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MINUS TOTAL DOMINATION IN GRAPHS

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Abstract. A three-valued function $f\colon V \to \{-1,0,1\}$ defined on the vertices of a graph $G=(V,E)$ is a minus total dominating function (MTDF) if the sum of its function values over any open neighborhood is at least one. That is, for every $v \in V$, $f(N(v)) \geqslant 1$, where $N(v)$ consists of every vertex adjacent to v. The weight of an MTDF is $f(V)=\sum f(v)$, over all vertices $v \in V$. The minus total domination number of a graph G, denoted $\gamma_t^-(G)$, equals the minimum weight of an MTDF of G. In this paper, we discuss some properties of minus total domination on a graph G and obtain a few lower bounds for $\gamma_t^-(G)$.

Keywords: minus domination, total domination, minus total domination

MSC 2010: 05C69

1. Introduction

Let $G=(V,E)$ be a simple graph and v be a vertex in V. The open neighborhood of v, denoted by $N(v)$, is the set of vertices adjacent to v, i.e., $N(v)=\{u \in V\colon uv \in E\}$. The closed neighborhood of v is the set $N[v]=N(v)\cup\{v\}$. The degree of v in G is $d_G(v)=|N(v)|$. A vertex v of a tree T is called a leaf of T if $d_T(v)=1$. $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of the vertices of G. When no ambiguity can occur, we often simply write $d(v)$, δ, Δ instead of $d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively. Let $S \subseteq V$, $G[S]$ denote the subgraph of G induced by S. For $S \subseteq V$ and $v \in V$, the degree of v in S, denoted by $d_S(v)$, is the number of neighbors v has in S.

In the following we introduce a definition of a dominating function on a graph G.

Definition 1. Let $\mathbb{R}$ be the real numbers set and $Y \subseteq \mathbb{R}$. A function $f\colon V \to Y$ defined on the vertices of a graph $G=(V,E)$ is a (Y,α)-*dominating function* if f satisfies some condition α. For $S \subseteq V$, let $f(S)=\sum\limits_{v\in S} f(v)$. The weight of f is defined as $f(V)$. A (Y,α)-dominating function f is minimal (Y,α)-dominating function if there does not exist a (Y,α)-dominating function g, $g \neq f$, for which $g(v) \leqslant f(v)$ for every $v \in V$. The (Y,α)-*domination number* of G is $\gamma_{(Y,\alpha)}(G) = \min\{f(V)\colon f$ is a (Y,α)-dominating function of $G\}$.

From the above definition we can easily see the following facts:

(i) If $Y_1 = \{0,1\}$ and $\alpha_1 =$ "$f(N(v)) \geqslant 1$ for every $v \in V$", then a (Y_1,α_1)-dominating function is a *total dominating function* (TDF) of a graph G without isolated vertices and $\gamma_{(Y_1,\alpha_1)}(G) = \gamma_t(G)$ is the *total domination number* of G. (Total domination has been studied in [1]–[4], [8], [10], [11].)

(ii) If $Y_2 = \{-1,0,1\}$ and $\alpha_2 =$ "$f(N[v]) \geqslant 1$ for every $v \in V$", then a (Y_2,α_2)-dominating function is a *minus dominating function* (MDF) and $\gamma_{(Y_2,\alpha_2)}(G) = \gamma^-(G)$ is the *minus domination number* of G. (Minus domination has been studied in [5]–[7], [10], [13].)

(iii) If $Y_3 = \{-1,1\}$ and $\alpha_3 =$ "$f(N(v)) \geqslant 1$ for every $v \in V$", then a (Y_3,α_3)-dominating function is a *signed total dominating function* (STDF) of a graph G without isolated vertices and $\gamma_{(Y_3,\alpha_3)}(G) = \gamma_t^s(G)$ is the *signed total domination number* of G. (Signed total domination has been studied in [12], [14]–[16].)

(iv) If $Y_4 = \{-1,0,1\}$ and $\alpha_4 =$ "$f(N(v)) \geqslant 1$ for every $v \in V$", then a (Y_4,α_4)-dominating function is a *minus total dominating function* (MTDF) of a graph G without isolated vertices and $\gamma_{(Y_4,\alpha_4)}(G) = \gamma_t^-(G)$ is the *minus total domination number* of G. We call a MTDF of weight $\gamma_t^-(G)$ a $\gamma_t^-(G)$-function. (Minus total domination has been defined in [9].)

In this paper, we discuss some properties of minus total domination on a graph G and obtain a few lower bounds for $\gamma_t^-(G)$. To ensure existence of an MTDF, we henceforth restrict our attention to graphs without isolated vertices.

2. Properties on minus total domination

Theorem 1. *A MTDF f on a graph G is minimal if and only if for every vertex $v \in V$ with $f(v) \geqslant 0$, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$.*

Proof. Let f be a minimal MTDF and assume that there is a vertex v with $f(v) \geqslant 0$ and $f(N(u)) > 1$ for every vertex $u \in N(v)$. Define a new function $g\colon V \to \{-1,0,1\}$ by $g(v) = f(v) - 1$ and $g(u) = f(u)$ for all $u \neq v$. Then for all $u \in N(v)$, $g(N(u)) = f(N(u)) - 1 \geqslant 1$. For $w \notin N(v)$, $g(N(w)) = f(N(w)) \geqslant 1$. Thus g is an MTDF on G. Since $g < f$, the minimality of f is contradicted.

Conversely, let f be an MTDF on G such that for every $v \in V$ with $f(v) \geqslant 0$, there exists a vertex $u \in N(v)$ with $f(N(u)) = 1$. Assume f is not minimal, i.e., there is an MTDF g on G such that $g < f$. Then $g(w) \leqslant f(w)$ for all $w \in V$, and there is at least a vertex $v_0 \in V$ with $g(v_0) < f(v_0)$. Therefore, $f(v_0) \geqslant 0$, and by assumption, there exists a vertex $u_0 \in N(v_0)$ with $f(N(u_0)) = 1$. But since $g(w) \leqslant f(w)$ for all $w \in V$ and $g(v_0) < f(v_0)$, we know that $g(N(u_0)) < f(N(u_0)) = 1$. This contradicts the fact that g is a MTDF. Therefore f is a minimal MTDF. □

Consider the graph in Fig. 1. One can see that the function f given in Fig. 1(a) is a minimal TDF but is not a minimal MTDF (cf. Fig. 1(b)). Notice that the vertex v in Fig. 1(a) satisfies $f(v) \geqslant 0$ and $N(v) = \{u\}$, but $f(u) = 2 > 1$, so the minimality condition of Theorem 1 is not satisfied.

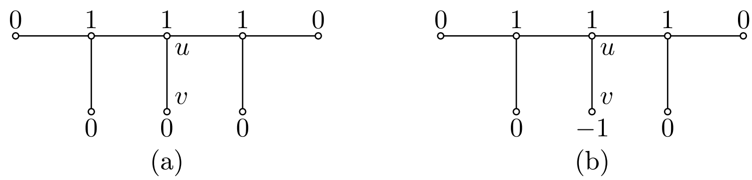

Fig. 1

From [14] we know that γ_t and γ_t^s are not comparable in general. Furthermore, every TDF (or STDF) on a graph is an MTDF. Therefore, the total domination number, signed total domination number and minus total domination number of a graph are related as follows.

Theorem 2. *For any graph G, $\gamma_t^-(G) \leqslant \min(\gamma_t(G), \gamma_t^s(G))$.*

Theorem 3. *For any positive integer k, there exists an outerplanar graph G with $\gamma_t^-(G) \leqslant -k$.*

Proof. Consider the class of outerplanar graphs G_k which can be constructed as in Fig. 2. Then $|V(G_k)| = 3(k+3) + 3 = 3k + 12$ and there are $2k + 8$ vertices of degree 1. By assigning to the $2(k+3)$ vertices of degree 1 the value -1 and to the remaining vertices the value 1, we produce an MTDF f of G_k of weight $(k+6) - 2(k+3) = -k$ as illustrated. □

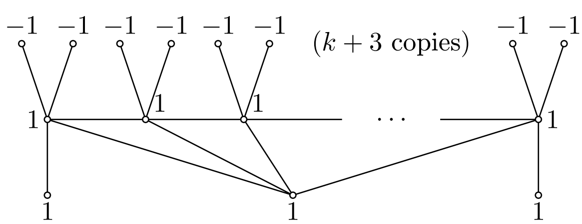

Fig. 2 An outerplanar graph G_k with $\gamma_t^-(G_k) \leqslant -k$.

We introduce the following notation which we shall frequently use in the proofs that follow. For a given MTDF f on a graph G, let $P_f = \{v \in V(G)\colon f(v) = 1\}$, $M_f = \{v \in V(G)\colon f(v) = -1\}$, and let $Q_f = \{v \in V(G)\colon f(v) = 0\}$.

Lemma 1. *Let f be an MTDF of a tree T of order $n \geqslant 2$. Then $|P_f| \geqslant |M_f| + 2$.*

Proof. *Case 1*: $T[P_f]$ is connected.

Since every vertex in M_f must have a neighbor in P_f, we have $\sum\limits_{v\in M_f} d_{P_f}(v) \geqslant |M_f|$. Since every vertex has higher degree in P_f than in M_f, it follows that $\sum\limits_{v\in P_f} d_{M_f}(v) \leqslant \sum\limits_{v\in P_f} (d_{P_f}(v) - 1)$. Thus $|M_f| \leqslant \sum\limits_{v\in M_f} d_{P_f}(v) = \sum\limits_{v\in P_f} d_{M_f}(v) \leqslant \sum\limits_{v\in P_f} (d_{P_f}(v) - 1)$. But $\sum\limits_{v\in P_f} d_{P_f}(v)$ is equal to twice the number of edges in the subgraph $T[P_f]$ induced by P_f. As $T[P_f]$ is connected, $T[P_f]$ is a subtree of T. Thus $|M_f| \leqslant \sum\limits_{v\in P_f} (d_{P_f}(v) - 1) = 2|E(T[P_f])| - |P_f| = 2(|P_f| - 1) - |P_f| = |P_f| - 2$. Hence $|P_f| \geqslant |M_f| + 2$.

Case 2: $T[P_f]$ is disconnected.

Then $T[P_f]$ is a forest. Assume that $P_1, P_2, \ldots, P_k$ are the components of $T[P_f]$. Then $|V(P_i)| \geqslant 2$ for $1 \leqslant i \leqslant k$. Let $M_i = \bigcup\limits_{v\in V(P_i)} (N(v) \cap M_f)$ and let $T_i = T[V(P_i) \cup M_i]$. Then T_i is a subtree of T. Similarly to Case 1, we have $|V(P_i)| \geqslant |M_i| + 2$. Therefore, $|P_f| = \sum\limits_{i=1}^{k} |V(P_i)| \geqslant \sum\limits_{i=1}^{k} (|M_i| + 2) \geqslant |M_f| + 2k \geqslant |M_f| + 2$. □

Theorem 4. *If T is a tree of order $n \geqslant 4$, then $\gamma_t(T) - \gamma_t^-(T) \leqslant \frac{1}{2}(n-4)$.*

Proof. Let f be a $\gamma_t^-(G)$-function of T. If $M_f = \emptyset$, then $\gamma_t(T) - \gamma_t^-(T) = 0 \leqslant \frac{1}{2}(n-4)$. So assume that $M_f \neq \emptyset$. Let $v \in M_f$. Since $f(N(v)) \geqslant 1$, there is a vertex $u \in P_f \cap N(v)$ such that $|N(u) \cap P_f| \geqslant 2$. Let P' be the component of $T[P_f]$ which contains the vertex u. Then P' is a subtree of T and $|V(P')| \geqslant 3$. Moreover, by Lemma 1, $|P_f| \geqslant |M_f| + 2$. Hence $|M_f| = n - |P_f| - |Q_f| \leqslant n - (|M_f| + 2) - |Q_f| = n - |M_f| - |Q_f| - 2$. Thus, $|M_f| \leqslant \frac{1}{2}(n - |Q_f| - 2)$.

Case 1: $|Q_f| \geqslant 2$.

Since P_f is a total domination set of T, $\gamma_t(T) \leqslant |P_f|$. Furthermore, $\gamma_t^-(T) = |P_f| - |M_f|$. Thus $\gamma_t(T) - \gamma_t^-(T) \leqslant |P_f| - (|P_f| - |M_f|) = |M_f| \leqslant \frac{1}{2}(n - |Q_f| - 2) \leqslant \frac{1}{2}(n-4)$.

Case 2: $|Q_f| \leqslant 1$.

Since P' is a subtree of T and $|V(P')| \geqslant 3$, there are at least two leaves in P'. Let w be a leaf of P' such that $N(w) \cap Q_f = \emptyset$. Since w is not adjacent to any vertex in M_f. it follows that $P_f - \{w\}$ is a total domination set of T. Hence $\gamma_t(T) \leqslant |P_f| - 1$. Thus $\gamma_t(T) - \gamma_t^-(T) \leqslant (|P_f| - 1) - (|P_f| - |M_f|) = |M_f| - 1 \leqslant \frac{1}{2}(n-2) - 1 = \frac{1}{2}(n-4)$.

□

Theorem 5. *For any complete graph K_n on n ($n \geqslant 2$) vertices, $\gamma_t^-(K_n) = 2$.*

Proof. Let f be a $\gamma_t^-(G)$-function of K_n. Obviously, $|P_f| \geqslant 2$. Let $v \in P_f$. Since $f(N(v)) \geqslant 1$, $\gamma_t^-(K_n) = f(N[v]) = f(N(v)) + f(v) \geqslant 2$.

On the other hand, let g be the function of K_n defined as follows. Assign to a pair of vertices the value 1 and to the remaining vertices the value 0. It is easy to see that g is an MTDF of K_n and the weight $g(V) = 2$. Thus $\gamma_t^-(G) \leqslant g(V) = 2$. Consequently, $\gamma_t^-(K_n) = 2$. □

Theorem 6. *For any path P_n on n ($n \geqslant 2$) vertices,*

$$\gamma_t^-(P_n) = \gamma_t(P_n) = \begin{cases} \lceil \frac{1}{2}n \rceil, & n \equiv 0,1,3 \pmod 4, \\ \frac{1}{2}n + 1, & n \equiv 2 \pmod 4. \end{cases}$$

Proof. Let f be a $\gamma_t^-(G)$-function of P_n. We claim that for every vertex $V(P_n)$, $f(v) \geqslant 0$. If this is not the case, then there exists a vertex $v \in V(P_n)$ such that $f(v) = -1$. Let $u \in N(v)$. Then $f(N(u)) \leqslant 0$, a contradiction. Thus f is a total dominating function of P_n. Then $\gamma_t(P_n) \leqslant f(V(P_n)) = \gamma_t^-(P_n)$. On the other hand, by Theorem 2, we have $\gamma_t^-(P_n) \leqslant \gamma_t(P_n)$. Consequently, $\gamma_t^-(P_n) = \gamma_t(P_n)$. □

The proof of the following result is similar to that of Theorem 6 and is therefore omitted.

Theorem 7. *For any cycle C_n on n ($n \geqslant 2$) vertices,*

$$\gamma_t^-(C_n) = \gamma_t(C_n) = \begin{cases} \lceil \frac{1}{2}n \rceil, & n \equiv 0,1,3 \pmod 4, \\ \frac{1}{2}n + 1, & n \equiv 2 \pmod 4. \end{cases}$$

Theorem 8. *For any complete multipartite graph $G \cong K(m_1, m_2, \ldots, m_n)$, $\gamma_t^-(G) = 2$.*

Proof. Let f be a $\gamma_t^-(G)$-function on G and let $A_1, A_2, \ldots, A_n$ denote the partite sets of G. For $1 \leqslant i \leqslant n$, let $P_i = \{v \in A_i \colon f(v) = 1\}$ and $M_i = \{v \in A_i \colon f(v) = -1\}$. Obviously, there exists an integer j ($1 \leqslant j \leqslant n$) such that $|P_j| > |M_j|$ (otherwise for every $v \in V(G)$, $f(N(v)) \leqslant 0$). Let $v_0 \in A_j$. Since $f(N(v_0)) = \sum_{v \in V - A_j} f(v) \geqslant 1$, it follows that $\gamma_t^-(G) = f(V) = \sum_{v \in V} f(v) = f(N(v_0)) + \sum_{v \in A_j} f(v) \geqslant 1 + |P_j| - |M_j| \geqslant 2$.

On the other hand, assume that $v_1 \in A_1$ and $v_2 \in A_2$. Let g be the function on G defined as follows. Assign to the vertices v_1 and v_2 the value 1 and to the remaining vertices the value 0. It is easy to see that g is an MTDF of G and the weight $g(V) = 2$. Thus $\gamma_t^-(G) \leqslant g(V) = 2$. Consequently, $\gamma_t^-(G) = 2$. □

3. Lower bounds on minus total domination number

Theorem 9. *If T is a tree of order $n \geqslant 2$, then $\gamma_t^-(T) \geqslant 2$.*

Proof. Let f be a $\gamma_t^-(G)$-function of T. By Lemma 1, $|P_f| \geqslant |M_f| + 2$. Thus $\gamma_t^-(T) = |P_f| - |M_f| \geqslant 2$. □

Theorem 10. *For any graph G of order n, maximum degree Δ and minimum degree $\delta \geqslant 1$,*

$$\gamma_t^-(G) \geqslant \frac{\delta - \Delta + 2}{\delta + \Delta} n.$$

Proof. Let f be a $\gamma_t^-(G)$-function on G. Let P_f, M_f and Q_f be the sets of vertices in G that are assigned the values $+1$, -1 and 0 under f, respectively. Let $P_f = P_\Delta \cup P_\delta \cup P_\Theta$ where P_Δ and P_δ are the sets of all vertices of P_f with degree equal to Δ and δ, respectively, and P_Θ contains all other vertices in P_f, if any. Similarly, we define $M_f = M_\Delta \cup M_\delta \cup M_\Theta$ and $Q_f = Q_\Delta \cup Q_\delta \cup Q_\Theta$. Further, for $i \in \{\Delta, \delta, \Theta\}$, let V_i be defined by $V_i = P_i \cup M_i \cup Q_i$. Thus $n = |V_\Delta| + |V_\delta| + |V_\Theta|$.

Since for each $v \in V$, $f(N(v)) \geqslant 1$, we have $\sum\limits_{v \in V} f(N(v)) \geqslant |V| = n$. The sum $\sum\limits_{v \in V} f(N(v))$ counts the value $f(v)$ exactly $d(v)$ times for each vertex $v \in V$, i.e., $\sum\limits_{v \in V} f(N(v)) = \sum\limits_{v \in V} f(v)d(v)$. Thus, $\sum\limits_{v \in V} f(v)d(v) \geqslant n$. Breaking the sum up into the nine summations and replacing $f(v)$ by the corresponding value of 1, 0 or -1 yields

$$\sum_{v \in P_\Delta} d(v) + \sum_{v \in P_\delta} d(v) + \sum_{v \in P_\Theta} d(v) - \sum_{v \in M_\Delta} d(v) - \sum_{v \in M_\delta} d(v) - \sum_{v \in M_\Theta} d(v) \geqslant n.$$

We know that $d(v) = \Delta$ for all v in P_Δ or M_Δ, and $d(v) = \delta$ for all v in P_δ or M_δ. For any vertex v in either P_Θ or M_Θ, $\delta + 1 \leqslant d(v) \leqslant \Delta - 1$. Thus

$$\Delta|P_\Delta| + \delta|P_\delta| + (\Delta - 1)|P_\Theta| - \Delta|M_\Delta| - \delta|M_\delta| - (\delta + 1)|M_\Theta| \geqslant n.$$

For $i \in \{\Delta, \delta, \Theta\}$, we replace $|P_i|$ with $|V_i| - |M_i| - |Q_i|$ in the above inequality. Therefore, we have

$$\begin{aligned}&\Delta|V_\Delta| + \delta|V_\delta| + (\Delta - 1)|V_\Theta| \\ &\qquad \geqslant n + 2\Delta|M_\Delta| + 2\delta|M_\delta| + (\Delta + \delta)|M_\Theta| + \Delta|Q_\Delta| + \delta|Q_\delta| + (\Delta - 1)|Q_\Theta|.\end{aligned}$$

It follows that

$$\begin{aligned}(\Delta-1)n &\geqslant 2\Delta|M_\Delta|+2\delta|M_\delta|+(\Delta+\delta)|M_\Theta|+\Delta|Q_\Delta|+\delta|Q_\delta|+(\Delta-1)|Q_\Theta|\\ &\quad+(\Delta-\delta)(|P_\delta|+|Q_\delta|+|M_\delta|)+(|P_\Theta|+|Q_\Theta|+|M_\Theta|)\\ &=2\Delta|M_\Delta|+(\delta+\Delta)|M_\delta|+(\delta+\Delta+1)|M_\Theta|\\ &\quad+\Delta|Q_\Delta|+\Delta|Q_\delta|+\Delta|Q_\Theta|+(\Delta-\delta)|P_\delta|+|P_\Theta|\\ &\geqslant(\Delta+\delta)|M_\Delta|+(\Delta+\delta)|M_\delta|+(\Delta+\delta)|M_\Theta|+\Delta|Q_f|\\ &\geqslant(\Delta+\delta)|M_f|+\Delta|Q_f|\\ &\geqslant\frac{1}{2}(\Delta+\delta)(2|M_f|+|Q_f|).\end{aligned}$$

Thus $2|M_f|+|Q_f|\leqslant 2(\Delta-1)(\Delta+\delta)^{-1}n$.

Therefore, $\gamma_t^-(G)=n-(2|M_f|+|Q_f|)\geqslant n-(2\Delta-2)(\Delta+\delta)^{-1}n=(\delta-\Delta+2)(\Delta+\delta)^{-1}n$. □

Corollary 1. *If G is an r-regular graph of order n, then $\gamma_t^-(G)\geqslant n/r$, and the bound is sharp.*

Proof. Since G is an r-regular graph, $\Delta=\delta=r$. By Theorem 10, the result follows.

That the bound is sharp may be seen by considering a complete bipartite graph $K_{r,r}$ of order $n=2r$. By Theorem 8, $\gamma_t^-(K_{r,r})=2=n/r$. □

Corollary 2 ([12], [16])**.** *If G is an r-regular graph of order n, then $\gamma_t^s(G)\geqslant n/r$.*

In the following, we give a lower bound on the minus total domination number of a bipartite graph in terms of its order and characterize the graphs attaining this bound. For this purpose, we define a family $\mathscr{G}$ of bipartite graphs as follows.

For $s\geqslant 2$, let G_s be the bipartite graph obtained from the disjoint union of $2s$ stars $K_{1,s-1}$ with centers $\{x_1,x_2,\dots,x_s,y_1,y_2,\dots,y_s\}$ by adding all edges of the type x_iy_j, $1\leqslant i\leqslant j\leqslant s$. Then $|V(G_s)|=2s^2$ and $|E(G_s)|=3s^2-2s$. Let $\mathscr{G}=\{G_s\colon s\geqslant 2\}$.

Theorem 11. *If G is a bipartite graph of order n, then $\gamma_t^-(G)\geqslant 2\sqrt{2n}-n$, with equality if and only if $G\in\mathscr{G}$.*

Proof. Let f be a $\gamma_t^-(G)$-function on G and let X and Y be the partite sets of G. Further, let $X^+=\{v\in X\colon f(v)=1\}$, $X^-=\{v\in X\colon f(v)=-1\}$, $Y^+=\{v\in Y\colon f(v)=1\}$, $Y^-=\{v\in Y\colon f(v)=-1\}$. Then $P_f=X^+\cup Y^+$, $M_f=X^-\cup Y^-$. For convenience, let $x_1=|X^+|$, $x_2=|X^-|$, $y_1=|Y^+|$, $y_2=|Y^-|$, $p=|P_f|$, $m=|M_f|$, $q=|Q_f|$. Obviously, $x_1\geqslant 1$, $y_1\geqslant 1$. Then $x_1+y_1=p\geqslant 2$.

Since each vertex in X^- is adjacent to at least one vertex in Y^+, by the Pigeonhole Principle, at least one vertex v_0 of Y^+ is adjacent to at least $\lceil x_2/y_1 \rceil$ vertices of X^-. Since $1 \leqslant f(N(v_0)) = |N(v_0) \cap X^+| - |N(v_0) \cap X^-| \leqslant |N(v_0) \cap X^+| - \lceil x_2/y_1 \rceil$, it follows that $x_1 = |X^+| \geqslant |N(v_0) \cap X^+| \geqslant \lceil x_2/y_1 \rceil + 1 \geqslant x_2/y_1 + 1$. Thus $x_1 y_1 \geqslant x_2 + y_1$. Using a similar argument, we may show that $x_1 y_1 \geqslant y_2 + x_1$. Thus $2x_1 y_1 \geqslant x_1 + y_1 + x_2 + y_2 = n - q$. Furthermore, since $2x_1 y_1 \leqslant \frac{1}{2}(x_1 + y_1)^2 = \frac{1}{2}p^2$, we have $\frac{1}{2}p^2 \geqslant n - q$. Thus $p^2 + 2q \geqslant 2n$. Since $p = x_1 + y_1 \geqslant 2$, it follows that $(p + \frac{1}{2}q)^2 \geqslant 2n$. So $2p + q \geqslant 2\sqrt{2n}$. Therefore

$$\gamma_t^-(G) = p - m = p - (n - p - q) = (2p + q) - n \geqslant 2\sqrt{2n} - n.$$

If G is a bipartite graph of order n such that $\gamma_t^-(G) = 2\sqrt{2n} - n$, then $2p + q = 2\sqrt{2n}$ and $q = 0$. Further, $2x_1 y_1 = \frac{1}{2}(x_1 + y_1)^2$ and $x_1 y_1 = x_1 + y_2 = x_2 + y_1$. Thus $x_1 = y_1$ and $x_2 = y_2 = x_1(x_1 - 1)$. Furthermore, each vertex of X^- (respectively, Y^-) has degree 1 and is adjacent to a vertex of Y^+ (respectively, X^+), while each vertex of X^+ is adjacent to all x_1 vertices of Y^+ and to $x_1 - 1$ vertices of Y^- and each vertex of Y^+ is adjacent to all x_1 vertices of X^+ and to $x_1 - 1$ vertices of X^-. Thus, if $\gamma_t^-(G) = 2\sqrt{2n} - n$, then $G \in \mathscr{G}$.

On the other hand, suppose $G \in \mathscr{G}$. Then $G = G_s$ for some $s \geqslant 2$. So G_s has order $n = 2s^2$. Assigning to the $2s$ central vertices of stars the value 1, and to all other vertices the value -1, we produce an MTDF f of weight $f(V) = 2s - 2s(s - 1) = 2s - (n - 2s) = 4s - n = 2\sqrt{2n} - n$. Therefore, $\gamma_t^-(G) \leqslant f(V) = 2\sqrt{2n} - n$. Consequently, $\gamma_t^-(G) = 2\sqrt{2n} - n$. □

Let $F_2 = K_2$ and for $s \geqslant 3$, let F_s be the graph obtained from the disjoint union of s stars $K_{1,s-2}$ by adding all edges between the central vertices of the s stars. Let $\mathscr{F} = \{F_s | s \geqslant 2\}$.

Theorem 12. *If G is a graph of order n, then $\gamma_t^-(G) \geqslant \sqrt{4n+1} + 1 - n$, with equality if and only if $G \in \mathscr{F}$.*

Proof. Let f be a $\gamma_t^-(G)$-function on G and let $|P_f| = p$, $|M_f| = m$ and $|Q_f| = q$. Then $\gamma_t^-(G) = |P_f| - |M_f| = p - m = p - (n - p - q) = 2p + q - n$. Each vertex in M_f is adjacent to at least one vertex of P_f. Thus, by Pigeonhole Principle, at least one vertex v of P_f is adjacent to at least $\lceil |M_f|/|P_f| \rceil = \lceil m/p \rceil$ vertices of M_f. It follows, therefore, that $1 \leqslant f(N(v)) = |N(v) \cap P_f| - |N(v) \cap M_f| \leqslant (|P_f| - 1) - \lceil m/p \rceil = (p - 1) - \lceil m/p \rceil \leqslant p - 1 - m/p$, and so $p^2 - 2p - m \geqslant 0$. Hence, we have $p^2 - p + q - n \geqslant 0$. Thus $p \geqslant \frac{1}{2}\bigl(\sqrt{4(n-q)+1} + 1\bigr)$, and so $\gamma_t^-(G) = 2p + q - n \geqslant \sqrt{4(n-q)+1} + 1 - (n - q)$.

Let $g(x) = \sqrt{4x+1} + 1 - x$. Then $g'(x) = 2(4x+1)^{-1/2} - 1$. For $x \geqslant 1$, $g'(x) < 0$. That is, $g(x)$ is a monotone decreasing function when $x \geqslant 1$. Furthermore, since

$p = |P_f| \geqslant 2$, we have $n - q = p + m \geqslant 2$. Therefore, $g(n-q) \geqslant g(n)$. Consequently, $\gamma_t^-(G) \geqslant \sqrt{4(n-q)+1} + 1 - (n-q) \geqslant \sqrt{4n+1} + 1 - n$.

If G is a graph of order n such that $\gamma_t^-(G) = \sqrt{4n+1} + 1 - n$, then $2p + q = \sqrt{4n+1} + 1$ and $q = 0$. Thus $n = p(p-1)$ and $m = p(p-2)$. Furthermore, each vertex of M_f has degree 1 and is adjacent to a vertex of P_f, while each vertex of P_f is adjacent to all the other $p-1$ vertices of P_f and to $p-2$ vertices of M_f. It follows that $G \in \mathscr{F}$.

On the other hand, suppose $G \in \mathscr{F}$. Then $G = F_s$ for some $s \geqslant 2$. So F_s has order $n = s(s-1)$, and so $s = \frac{1}{2}(\sqrt{4n+1} + 1)$. Assigning to the s central vertices of stars the value 1, and to all other vertices the value -1, we produce an MTDF f of weight $f(V) = s - s(s-2) = s - (n-s) = 2s - n = \sqrt{4n+1} + 1 - n$. Therefore, $\gamma_t^-(G) \leqslant f(V) = \sqrt{4n+1} + 1 - n$. Consequently, $\gamma_t^-(G) = \sqrt{4n+1} + 1 - n$. □

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