Extendibility of Choquet rational preferences on generalized lotteries

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Abstract. Given a finite set of generalized lotteries, that is random quantities equipped with a belief function, and a partial preference relation on them, a necessary and sufficient condition (Choquet rationality) has been provided for its representation as a Choquet expected utility of a strictly increasing utility function. Here we prove that this condition assures the extension of the preference relation and it actually guides the decision maker in this process.

Keywords: Generalized lottery, preference relation, belief function, probability envelope, Choquet expected utility, Choquet rationality

1 Introduction

In the classical von Neumann-Morgenstern decision theory under risk [23, 18], the decision maker faces "one-shot" decisions [17] by specifying a preference relation on *lotteries*, i.e., random quantities endowed with a probability distribution. If the preference relation satisfies suitable axioms then the preference is representable by an *expected utility* (EU) and the decision maker behaves like an EU maximizer.

The assumptions behind the EU theory rely on a complete probabilistic description of the decisions, which is rarely met in practice. Indeed, in situations of incomplete and revisable information, uncertainty cannot be handled through a probability but it is unavoidable to refer to non-additive uncertainty measures, for which the EU model is no more appropriate.

Here, we refer to Dempster-Shafer *belief functions* [7, 19] as uncertainty measures and to *Choquet expected utility* (CEU) as decision model (see for instance [20, 21, 15, 1]). We recall that in some probabilistic inferential problems belief functions can be obtained as lower envelopes of a family of probabilities, possibly arising as coherent extensions of a probability assessed on a set of events different from those of interest (see for instance [7, 5, 10, 14, 6]).

Another issue typical of real problems is the partial observability of the world which leads the decision maker to act under partial knowledge. Both in the classical expected utility and in the Choquet expected utility frameworks it can be difficult to construct the utility function u and even to test if the preferences agree with an EU (or a CEU). In fact, to find the utility u the classical methods ask for comparisons between "lotteries" and "certainty equivalent" or, in any case, comparisons among particular large classes of lotteries (for a discussion in the EU framework see [13]). For that, the decision maker is often forced to make comparisons which have little or nothing to do with the given problem, having to choose between risky prospects and certainty.

In [4], referring to the EU model, a different approach (based on a "rationality principle") is proposed: it does not need all these non-natural comparisons but, instead, it can work by considering only the (few) lotteries and comparisons of interest. Moreover, when new information is introduced, the same principle assures that the preference relation can be extended maintaining rationality, and, even more, the principle suggests how to extend it.

In [2] and in an extended version [3], we proposed a similar approach for the CEU model by generalizing the usual definition of lottery. In detail, a *generalized lottery* L (or *g-lottery* for short) is a random quantity with a finite support X_L endowed with a Dempster-Shafer belief function Bel_L [7, 19, 22] (or, equivalently, a *basic assignment* m_L) defined on the power set $\wp(X_L)$.

Assuming that the elements of the set $X = \{x_1, \ldots, x_n\}$ resulting by the union of the supports of the considered g-lotteries is totally ordered as $x_1 < \ldots < x_n$ (which is quite natural, thinking at elements of X as money payoffs), then for every g-lottery L the Choquet integral of any strictly increasing utility function $u : X \to \mathbb{R}$, not only is a weighted average (as observed in [12]), but the weights have a clear meaning. In fact, this allows to map every g-lottery L to a "standard" lottery whose probability distribution is constructed (following a pessimistic approach) through the *aggregated basic assignment* M_L.

The "Choquet rationality principle" (namely, condition **(g-CR)**) requires that *it is not possible to obtain two g-lotteries* L *and* L' *with* $M_L = M_{L'}$, *by combining in the same way the aggregated basic assignments of two groups of g-lotteries, if every g-lottery of the first group is not preferred to the corresponding one of the second group, and at least a preference is strict.*

Condition **(g-CR)** turns out to be necessary and sufficient for the existence of a strictly increasing $u : X \to \mathbb{R}$ whose CEU represents our preferences on a finite set $\mathcal{L}$ of g-lotteries, under a natural assumption of agreement of the preference relation with the order of X.

In this paper we show that condition **(g-CR)** assures also the extendibility of a preference relation and actually "guides" the decision maker in this process. An algorithm for the extension of a preference relation to a new pair of g-lotteries is also provided. Such algorithm relies on the solution of at most three linear programming problems and can be used "interactively" by the decision maker in a step by step enlargement of his preferences.

The paper is structured as follows. In Section 2 some preliminary notions are given, while Section 3 copes with preferences on g-lotteries and introduces the condition **(g-CR)**. Finally, Subsection 3.1 presents a motivating example, and Subsection 3.2 deals with the extendibility of a Choquet rational preference relation providing an algorithm for this task.

2 Numerical model of reference

Let X be a finite set of states of nature and denote by $\wp(X)$ the power set of X. We recall that a *belief function* Bel [7, 19, 22] on an algebra of events $\mathcal{A} \subseteq \wp(X)$ is a function such that $Bel(\emptyset) = 0$, $Bel(X) = 1$ and satisfying the n-monotonicity property for every $n \geq 2$, i.e., for every $A_1, \ldots, A_n \in \mathcal{A}$,

$$Bel\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} A_i\right). \tag{1}$$

A belief function Bel on $\mathcal{A}$ is completely singled out by its Möbius inverse, defined for every $A \in \mathcal{A}$ as

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B).$$

Such a function, usually called *basic (probability) assignment*, is a function $m : \mathcal{A} \to [0,1]$ satisfying $m(\emptyset) = 0$ and $\sum_{A \in \mathcal{A}} m(A) = 1$, and is such that for every $A \in \mathcal{A}$

$$Bel(A) = \sum_{B \subseteq A} m(B). \tag{2}$$

A set A in $\mathcal{A}$ is a *focal element* for m (and so also for the corresponding Bel) whenever $m(A) > 0$.

Given a set $X = \{x_1, \ldots, x_n\}$ and a normalized capacity $\varphi : \wp(X) \to [0,1]$ (i.e., a function monotone with respect to the inclusion, and satisfying $\varphi(\emptyset) = 0$ and $\varphi(X) = 1$), the *Choquet integral* of a function $f : X \to \mathbb{R}$, with $f(x_1) \leq \ldots \leq f(x_n)$ is defined as

$$\oint f \, \mathrm{d}\varphi = \sum_{i=1}^{n} f(x_i)(\varphi(E_i) - \varphi(E_{i+1})) \tag{3}$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$, and $E_{n+1} = \emptyset$ [8].

In the classical von Neumann-Morgenstern theory [23] a *lottery* L consists of a *probability distribution* on a finite *support* X_L, which is an arbitrary finite set of *prizes* or *consequences*.

In this paper we adopt a generalized notion of lottery L, by assuming that a *belief function* Bel_L is assigned on the power set $\wp(X_L)$ of X_L.

Definition 1. *A* **generalized lottery***, or* **g-lottery** *for short, on a finite set* X_L *is a pair* $L = (\wp(X_L), Bel_L)$ *where* Bel_L *is a belief function on* $\wp(X_L)$*.*

Let us notice that, a g-lottery $L = (\wp(X_L), Bel_L)$ could be equivalently defined as $L = (\wp(X_L), m_L)$, where m_L is the basic assignment associated to Bel_L. We stress that this definition of g-lottery generalizes the classical one in which $m_L(A) = 0$ for every $A \in \wp(X_L)$ with $\mathrm{card}\, A > 1$.

For example, a g-lottery L on $X_L = \{x_1, x_2, x_3\}$ can be expressed as

$$L = \begin{pmatrix} \{x_1\} & \{x_2\} & \{x_3\} & \{x_1,x_2\} & \{x_1,x_3\} & \{x_2,x_3\} & \{x_1,x_2,x_3\} \\ b_1 & b_2 & b_3 & b_{12} & b_{13} & b_{23} & b_{123} \end{pmatrix}$$

where the belief function Bel_L on $\wp(X_L)$ is such that $b_I = Bel_L(\{x_i : i \in I\})$ for every $I \subseteq \{1,2,3\}$. Notice that as one always has $Bel_L(\emptyset) = m_L(\emptyset) = 0$, the empty set is not reported in the tabular expression of L. An equivalent representation of previous g-lottery is obtained through the basic assignment m_L associated to Bel_L (where $m_I = m_L(\{x_i : i \in I\})$ for every $I \subseteq \{1,2,3\}$)

$$L = \begin{pmatrix} \{x_1\} & \{x_2\} & \{x_3\} & \{x_1,x_2\} & \{x_1,x_3\} & \{x_2,x_3\} & \{x_1,x_2,x_3\} \\ m_1 & m_2 & m_3 & m_{12} & m_{13} & m_{23} & m_{123} \end{pmatrix}.$$

Given a finite set $\mathcal{L}$ of g-lotteries, let $X = \bigcup\{X_L : L \in \mathcal{L}\}$. Then, any g-lottery L on X_L with belief function Bel_L can be rewritten as a g-lottery on X by defining a suitable extension Bel'_L of Bel_L.

Proposition 1. *Let $L = (\wp(X_L), Bel_L)$ be a g-lottery on X_L. Then for any finite $X \supseteq X_L$ there exists a unique belief function Bel'_L on $\wp(X)$ with the same focal elements of Bel_L and such that $Bel'_{L|\wp(X_L)} = Bel_L$.*

Given $L_1, \ldots, L_t \in \mathcal{L}$, all rewritten on X, and a real vector $\mathbf{k} = (k_1, \ldots, k_t)$ with $k_i \geq 0$ $(i = 1, \ldots, t)$ and $\sum_{i=1}^t k_i = 1$, the *convex combination* of $L_1, \ldots, L_t$ according to $\mathbf{k}$ is defined as

$$\mathbf{k}(L_1, \ldots, L_t) = \begin{pmatrix} A \\ \sum_{i=1}^t k_i m_{L_i}(A) \end{pmatrix} \quad \text{for every } A \in \wp(X) \setminus \{\emptyset\}. \tag{4}$$

Since the convex combination of belief functions (basic assignments) on $\wp(X)$ is a belief function (basic assignment) on $\wp(X)$, $\mathbf{k}(L_1, \ldots, L_t)$ is a g-lottery on X.

For every $A \in \wp(X) \setminus \{\emptyset\}$, there exists a *degenerate g-lottery* δ_A on X such that $m_{\delta_A}(A) = 1$, and, moreover, every g-lottery L with focal elements $A_1, \ldots, A_k$ can be expressed as $\mathbf{k}(\delta_{A_1}, \ldots, \delta_{A_k})$ with $\mathbf{k} = (m_L(A_1), \ldots, m_L(A_k))$.

3 Preferences over a set of generalized lotteries

Consider a set $\mathcal{L}$ of g-lotteries with $X = \bigcup\{X_L : L \in \mathcal{L}\}$ and assume X is *totally ordered* by the relation $\leq$, which is a quite natural condition thinking at elements of X as money payoffs. Denote with $<$ the total strict order on X induced by $\leq$.

In what follows the set X is always assumed to be finite, i.e., $X = \{x_1, \ldots, x_n\}$ with $x_1 < \ldots < x_n$. Under previous assumption, we can define the *aggregated basic assignment* of a g-lottery L, for every $x_i \in X$, as

$$M_L(x_i) = \sum_{x_i \in B \subseteq E_i} m_L(B), \tag{5}$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$. Note that $M_L(x_i) \geq 0$ for every $x_i \in X$ and $\sum_{i=1}^{n} M_L(x_i) = 1$, thus M_L determines a probability distribution on X.

Let $\precsim$ be a *preference/indifference* relation on $\mathcal{L}$. For every $L, L' \in \mathcal{L}$ the assertion that "L is indifferent to L'", denoted by $L \sim L'$, summarizes the two assertions $L \precsim L'$ and $L' \precsim L$. Observe that not all the pairs of g-lotteries are necessarily compared. An additional strict preference relation can be elicited by assertions such as "L is strictly preferred to L'", denoted by $L \prec L'$. Let $\prec^\bullet$ be the asymmetric relation formally deduced from $\precsim$, namely $\prec^\bullet = \precsim \setminus \sim$. If the pair of relations $(\precsim, \prec)$ represents the opinion of the decision maker, then it is natural to have $\prec \subset \prec^\bullet$: in fact, it is possible that, at an initial stage of judgement, the decision maker has not decided yet if $L \prec L'$ or $L \sim L'$ and he expresses his opinion only by $L \precsim L'$. Obviously if $\precsim$ is complete then $\prec = \prec^\bullet$ and so for every $L, L' \in \mathcal{L}$ either $L \prec L'$ or $L' \prec L$ or $L \sim L'$.

Remark 1. Since the set X is totally ordered by $\leq$, it is natural to require that the partial preference relation $(\precsim, \prec)$ agrees with $\leq$ on degenerate g-lotteries $\delta_{\{x\}}$, for $x \in X$, that correspond to decisions under certainty. For this, $\mathcal{L}$ must contain the set of degenerate g-lotteries on singletons $\mathcal{L}_0 = \{\delta_{\{x\}} : x \in X\}$ and it must be $x \leq x'$ if and only if $\delta_{\{x\}} \precsim \delta_{\{x'\}}$, for $x, x' \in X$. Actually, the decision maker is not asked to provide such a set of preferences, but in this case the initial partial preference $(\precsim, \prec)$ on $\mathcal{L}$ must be extended in order to reach this technical condition and, of course, the decision maker is asked to accept such an extension.

We call the pair $(\precsim, \prec)$ *strengthened preference relation* if $\prec$ is not empty, moreover, we say that a function $U : \mathcal{L} \to \mathbb{R}$ *represents* (or *agrees with*) $(\precsim, \prec)$ if, for every $L, L' \in \mathcal{L}$

$$L \precsim L' \Rightarrow U(L) \leq U(L') \text{ and } L \prec L' \Rightarrow U(L) < U(L'). \tag{6}$$

In analogy with [4], given $(\precsim, \prec)$ on $\mathcal{L}$, our aim is to find a necessary and sufficient condition for the existence of a utility function $u : X \to \mathbb{R}$ such that the *Choquet expected utility* of g-lotteries in $\mathcal{L}$, defined for every $L \in \mathcal{L}$ as

$$\mathrm{CEU}(L) = \oint u \, \mathrm{d}Bel_L, \tag{7}$$

represents $(\precsim, \prec)$. In particular, since X is totally ordered by $\leq$ and $\mathrm{CEU}(\delta_{\{x\}}) = u(x)$ for every $x \in X$, we search for a *strictly increasing* u.

The next axiom requires that it is not possible to obtain two g-lotteries having the same aggregated basic assignment, by combining in the same way the aggregated basic assignments of two groups of g-lotteries, if each g-lottery in the first group is not preferred to the corresponding one in the second group, and at least a preference is strict.

Definition 2. *A strengthened preference relation* $(\precsim, \prec)$ *on a set* $\mathcal{L}$ *of g-lotteries is said to be* **Choquet rational** *if it satisfies the following condition:*

(g-CR) *For all $h \in \mathbb{N}$ and $L_i, L'_i \in \mathcal{L}$ with $L_i \precsim L'_i$ $(i = 1, \ldots, h)$, if*

$$\mathbf{k}(M_{L_1}, \ldots, M_{L_h}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_h})$$

with $\mathbf{k} = (k_1, \ldots, k_h)$, $k_i > 0$ $(i = 1, \ldots, h)$ and $\sum_{i=1}^{h} k_i = 1$, then it can be $L_i \prec L'_i$ for no $i = 1, \ldots, h$. In particular, if $\precsim$ is complete, it must be $L_i \sim L'_i$ for every $i = 1, \ldots, h$.

Note that the convex combination referred to in condition **(g-CR)** is the usual one involving probability distributions on X. Moreover, it is easily proven that if $\mathbf{k}(L_1, \ldots, L_h) = \mathbf{k}(L'_1, \ldots, L'_h)$, then it also holds $\mathbf{k}(M_{L_1}, \ldots, M_{L_h}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_h})$ but the converse is generally not true.

The following theorem, proved in [2], shows that **(g-CR)** is a necessary and sufficient condition for the existence of a strictly increasing utility function u whose Choquet expected value on g-lotteries represents $(\precsim, \prec)$.

Theorem 1. *Let $\mathcal{L}$ be a finite set of g-lotteries, $X = \bigcup\{X_L : L \in \mathcal{L}\}$ with X totally ordered by $\leq$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$. Assume $\mathcal{L}_0 \subseteq \mathcal{L}$ and for every $x, x' \in X$, $x \leq x'$ if and only if $\delta_{\{x\}} \precsim \delta_{\{x'\}}$. The following statements are equivalent:*

(i) $(\precsim, \prec)$ is Choquet rational (i.e., it satisfies **(g-CR)***);*
(ii) there exists a strictly increasing function $u : X \to \mathbb{R}$ (unique up to a positive linear transformation), whose Choquet expected utility (CEU) on $\mathcal{L}$ represents $(\precsim, \prec)$.

The proof of previous result provides an operative procedure to compute a strictly increasing utility function u on X in case **(g-CR)** is satisfied. For this, introduce the collections $S = \{(L_j, L'_j) : L_j \prec L'_j, L_j, L'_j \in \mathcal{L}\}$ and $R = \{(G_h, G'_h) : G_h \precsim G'_h, G_h, G'_h \in \mathcal{L}\}$ with $s = \operatorname{card} S$ and $r = \operatorname{card} R$. Then condition **(g-CR)** is equivalent to the *non-existence* of a row vector $\mathbf{k}$ of size $(1 \times s + r)$ with $k_i > 0$ for at least a pair $(L_i, L'_i) \in S$ and $\sum_{i=1}^{s+r} k_i = 1$ such that

$$\mathbf{k}(M_{L_1}, \ldots, M_{L_s}, M_{G_1}, \ldots, M_{G_r}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_s}, M_{G'_1}, \ldots, M_{G'_r}).$$

In turn, setting $\mathbf{k} = (\mathbf{y}, \mathbf{z})$, previous condition is equivalent to the *non-solvability* of the following linear system (in which $|| \cdot ||_1$ denotes the L^1-norm)

$$\mathcal{S}' : \begin{cases} \mathbf{y}A + \mathbf{z}B = \mathbf{0} \\ \mathbf{y}, \mathbf{z} \geq \mathbf{0} \\ \mathbf{y} \neq \mathbf{0} \\ ||\mathbf{y}||_1 + ||\mathbf{z}||_1 = 1 \end{cases} \tag{8}$$

where $A = (a^j)$ and $B = (b^h)$ are, respectively, $(s \times n)$ and $(r \times n)$ real matrices with rows $a^j = M_{L'_j} - M_{L_j}$ for $j = 1, \ldots, s$, and $b^h = M_{G'_h} - M_{G_h}$ for $h = 1, \ldots, r$, and $\mathbf{y}$ and $\mathbf{z}$ are, respectively, $(1 \times s)$ and $(1 \times r)$ unknown row vectors.

By virtue of a well-known alternative theorem (see, e.g., [11]), in [2] the non-solvability of $\mathcal{S}'$ has been proven to be equivalent to the *solvability* of the following system

$$\mathcal{S} : \begin{cases} A\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \tag{9}$$

where $\mathbf{w}$ is a $(n \times 1)$ unknown column vector. In detail, setting $u(x_i) = w_i$, $i = 1, \dots, n$, the solution $\mathbf{w}$ induces a utility function u on X which, taking into account Remark 1, is strictly increasing and whose CEU represents $(\precsim, \prec)$.

3.1 A paradigmatic example

To motivate the topic dealt with in this paper we introduced the following example, which is inspired to the well-known Ellsberg's paradox [9].

Example 1. Consider the following hypothetical experiment. Let us take two urns, say U_1 and U_2, from which we are asked to draw a ball each. U_1 contains $\frac{1}{3}$ of white (w) balls and the remaining balls are black (b) and red (r), but in a ratio entirely unknown to us, analogously, U_2 contains $\frac{1}{4}$ of green (g) balls and the remaining balls are yellow (y) and orange (o), but in a ratio entirely unknown to us.

In light of the given information, the composition of U_1 singles out a class of probability measures $\mathbf{P}^1 = \{P^\theta\}$ on the power set $\wp(S_1)$ of $S_1 = \{w, b, r\}$ s.t. $P^\theta(\{w\}) = \frac{1}{3}$, $P^\theta(\{b\}) = \theta$, $P^\theta(\{r\}) = \frac{2}{3} - \theta$, with $\theta \in \left[0, \frac{2}{3}\right]$. Analogously, for the composition of U_2 we have the class $\mathbf{P}^2 = \{P^\lambda\}$ on $\wp(S_2)$ with $S_2 = \{g, y, o\}$ s.t. $P^\lambda(\{g\}) = \frac{1}{4}$, $P^\lambda(\{y\}) = \lambda$, $P^\lambda(\{o\}) = \frac{3}{4} - \lambda$, with $\lambda \in \left[0, \frac{3}{4}\right]$.

Concerning the ball drawn from U_1 and the one drawn from U_2, the following gambles are considered:

	w	b	r
L_1	100€	0€	0€
L_2	0€	0€	100€
L_3	0€	100€	100€
L_4	100€	100€	0€

	g	y	o
G_1	100€	10€	10€
G_2	10€	10€	100€
G_3	10€	100€	100€
G_4	100€	100€	10€

If we express the strict preferences $L_2 \prec L_1$, $L_4 \prec L_3$, then for no value of θ there exists a function $u : \{0, 100\} \to \mathbb{R}$ s.t. its expected value on the L_i's w.r.t. P^θ represents our preferences on the L_i's. Indeed, putting $w_1 = u(0)$ and $w_2 = u(100)$, both the following inequalities must hold $\frac{1}{3}w_1 + \theta w_1 + \left(\frac{2}{3} - \theta\right) w_2 < \frac{1}{3}w_2 + \theta w_1 + \left(\frac{2}{3} - \theta\right) w_1$ and $\frac{1}{3}w_2 + \theta w_2 + \left(\frac{2}{3} - \theta\right) w_1 < \frac{1}{3}w_1 + \theta w_2 + \left(\frac{2}{3} - \theta\right) w_2$, from which, summing memberwise, we get $w_1 + w_2 < w_1 + w_2$, i.e., a contradiction. The same can be proven if we express the strict preferences $G_2 \prec G_1$, $G_4 \prec G_3$.

Now take $\underline{P}^1 = \min \mathbf{P}^1$ and $\underline{P}^2 = \min \mathbf{P}^2$, where the minimum is intended pointwise on the elements of $\wp(S_1)$ and $\wp(S_2)$, obtaining:

$\wp(S^1)$	$\emptyset$	$\{w\}$	$\{b\}$	$\{r\}$	$\{w,b\}$	$\{w,r\}$	$\{b,r\}$	S_1
$\underline{P}^1$	0	$\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1

$\wp(S_2)$	$\emptyset$	$\{g\}$	$\{y\}$	$\{o\}$	$\{g,y\}$	$\{g,o\}$	$\{y,o\}$	S_2
$\underline{P}^2$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	1

It is easily verified that both $\underline{P}^1$ and $\underline{P}^2$ are belief functions.

The gambles L_i's and G_i's allow to transport the belief functions $\underline{P}^1$ and $\underline{P}^2$ to the whole set of prizes $\{0,10,100\}$, obtaining the following g-lotteries with the corresponding aggregated basic assignments

	$\{0\}$	$\{10\}$	$\{100\}$	$\{0,10\}$	$\{0,100\}$	$\{10,100\}$	$\{0,10,100\}$
L_1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	1
L_2	$\frac{1}{3}$	0	0	$\frac{1}{3}$	1	0	1
L_3	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	1
L_4	0	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	1
G_1	0	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	1	1
G_2	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	1	1
G_3	0	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	1	1
G_4	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	1	1

	0	10	100
M_{L_1}	$\frac{2}{3}$	0	$\frac{1}{3}$
M_{L_2}	1	0	0
M_{L_3}	$\frac{1}{3}$	0	$\frac{2}{3}$
M_{L_4}	$\frac{2}{3}$	0	$\frac{1}{3}$
M_{G_1}	0	$\frac{3}{4}$	$\frac{1}{4}$
M_{G_2}	0	1	0
M_{G_3}	0	$\frac{1}{3}$	$\frac{3}{4}$
M_{G_4}	0	$\frac{3}{4}$	$\frac{1}{4}$

It is easily proven that for every strictly increasing $u : \{0,10,100\} \to \mathbb{R}$ the strict preferences $L_2 \prec L_1$, $L_4 \prec L_3$, $G_2 \prec G_1$, $G_4 \prec G_3$ are represented by their Choquet expected utility. Indeed, putting $w_1 = u(0)$, $w_2 = u(10)$, $w_3 = u(100)$, the following system

$$\mathcal{S} : \begin{cases} w_1 < \frac{2}{3}w_1 + \frac{1}{3}w_3 \\ \frac{2}{3}w_1 + \frac{1}{3}w_3 < \frac{1}{3}w_1 + \frac{2}{3}w_3 \\ w_2 < \frac{3}{4}w_2 + \frac{1}{4}w_3 \\ \frac{3}{4}w_2 + \frac{1}{4}w_3 < \frac{1}{4}w_2 + \frac{3}{4}w_3 \\ w_1 < w_2 < w_3 \end{cases}$$

is such that any choice of values satisfying $w_1 < w_2 < w_3$ is a solution.

Now, suppose to toss a fair coin and to choose among L_1 and G_1 depending on the face shown by the coin. In analogy, suppose to choose among L_2 and G_1 with a totally similar experiment. Let us denote with F_1 and F_2 the results of the two experiments. This implies that F_1 and F_2 can be defined as the convex combinations $F_1 = \frac{1}{2}L_1 + \frac{1}{2}G_1$ and $F_2 = \frac{1}{2}L_2 + \frac{1}{2}G_1$, obtaining the g-lotteries with the corresponding aggregated basic assignments

	$\{0\}$	$\{10\}$	$\{100\}$	$\{0,10\}$	$\{0,100\}$	$\{10,100\}$	$\{0,10,100\}$
F_1	$\frac{8}{24}$	$\frac{9}{24}$	$\frac{7}{24}$	$\frac{17}{24}$	$\frac{15}{24}$	$\frac{16}{24}$	1
F_2	$\frac{4}{24}$	$\frac{9}{24}$	$\frac{3}{24}$	$\frac{13}{24}$	$\frac{15}{24}$	$\frac{12}{24}$	1

	0	10	100
M_{F_1}	$\frac{8}{24}$	$\frac{9}{24}$	$\frac{7}{24}$
M_{F_2}	$\frac{12}{24}$	$\frac{9}{24}$	$\frac{3}{24}$

If we add to previous preferences the further strict preference $F_1 \prec F_2$ then there is no strictly increasing $u : \{0,10,100\} \to \mathbb{R}$ whose Choquet expected utility represents our preferences. Indeed, in this case, the extended system

$$\mathcal{S} : \begin{cases} w_1 < \frac{2}{3}w_1 + \frac{1}{3}w_3 \\ \frac{2}{3}w_1 + \frac{1}{3}w_3 < \frac{1}{3}w_1 + \frac{2}{3}w_3 \\ w_2 < \frac{3}{4}w_2 + \frac{1}{4}w_3 \\ \frac{3}{4}w_2 + \frac{1}{4}w_3 < \frac{1}{4}w_2 + \frac{3}{4}w_3 \\ \frac{8}{24}w_1 + \frac{9}{24}w_2 + \frac{7}{24}w_3 < \frac{12}{24}w_1 + \frac{9}{24}w_2 + \frac{3}{24}w_3 \\ w_1 < w_2 < w_3 \end{cases}$$

admits no solution. Notice that, taking into account Remark 1, condition **(g-CR)** fails since it holds

$$\frac{3}{4}M_{F_1} + \frac{1}{8}M_{\delta_{\{0\}}} + \frac{1}{8}M_{\delta_{\{10\}}} = \frac{3}{4}M_{F_2} + \frac{1}{8}M_{\delta_{\{10\}}} + \frac{1}{8}M_{\delta_{\{100\}}}.$$

3.2 Extension of Choquet rational preferences

In previous section it has been shown that condition **(g-CR)** is equivalent to the existence of a strictly increasing utility function u on X, whose CEU represents $(\precsim, \prec)$, moreover, such a u can be explicitly determined by solving the linear system $\mathcal{S}$ defined in (9). It is straightforward that once a utility u has been fixed, a complete preference relation on $\mathcal{L}$ (or any finite superset $\mathcal{L}'$ of g-lotteries on the same finite set X) extending $(\precsim, \prec)$ is induced by the corresponding CEU functional.

Nevertheless, system $\mathcal{S}$ has generally infinite solutions which can give rise to possibly very different complete preference relations, thus any choice of a utility function causes a loss of information, moreover, it is not clear why one should choose a utility function in place of another.

This is why it is preferable to face the extension in a qualitative setting by considering the entire class of utility functions whose CEU represents the preference $(\precsim, \prec)$ and suggesting to the decision maker those pairs of g-lotteries where all the utility functions unanimously agree. In this view, the following Theorem 2 proves the extendibility of a Choquet rational relation and shows how condition **(g-CR)** guides the decision maker in assessing his preferences.

Theorem 2. *Let X be a finite set totally ordered by $\leq$, $\mathcal{L}$ and $\mathcal{L}'$ finite sets of g-lotteries on X, with $\mathcal{L} \subseteq \mathcal{L}'$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$. Assume $\mathcal{L}_0 \subseteq \mathcal{L}$ and for every $x, x' \in X$, $x \leq x'$ if and only if $\delta_{\{x\}} \precsim \delta_{\{x'\}}$. Then if $(\precsim, \prec)$ satisfies condition* **(g-CR)** *there exists a family $\{\precsim^\gamma : \gamma \in \Gamma\}$ of complete relations on $\mathcal{L}'$ satisfying* **(g-CR)** *which extend $(\precsim, \prec)$. Moreover, denoting with $\prec^\gamma$ and $\sim^\gamma$, respectively, the strict and symmetric parts of $\precsim^\gamma$, for $\gamma \in \Gamma$, condition* **(g-CR)** *singles out the relations*

$$\prec^\star = \bigcap\{\prec^\gamma : \gamma \in \Gamma\} \quad \textit{and} \quad \sim^\star = \bigcap\{\sim^\gamma : \gamma \in \Gamma\}.$$

Proof. Suppose $X = \{x_1, \ldots, x_n\}$ with $x_1 < \ldots < x_n$. By the proof of Theorem 1 (see [2]), $(\precsim, \prec)$ satisfies condition **(g-CR)** if and only if system $\mathcal{S}$ defined in (9) admits a $(n \times 1)$ column vector $\mathbf{w}$ as solution. In turn, setting $u(x_i) = w_i$, for $i = 1, \ldots, n$, we get a strictly increasing utility function u on X whose Choquet expected value represents $(\precsim, \prec)$ on $\mathcal{L}$. Defining for every $L, L' \in \mathcal{L}'$

$$L \precsim^\gamma L' \Leftrightarrow \mathrm{CEU}(L) \leq \mathrm{CEU}(L'),$$

we get a relation $\precsim^\gamma$ on $\mathcal{L}'$ which is complete and satisfies **(g-CR)** by virtue of Theorem 1. This implies that the family $\{\precsim^\gamma : \gamma \in \Gamma\}$ is not empty and all its members are obtained varying the solution $\mathbf{w}$ of system $\mathcal{S}$. The correspondence between the set of solutions and the family of relations $\{\precsim^\gamma : \gamma \in \Gamma\}$ is onto but not one-to-one, as every positive linear transformation of a solution $\mathbf{w}$ gives rise to the same relation $\precsim^\gamma$.

The relations $\prec^\star$ and $\sim^\star$ express, respectively, the pairs of g-lotteries in $\mathcal{L}'$ on which all the strict $\prec^\gamma$ and symmetric $\sim^\gamma$ parts, for $\gamma \in \Gamma$, agree. It trivially holds that $\prec^\star$ and $\sim^\star$ extend the relations $\prec$ and $\sim$ obtained from $(\precsim, \prec)$, moreover, in order to determine $\prec^\star$ and $\sim^\star$, for every $F, G \in \mathcal{L}'$ such that it does not hold $F \prec G$ or $G \prec F$ or $F \sim G$ it is sufficient to test the solvability of the three linear systems

$$\mathcal{S}^{\prec^\star} : \begin{cases} A'\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \qquad \mathcal{S}^{\succ^\star} : \begin{cases} A''\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \qquad \mathcal{S}^{\sim^\star} : \begin{cases} A\mathbf{w} > \mathbf{0} \\ B'\mathbf{w} \geq \mathbf{0} \end{cases}$$

where $\mathbf{w}$ is an unknown $(n \times 1)$ column vector, A and B are, respectively, $(s \times n)$ and $(r \times n)$ real matrices defined as in (8), A' is a $((s+1) \times n)$ real matrix obtained adding to A the $(s+1)$-th row $a^{(s+1)} = M_G - M_F$, A'' is a $((s+1) \times n)$ real matrix obtained adding to A the $(s+1)$-th row $a^{(s+1)} = M_F - M_G$, and B' is a $((r+2) \times n)$ real matrix obtained adding to B the $(r+1)$-th row $b^{(r+1)} = M_G - M_F$ and the $(r+2)$-th row $b^{(r+2)} = M_F - M_G$.

Depending on the solvability of systems $\mathcal{S}^{\prec^\star}, \mathcal{S}^{\succ^\star}, \mathcal{S}^{\sim^\star}$ we can have the following situations:

(a) $F \prec^\star G$ if and only if $\mathcal{S}^{\prec^\star}$ is solvable and $\mathcal{S}^{\succ^\star}, \mathcal{S}^{\sim^\star}$ are not, as this happens if and only if $\mathrm{CEU}(F) < \mathrm{CEU}(G)$ for every u given by a solution of $\mathcal{S}$;
(b) $G \prec^\star F$ if and only if $\mathcal{S}^{\succ^\star}$ is solvable and $\mathcal{S}^{\prec^\star}, \mathcal{S}^{\sim^\star}$ are not, as this happens if and only if $\mathrm{CEU}(G) < \mathrm{CEU}(F)$ for every u given by a solution of $\mathcal{S}$;
(c) $F \sim^\star G$ if and only if $\mathcal{S}^{\sim^\star}$ is solvable and $\mathcal{S}^{\prec^\star}, \mathcal{S}^{\succ^\star}$ are not, as this happens if and only if $\mathrm{CEU}(F) = \mathrm{CEU}(G)$ for every u given by a solution of $\mathcal{S}$.

In all the remaining cases, the Choquet expected utilities determined by solutions of $\mathcal{S}$ do not unanimously agree in ordering the pair F and G. □

Relations $\prec^\star$ and $\sim^\star$ determined in the proof of previous theorem express "forced" preferences that the decision maker has to accept in order to maintain Choquet rationality. On the other hand, pairs of g-lotteries not ruled by $\prec^\star$ and $\sim^\star$ are subject to a choice by the decision maker. In the latter situation, a subjective elicitation is required or, in case of a software agent [17], a suitable automatic choice criterion can be adopted.

We stress that each choice made by the decision maker imposes a new constraint in system $\mathcal{S}$, thus the set of utility functions whose CEU represents the current strengthened preference $(\precsim, \prec)$ is possibly reduced.

Previous discussion suggests the following Algorithm 1 which is thought to guide the decision maker in enlarging a Choquet rational preference relation $(\precsim, \prec)$ to a (possibly new) pair of g-lotteries F and G: the extended preference is still denoted as $(\precsim, \prec)$. In particular, Algorithm 1 returns to the decision maker what he must do or he cannot do in order to maintain **(g-CR)**.

Notice that possibly $F, G \in \mathcal{L}$, thus previous algorithm can be used to produce a step by step completion of the preference relation $(\precsim, \prec)$ on $\mathcal{L}$.

Algorithm 1 Extension of a Choquet rational relation

function EXTENSION($(\precsim, \prec)$, F, G)
  if $\mathcal{S}^{\prec^\star}$ and $\mathcal{S}^{\succ^\star}$ are solvable **then** free preference between F and G
  else if $\mathcal{S}^{\prec^\star}$ is solvable and $\mathcal{S}^{\sim^\star}$ is not **then** it must be $F \prec G$
  else if $\mathcal{S}^{\succ^\star}$ is solvable and $\mathcal{S}^{\sim^\star}$ is not **then** it must be $G \prec F$
  else if $\mathcal{S}^{\prec^\star}$ and $\mathcal{S}^{\sim^\star}$ are solvable **then** it cannot be $G \prec F$
  else if $\mathcal{S}^{\succ^\star}$ and $\mathcal{S}^{\sim^\star}$ are solvable **then** it cannot be $F \prec G$
  else it must be $F \sim G$
end function

Algorithm 1 requires as input a Choquet rational preference relation $(\precsim, \prec)$ on a set of g-lotteries $\mathcal{L}$, and two (possibly new) g-lotteries F and G, all rewritten on $X = \{x_1, \ldots, x_n\}$ with $x_1 < \ldots < x_n$. The g-lotteries in $\mathcal{L} \cup \{F, G\}$ can be simply regarded as basic assignments on $\wp(X)$, i.e., as real $(1 \times q)$ row vectors with $q = 2^n - 1$. The formation of matrices A, A', A'', B, B' requires the computation of the aggregated basic assignment M_L for every $L \in \mathcal{L} \cup \{F, G\}$, which can be done in polynomial time with respect to q.

The extension is faced through the solution of at most three linear programming problems, whose solution has time complexity which is a polynomial in $n = \log_2(q+1)$ and the digital size of the coefficients in matrices A', B or A'', B or A, B', respectively [16].

The following example shows an application of Algorithm 1.

Example 2. Consider the situation described in Example 1. It has already been observed that adding the further strict preference $F_1 \prec F_2$ implies that the global preference relation has no more a Choquet expected utility representation. We use Algorithm 1 to guide the decision maker in judging his preference between F_1 and F_2 in order to preserve Choquet rationality. It is easily seen that only system

$$\mathcal{S}^{\succ^\star} : \begin{cases} w_1 < \frac{2}{3}w_1 + \frac{1}{3}w_3 \\ \frac{2}{3}w_1 + \frac{1}{3}w_3 < \frac{1}{3}w_1 + \frac{2}{3}w_3 \\ w_2 < \frac{3}{4}w_2 + \frac{1}{4}w_3 \\ \frac{3}{4}w_2 + \frac{1}{4}w_3 < \frac{1}{4}w_2 + \frac{3}{4}w_3 \\ \frac{12}{24}w_1 + \frac{9}{24}w_2 + \frac{3}{24}w_3 < \frac{8}{24}w_1 + \frac{9}{24}w_2 + \frac{7}{24}w_3 \\ w_1 < w_2 < w_3 \end{cases}$$

is solvable while $\mathcal{S}^{\prec^\star}$ and $\mathcal{S}^{\sim^\star}$ are not. In turn, this implies that $F_2 \prec^\star F_1$ and so the decision maker is forced to strictly prefer F_1 to F_2 to respect condition **(g-CR)**.

On the other hand, considering the g-lotteries L_1 and G_1, both systems

$$\mathcal{S}^{\prec^\star} : \begin{cases} w_1 < \frac{2}{3}w_1 + \frac{1}{3}w_3 \\ \frac{2}{3}w_1 + \frac{1}{3}w_3 < \frac{1}{3}w_1 + \frac{2}{3}w_3 \\ w_2 < \frac{3}{4}w_2 + \frac{1}{4}w_3 \\ \frac{3}{4}w_2 + \frac{1}{4}w_3 < \frac{1}{4}w_2 + \frac{3}{4}w_3 \\ \frac{2}{3}w_1 + \frac{1}{3}w_3 < \frac{3}{4}w_2 + \frac{1}{4}w_3 \\ w_1 < w_2 < w_3 \end{cases} \qquad \mathcal{S}^{\succ^\star} : \begin{cases} w_1 < \frac{2}{3}w_1 + \frac{1}{3}w_3 \\ \frac{2}{3}w_1 + \frac{1}{3}w_3 < \frac{1}{3}w_1 + \frac{2}{3}w_3 \\ w_2 < \frac{3}{4}w_2 + \frac{1}{4}w_3 \\ \frac{3}{4}w_2 + \frac{1}{4}w_3 < \frac{1}{4}w_2 + \frac{3}{4}w_3 \\ \frac{3}{4}w_2 + \frac{1}{4}w_3 < \frac{2}{3}w_1 + \frac{1}{3}w_3 \\ w_1 < w_2 < w_3 \end{cases}$$

are solvable, thus in this case the decision maker is totally free to choose his preference between L_1 and G_1.

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