Czechoslovak Mathematical Journal

Hailong Liu; Liang Sun

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*Czechoslovak Mathematical Journal*, Vol. 54 (2004), No. 4, 883–887

Persistent URL: http://dml.cz/dmlcz/127937

Czechoslovak Mathematical Journal, 54 (129) (2004), 883–887

# ON THE MINUS DOMINATION NUMBER OF GRAPHS

HAILONG LIU and LIANG SUN, Beijing

(Received November 8, 2001)

*Abstract.* Let $G = (V, E)$ be a simple graph. A 3-valued function $f\colon V(G) \to \{-1, 0, 1\}$ is said to be a minus dominating function if for every vertex $v \in V$, $f(N[v]) = \sum\limits_{u \in N[v]} f(u) \geqslant 1$, where $N[v]$ is the closed neighborhood of $v$. The weight of a minus dominating function $f$ on $G$ is $f(V) = \sum\limits_{v \in V} f(v)$. The minus domination number of a graph $G$, denoted by $\gamma^-(G)$, equals the minimum weight of a minus dominating function on $G$. In this paper, the following two results are obtained.

(1) If $G$ is a bipartite graph of order $n$, then

$$\gamma^-(G) \geqslant 4\big(\sqrt{n+1} - 1\big) - n.$$

(2) For any negative integer $k$ and any positive integer $m \geqslant 3$, there exists a graph $G$ with girth $m$ such that $\gamma^-(G) \leqslant k$. Therefore, two open problems about minus domination number are solved.

*Keywords*: minus dominating function, minus domination number

*MSC 2000*: 05C69

## 1. INTRODUCTION

Let $G = (V, E)$ be a simple graph. The girth of $G$ is the length of a shortest cycle in $G$. For a vertex $v$ of $G$, the closed neighborhood of $v$ is the set $N[v]$ consisting of $v$ together with all vertices of $G$ adjacent to $v$. Let $f$ be a real valued function on $V$. For a non-empty subset $S$ of $V$, we define $f(S) = \sum\limits_{v \in S} f(v)$. The minus dominating function is a function $f\colon V(G) \to \{-1, 0, 1\}$ such that $f(N[v]) \geqslant 1$ for

The project was partially supported by NSFC 19871036.

all $v \in V(G)$. The minus domination number for a graph $G$ is $\gamma^-(G) = \min\{f(V) : f$ is a minus dominating function on $G\}$. The problem of finding $\gamma^-(G)$ seems to be very difficult. Even if we restrict $G$ to be bipartite, the corresponding decision problem is also NP-complete. In [3], the following two open problems about the minus domination number of a graph were posed.

**Conjecture 1** ([3]). *If $G$ is a bipartite graph of order $n$, then*

$$\gamma^-(G) \geqslant 4\big(\sqrt{n+1} - 1\big) - n.$$

**Problem 1** ([3]). For every negative integer $k$ and positive integer $m$, does there exist a graph $G$ with girth $m$ and $\gamma^-(G) \leqslant k$?

In Section 2, we will prove that Conjecture 1 is true. And in Section 3 we will give a positive answer to Problem 1.

## 2. Minus domination of bipartite graphs

In this section, we will give a proof for Conjecture 1. A bipartite graph $B = (X, Y)$ is an $(a, b)$-bipartite graph if every vertex in $X$ has degree $a$ and every vertex in $Y$ has degree $b$. If $B = (X, Y)$ is an $(a, b)$-bipartite graph, then $a|X| = b|Y|$.

Let $\mathscr{F}_s$ be a family of bipartite graphs of order $n = 4s(s + 1)$ in which each bipartite graph $B = (X, Y)$ satisfies the following two properties:

(1) $X = X_1 \cup X_2$ is a partition of $X$ such that $|X_1| = 2s$ and $|X_2| = 2s^2$, and $Y = Y_1 \cup Y_2$ is a partition of $Y$ such that $|Y_1| = 2s$ and $|Y_2| = 2s^2$.

(2) Both $G[X_1 \cup Y_2]$ and $G[Y_1 \cup X_2]$ are $(2s, 2)$-bipartite graphs, $G[X_1 \cup Y_1] = K_{2s,2s}$ is an $(2s, 2s)$-bipartite graph, and $G[X_2 \cup Y_2]$ contains no edges.

Since $K_{2,2s}$ is a $(2s, 2)$-bipartite graph, the family $\mathscr{F}_s$ is not empty for any positive integer $s$.

It is easy to prove the following lemma.

**Lemma 1.** *For all positive integers $n$, the inequality $4\big(\sqrt{n+1} - 1\big) - n \leqslant 1$ holds and it becomes an equality only for $n = 3$.*

**Theorem 1.** *If $G$ is a bipartite graph of order $n$, then*

$$\gamma^-(G) \geqslant 4\big(\sqrt{n+1} - 1\big) - n.$$

*Further, a bipartite graph $G$ satisfies $\gamma^-(G) = 4\big(\sqrt{n+1} - 1\big) - n$ if and only if $G$ is $K_{1,2}$ or $G$ is a bipartite graph in $\mathscr{F}_s$ where $n = 4s(s + 1)$.*

Proof. Let $f$ be a minimum minus dominating function on $G$. Let $X$ and $Y$ be the bipartite sets of $G$. Denote $X^+ = \{v \in X \mid f(v) = 1\}$. $X^- = \{v \in X \mid f(v) = -1\}$ and $X^0 = \{v \in X \mid f(v) = 0\}$. Denote $Y^+ = \{v \in Y \mid f(v) = 1\}$, $Y^- = \{v \in Y \mid f(v) = -1\}$ and $Y^0 = \{v \in Y \mid f(v) = 0\}$. Let $P = X^+ \cup Y^+$, $M = X^- \cup Y^-$ and $W = V(G) - P - M = X^0 \cup Y^0$. Furthermore, let $|X^+| = x_1$, $|X^-| = x_2$, $|Y^+| = y_1$, $|Y^-| = y_2$, $|P| = p$, $|M| = m$ and $|W| = w = n - p - m$. It is obvious that $x_1 + y_1 = p > 0$, and $w \geqslant 0$.

*Case 1*: $x_1 = 0$ or $y_1 = 0$.

If $x_1 = 0$, then we have that $y_1 > 0$ and $y_2 = 0$. Furthermore, we have $x_2 = 0$. Otherwise, we assume that there exists a vertex $u \in X^- \neq \emptyset$. Since $f(N[u]) \geqslant 1$, we have $N[u] \cap Y^+ \neq \emptyset$. For any $v \in N[u] \cap Y^+$, since $X^+ = \emptyset$, we have $f(N[v]) \leqslant 0$. This contradicts that $f$ is a minus dominating function. Therefore, by Lemma 1, we have $\gamma^-(G) = p - m = x_1 + y_1 - (x_2 + y_2) = y_1 \geqslant 1 \geqslant 4\bigl(\sqrt{n+1} - 1\bigr) - n$. For the case $y_1 = 0$, the proof is completely similar. Furthermore, if a bipartite graph $G$ of order $n$ satisfies that $\gamma^-(G) = 1 = 4\bigl(\sqrt{n+1} - 1\bigr) - n$, then $n = 3$ and $G = K_{1,2}$.

*Case 2*: $x_1 > 0$ and $y_1 > 0$.

Since every vertex in $X^-$ must be adjacent to at least two vertices in $Y^+$, by the pigeon-hole principle, there is a vertex $v_0$ of $Y^+$ such that $v_0$ is adjacent to at least $\lceil 2x_2/y_1 \rceil$ vertices of $X^-$. Since $1 \leqslant f(N[v_0]) = 1 - |N(v_0) \cap X^-| + |N(v_0) \cap X^+| \leqslant 1 - \lceil 2x_2/y_1 \rceil + |N(v_0) \cap X^+|$, we have that

$$x_1 = |X^+| \geqslant |N(v_0) \cap X^+| \geqslant \lceil 2x_2/y_1 \rceil \geqslant 2x_2/y_1.$$

Thus we obtain that $x_1 y_1 \geqslant 2x_2$. Similarly, we have that $x_1 y_1 \geqslant 2y_2$. Therefore $x_1 y_1 \geqslant x_2 + y_2 = n - p - w$. Since $x_1 y_1 \leqslant \frac{1}{4}(x_1 + y_1)^2 = \frac{1}{4}p^2$, we have that $\frac{1}{4}p^2 \geqslant n - p - w$. Thus we have that $\frac{1}{4}p^2 + p \geqslant n - w$. Since $p = x_1 + y_1 \geqslant 2$ and $w \geqslant 0$, we have that $w(w + 4p - 8) \geqslant 0$. Thus we can obtain that

$$\frac{p^2}{4} + p \geqslant n - w \geqslant n - \frac{(p+2)w}{4} - \frac{w^2}{16}.$$

This follows that

$$\Bigl(\frac{2p+w}{4} + 1\Bigr)^2 \geqslant n + 1.$$

Thus we have that

$$2p + w \geqslant 4\bigl(\sqrt{n+1} - 1\bigr).$$

Therefore,

$$\gamma^-(G) = p - m = 2p - (n - w) = (2p + w) - n \geqslant 4\bigl(\sqrt{n+1} - 1\bigr) - n.$$

Now we assume that $G$ is a bipartite graph of order $n$ such that $\gamma^-(G) = 4\bigl(\sqrt{n+1}-1\bigr)-n$. Then $2p+w = 4\bigl(\sqrt{n+1}-1\bigr)$ and $w(w+4p-8)=0$. Since $p \geqslant 2$, we have that $w=0$ and $\frac{1}{4}p^2+p=n$. Thus $x_1y_1 = \frac{1}{4}(x_1+y_1)^2$ and $x_1y_1 = x_2+y_2$. Therefore, the following properties of $G$ can be obtained:

(1) $x_1 = y_1 = \frac{1}{2}p = \sqrt{n+1}-1$,
(2) $x_2 = y_2 = \frac{1}{2}x_1y_1 = \frac{1}{8}p^2 = \frac{1}{2}\bigl(n-2\sqrt{n+1}+2\bigr)$,
(3) every vertex in $X_2 \cup Y_2$ has degree 2,
(4) every vertex in $X_1$ ($Y_1$) is adjacent to $\sqrt{n+1}-1$ vertices in $Y_2$ ($X_2$), and
(5) $G[X_1 \cup Y_1]$ is a $\bigl(\sqrt{n+1}-1, \sqrt{n+1}-1\bigr)$ bipartite graph and $G[X_2 \cup Y_2]$ contains no edges.

Since $\sqrt{n+1}$ is an integer and $n$ is even, there exists an $s$ such that $n = 4s(s+1)$. Thus $G$ is a bipartite graph in $\mathscr{F}_s$. Now for any graph $G$ in $\mathscr{F}_s$, we let $f(v) = -1$ if $v \in X_2 \cup Y_2$ and $f(v) = 1$ if $v \in X_1 \cup Y_1$. Then $f$ is a minus dominating function on $G$. Thus $\gamma^-(G) \leqslant f(V(G)) = |X_1|+|Y_1|-|X_2|-|Y_2| = 4\bigl(\sqrt{n+1}-1\bigr)-n$. Therefore any graph $G$ in $\mathscr{F}_s$ satisfies that $\gamma^-(G) = 4\bigl(\sqrt{n+1}-1\bigr)-n$. This completes the proof. □

## 3. Graphs with negative minus domination number and large girth

In this section, we are going to give a positive answer to Problem 1. An $s$-regular graph with girth $m$ is called an $(s,\, m)$-graph.

**Lemma 2** ([8, p. 81]). *For any positive integers $s \geqslant 2$, $m \geqslant 3$ and $n \geqslant 3$, there exists a connected $(s,m)$-graph $G$ such that the order of $G$ is at least $n$.*

An $s$-factor of $G$ is an $s$-regular spanning subgraph of $G$, and $G$ is $s$-factorable if there are edge-disjoint $s$-factors $H_1, H_2, \ldots, H_r$ such that $G = H_1 \cup H_2 \cup \ldots \cup H_r$.

**Lemma 3.** *For any positive integer $r$, if $G$ is a $4r$-regular graph, then $G$ is 4-factorable.*

Proof. By a famous theorem of Petersen [7], we have that any regular graph with even degree is 2-factorable. Thus $G$ can be factored into $2r$ 2-factors $F_1, \ldots, F_{2r}$. Let $H_j = F_{2j-1} \cup F_{2j}$, $j = 1, \ldots, r$. Then $H_1, \ldots, H_r$ are $r$ pair-wise edge disjoint 4-factors of $G$. □

**Theorem 2.** *For any negative integer $k$ and positive integer $m \geqslant 3$, there exists a graph $G$ with girth $m$ and $\gamma^-(G) \leqslant k$.*

Proof. Assume that $k$ is a negative integer and $m \geqslant 3$ is a positive integer. Let $n$ be a positive integer such that $m - n \leqslant k$. By Lemma 2, there exists a connected $(8, m)$-graph $H$ with order at least $n$. By Lemma 3, $H$ can be factored into two edge disjoint 4-factors $H_1$ and $H_2$. Let $C$ be an $m$-cycle in $H$. By subdividing all edges in $E(H_1) - E(C)$ we obtain a new graph $G$ from $H$. Then $G$ is a connected graph with girth $m$. We denote by $T$ the set of all vertices with degree 2 in $G$. Then $t = |T| \geqslant 2n - m$, and the order of $G$ is $n + t$. We define a mapping $f\colon V(G) \to \{-1, 0, 1\}$ such that $f(v) = 1$ if $v \in V(G) - T$ and $f(v) = -1$ if $v \in T$. Then it is easy to verify that $f$ is a minus dominating function on $G$. Thus $\gamma^-(G) \leqslant f(V(G)) = n - t \leqslant n - (2n - m) = m - n \leqslant k$. Therefore, $G$ is a graph satisfying all the conditions of the theorem. This completes the proof. □

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*Authors' address*: Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China, e-mails: liu_hailong@163.com, sun30318@yahoo.com.