencoe, Illinois

Dear Sir:

In the article "On the Periodicity of the Last Digits of the Fibonacci Numbers" Vol. 1 No. 4, it was proved that for $n \gtrsim 3$ the n-th digit (from the right) had a period of $1.5 \cdot 10^n$ thus accounting for the observation made at the University of Alaska on an IBM 1620; that the last Fibonacci digit cycles every 60 numbers; the second to last digit, every 300 numbers; the third, every 1500; the fourth, 15000; the fifth, 150000.

I,too, have observed the periodicity of the last Fibonacci digits on an IBM 709 at Northwestern University (before discovering the Fibonacci Quarterly). However, I also considered the so called:

Tribonacci Series

1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201, 2209, 4063, 7473...

and found that its last digit repeats every 31 numbers, its second to last digit repeats every 620 numbers and its third to last digit repeats every 6200 numbers;

Tetranacci Series

1, 1, 1, 1, 4, 7, 13, 25, 49, 94, 181, 349, 673, 1297, 2500, 4819, 9289...

and found that the last digit repeats every 1560 numbers as does the second to the last digit. That is the period of the last and the second to the last is the same. The period of the third to last digit is 7800 and I believe the period of the fourth to last digit is also 7800 but I can not say for sure with my present results (I got all my data from one program which truncated at the fourth digit, at the time I was only thinking about the very last digit. However, it will be easy to find out and I shall do so when I get a chance. Actually, this sort of problem is a programmer's dream, because one may lose the most significant part of his calculations with impunity.)

Pentanacci Series

1, 1, 1, 1, 1, 5, 9, 17, 33, 65, 129, 253, 497, 977, 1921, 3777, 7425, 14597...

(Continued on page 302.)