Online Appendix

for "Nonlinear Pricing with Under-Utilization: A Theory of Multi-Part Tariffs" by Corrao, Flynn, and Sastry

A Proofs of Main Results

In this appendix, we provide the proofs of the main results. In Section A.1, we define and characterize implementable consumption functions under free disposal and characterize optimal contracts, proving Proposition 1. In Section A.2, we characterize the occurrence of multi-part tariffs by proving Proposition 2 and the corresponding corollaries. Finally, in Section A.3, we derive comparative statics for welfare, proving Propositions 3 and 4.

A.1 Implementation and Proof of Proposition 1

We say that consumption function ϕ is implementable if there exist a purchase function ξ and a price schedule T such that (ϕ, ξ, T) jointly satisfy the constraints (O), (IC), and (IR) of Problem 1. In this case, we say that ϕ is supported by (ξ, T). The following intermediate results fully characterize implementable consumption functions in terms of their functional properties. We say real functions are *monotone* when they are monotone non-decreasing.

Lemma 1. *Fix a consumption function ϕ that is monotone and such that $\phi \leq \phi^A$. Define the transfer function $t : \Theta \to \mathbb{R}$ as*

$$t(\theta) = C + u(\phi(\theta), \theta) - \int_0^\theta u_\theta(\phi(s), s)\, \mathrm{d}s \tag{23}$$

for some $C \leq 0$, and define the price schedule $T : X \to \overline{\mathbb{R}}$ as

$$T(x) = \inf_{\theta' \in \Theta} \{t(\theta') : x \leq \phi(\theta')\} \tag{24}$$

Then t and T are monotone.

Proof. Fix $\theta', \theta \in \Theta$ such that $\theta' \geq \theta$. Given that ϕ is monotone, it is almost everywhere differentiable with derivative denoted by ϕ' when defined. By the Fundamental Theorem of calculus, we can re-write the transfer function as

$$t(\theta) = C + u(\phi(0), 0) + \int_0^\theta (u_x(\phi(s), s)\phi'(s) + u_\theta(\phi(s), s))\, \mathrm{d}s - \int_0^\theta u_\theta(\phi(s), s)\, \mathrm{d}s \tag{25}$$

Subtracting $t(\theta)$ from $t(\theta')$, we get

$$t(\theta') - t(\theta) = \int_{\theta}^{\theta'} u_x(\phi(s), s)\phi'(s)\,\mathrm{d}s \tag{26}$$

Given that $\phi \leq \phi^A$, and that u is strictly quasiconcave in x, it follows that $u_x(\phi(s), s) \geq 0$ for all $s \in [0, \theta']$. Moreover, given that ϕ is monotone, it follows that $\phi'(s) \geq 0$ for almost all $s \in [0, \theta']$. Given that $\theta' \geq \theta$, Equation 26 implies that $t(\theta') \geq t(\theta)$. Given that θ', θ were arbitrarily chosen, it follows that t is monotone.

Next, fix $x, y \in X$ such that $y \leq x$. Given that ϕ is monotone, the definition of T implies that $T(y) \leq T(x)$. We then conclude that T is monotone. □

Lemma 2. *A consumption function ϕ is implementable if and only if ϕ is monotone and such that $\phi \leq \phi^A$. In this case, ϕ is supported by (ϕ, T), where T is defined as in Equation* (24) *for some $C \leq 0$.*[27]

Proof. (Only if). If ϕ is implementable, then there exists (ξ, T) that support ϕ. By Incentive Compatibility and by the taxation principle, there exists a transfer function $t : \Theta \to \mathbb{R}$ such that $u(\phi(\theta), \theta) - t(\theta) \geq u(\phi(\theta'), \theta) - t(\theta')$ for all $\theta, \theta' \in \Theta$. By a standard implementation result (see, e.g., Proposition 1 in Rochet, 1987), this implies that ϕ is monotone. Finally, if there exists $\theta \in \Theta$ such that $\phi(\theta) > \phi^A(\theta)$, then we would contradict Obedience for type θ since $u(\phi^A(\theta), \theta) > u(\phi(\theta), \theta)$ and $\phi^A(\theta)$ would be feasible given $\phi(\theta)$ by construction.

(If). Now suppose that ϕ is monotone and such that $\phi(\theta) \leq \phi^A(\theta)$ for all $\theta \in \Theta$. Define t and T given ϕ as in Equations 23 and 24 respectively. We next prove that (ϕ, ϕ, T) satisfies (O), (IC), and (IR).

First, for every $\theta \in \Theta$, we have

$$u(\phi(\theta), \theta) - T(\phi(\theta)) \geq u(\phi(\theta), \theta) - t(\theta) = \int_0^{\theta} u_\theta(\phi(s), s)\,\mathrm{d}s - C \geq 0 \tag{27}$$

where the first inequality follows from the definition of T and the last inequality follows from $C \leq 0$ and $u_\theta(\phi(\theta), \theta) \geq 0$ for all $\theta \in \Theta$ (u is monotone increasing in θ). This proves Individual Rationality.

Next, assume toward a contradiction that Obedience does not hold. That is, there exist $\theta \in \Theta$ and $y < \phi(\theta) \leq \phi^A(\theta)$ such that $u(y, \theta) > u(\phi(\theta), \theta)$. However, this yields a contradiction with strict quasiconcavity of u in x. Therefore, Obedience holds.

We are left to prove that (ϕ, ϕ, T) satisfy Incentive Compatibility. Fix $\theta', \theta \in \Theta$ such that

[27] Observe that here the purchase function is $\xi = \phi$.

$\theta' \neq \theta$. We first prove that, for all θ, θ', we have

$$u(\phi(\theta), \theta) - t(\theta) \geq \max_{x \leq \phi(\theta')} u(x, \theta) - t(\theta') \tag{28}$$

This is a variation of the standard reporting problem under consumption function ϕ and transfers t, where each agent, on top of misreporting their type, can freely dispose of the allocated quantity. Violations of this condition can take two forms. First, an agent of type θ could report type θ' and consume $x = \phi(\theta')$. We call this a single deviation. Second, an agent of type θ could report type θ' and consume $x < \phi(\theta')$. We call this a double deviation. Under our construction of transfers t and monotonicity of ϕ, by a standard mechanism-design argument (e.g., Proposition 1 in Rochet, 1987), there is no strict gain to any agent of reporting θ' and consuming $x = \phi(\theta')$. Thus, there are no profitable single deviations under (ϕ, t).

We now must rule out double deviations. Define the value function $V : \Theta \to \mathbb{R}$ under ϕ and t as

$$V(\theta) = u(\phi(\theta), \theta) - t(\theta) = \int_0^\theta u_\theta(\phi(s), s)\, \mathrm{d}s - C \tag{29}$$

Suppose, toward a contradiction, that there exists a double deviation in which type θ reports type θ'. We separate the argument by various cases comparing $(\theta, \phi(\theta), \phi^A(\theta))$ and $(\theta', \phi(\theta'), \phi^A(\theta'))$.

1. $\theta' < \theta$: Given that ϕ is monotone, it must be that $\phi(\theta') \leq \phi(\theta)$. Moreover, as (O) holds, we have that $\phi(\theta') < \phi(\theta)$. For the same reason, we have that $\phi(\theta')$ is optimal for type θ' when they could choose any $x \leq \phi(\theta')$. Moreover, by strict single-crossing of u and strict quasiconcavity of $u(\cdot, \theta)$, it is optimal for type θ to consume some $x \geq \phi(\theta')$. But, we know that $x \leq \phi(\theta')$; thus $x = \phi(\theta')$ is optimal. Hence, if there is a double deviation with $\theta' < \theta$, there is also a single deviation. This is a contradiction as we already showed that there are no strictly profitable single deviations.
2. $\theta' > \theta$ and $\phi^A(\theta) \geq \phi(\theta')$: the optimal choice of consumption for agent θ in $[0, \phi(\theta')]$ is given by $\phi(\theta')$ by strict quasiconcavity of u. Thus, there is a profitable single deviation, which is a contradiction.
3. $\theta' > \theta$ and $\phi^A(\theta) < \phi(\theta')$: We know $x = \phi^A(\theta)$ is most attractive following the misreport θ'. Suppose that there exists some $\hat{\theta} \in (\theta, \theta']$ such that $\phi(\hat{\theta}) = \phi^A(\theta)$. Given that t is monotone by Lemma 1, we know that a single deviation to $\hat{\theta}$ is weakly more attractive than a double deviation to $x \leq \phi(\theta')$. As no single deviations exist, this is a contradiction. As $\phi(\theta) \leq \phi^A(\theta) < \phi(\theta')$, it follows that no type $\hat{\theta} \in (\theta, \theta']$ receives

$\phi^A(\theta)$. We know that the most attractive misreport is the smallest type θ' such that $\phi(\theta') \geq \phi^A(\theta)$. It follows that $\phi^A(\theta) \leq \phi(\theta') \leq \phi^A(\theta')$ and therefore that there exists some $\hat{\theta}$ such that $\phi^A(\hat{\theta}) = \phi(\theta')$, by continuity of ϕ^A.

We now work toward a contradiction. By the hypothesis of a double deviation for type θ:

$$u(\phi^A(\theta), \theta) - t(\theta') > u(\phi(\theta), \theta) - t(\theta) \tag{30}$$

Define for any type θ, the value of optimal autarkic consumption as $V^*(\theta) = u(\phi^A(\theta), \theta)$. We can write $V^*(\theta) - V(\theta) > t(\theta')$. As we have ruled out single deviations, we know that:

$$u(\phi^A(\hat{\theta}), \hat{\theta}) - t(\theta') \leq u(\phi(\hat{\theta}), \hat{\theta}) - t(\hat{\theta}) \tag{31}$$

Thus $V^*(\hat{\theta}) - V(\hat{\theta}) \leq t(\theta')$. Together, we then have that $V(\hat{\theta}) - V(\theta) > V^*(\hat{\theta}) - V^*(\theta)$. From the definition of V in Equation 29, the left-hand-side is $V(\hat{\theta}) - V(\theta) = \int_\theta^{\hat{\theta}} u_\theta(\phi(s), s)\,\mathrm{d}s$. From the envelope theorem applied to the autarkic consumption problem, the right-hand-side is $V^*(\hat{\theta}) - V^*(\theta) = \int_\theta^{\hat{\theta}} u_\theta(\phi^A(s), s)\,\mathrm{d}s$. Combining these substitutions with the original inequality,

$$\int_\theta^{\hat{\theta}} u_\theta(\phi(s), s)\,\mathrm{d}s > \int_\theta^{\hat{\theta}} u_\theta(\phi^A(s), s)\,\mathrm{d}s \tag{32}$$

But we know that $\phi^A(s) \geq \phi(s)$ for all $s \in [\theta, \hat{\theta}]$, and this implies by single-crossing of u that $u_\theta(\phi^A(s), s) \geq u_\theta(\phi(s), s)$, which contradicts the inequality above.

We have ruled out double deviations in all cases and thereby completed the proof of the claim in Equation 28. We next prove that Equation 28 implies that (ϕ, ϕ, T) satisfy Incentive Compatibility. For all $\theta \in \Theta$, we have

$$\begin{aligned} & u(\phi(\theta), \theta) - T(\phi(\theta)) \geq u(\phi(\theta), \theta) - t(\theta) \\ & \geq \sup_{\theta' \in \Theta} \left\{ \sup_{x \in X : x \leq \phi(\theta')} \{u(x, \theta)\} - t(\theta') \right\} = \sup_{x \in X} \left\{ \sup_{\theta' \in \Theta : x \leq \phi(\theta')} \{u(x, \theta) - t(\theta')\} \right\} \\ & = \sup_{x \in X} \{u(x, \theta) - T(x)\} \end{aligned} \tag{33}$$

yielding Incentive Compatibility. This concludes the proof of the implication.

The second part of the statement directly follows from the proof of sufficiency. $\square$

We now show that optimizing over the set of implementable allocations is equivalent to maximizing virtual surplus subject to the implementation constraints from Lemma 2.

Lemma 3. *A consumption function ϕ^* is part of a solution to Problem 1 if any only if it solves*

$$\begin{aligned} &\max_{\phi} \quad \int_{\Theta} J(\phi(\theta), \theta)\, \mathrm{d}F(\theta) \\ &\textit{s.t.} \quad \phi(\theta') \geq \phi(\theta), \quad \phi(\theta) \leq \phi^A(\theta), \quad \theta, \theta' \in \Theta : \theta' \geq \theta \end{aligned} \tag{34}$$

Proof. We begin by eliminating the proposed allocation and transfers from the objective function of the seller. From the proof of Lemma 2, we have that every implementable ϕ is supported by $\xi = \phi$ and by a price schedule T defined as in Equation 24 where the transfer function t is defined in Equation 23 for some constant $C \leq 0$. Given that any ξ that supports ϕ leads to the same seller payoff, we can then set $\xi = \phi$ without loss of optimality. Moreover, we know that ϕ being implementable is equivalent to ϕ being monotone increasing and $\phi \leq \phi^A$ (given that $C \leq 0$). Finally, it is not optimal for the seller to exclude any agent from the mechanism as it is without loss to allocate any agent $x = 0$ rather than exclude them owing to the fact that $\pi(0, \cdot) = 0$, $u(0, \cdot) = 0$, $u(x, \cdot)$ is monotone increasing over Θ, and u has strict single-crossing in (x, θ). In particular, for any incentive compatible allocation that excludes some type θ, it is without loss of optimality to set $\phi(\theta) = \xi(\theta) = t(\theta) = 0$. Each agent is indifferent between participation and not, and this does not change the principal's payoff.

Plugging in the expression (23), we can simplify the expression for the seller's total transfer revenue as the following:

$$\begin{aligned} \int_{\Theta} t(\theta)\, \mathrm{d}F(\theta) &= \int_{\Theta} \left(C + u(\phi(\theta), \theta) - \int_0^{\theta} u_{\theta}(\phi(s), s)\, \mathrm{d}s \right) \mathrm{d}F(\theta) \\ &= \int_{\Theta} \left(C + u(\phi(\theta), \theta) - \frac{(1 - F(\theta))}{f(\theta)} u_{\theta}(\phi(\theta), \theta) \right) \mathrm{d}F(\theta) \end{aligned} \tag{35}$$

where the final equality follows by applying the standard integration-by-parts argument.

Plugging into the seller's objective, we find that the principal solves:

$$\begin{aligned} &\max_{\phi, C} \int_{\Theta} \left(J(\phi(\theta), \theta) + C \right) \mathrm{d}F(\theta) \\ &\text{s.t.} \quad C \leq 0,\ \phi(\theta') \geq \phi(\theta),\ \phi(\theta) \leq \phi^A(\theta) \quad \forall \theta, \theta' \in \Theta : \theta' \geq \theta \end{aligned} \tag{36}$$

It follows that it is optimal to set $C = 0$, completing the proof. □

Proof of Proposition 1. By Lemma 3, any optimal consumption function must solve Problem 34. Consider now the family of problems $\max_{x\in[0,\phi^A(\theta)]} J(x,\theta)$, indexed by $\theta \in \Theta$. As J is strictly quasiconcave in x, there is a unique maximum in this problem, which we call $\phi^*(\theta)$. Moreover, whenever $\phi^P(\theta) < \phi^A(\theta)$, we know that $\phi^*(\theta) = \phi^P(\theta)$. Otherwise $\phi^*(\theta) = \phi^A(\theta)$, by strict quasiconcavity of J in x. Thus, the solution of this pointwise problem is $\phi^*(\theta) = \min\left\{\phi^A(\theta), \phi^P(\theta)\right\}$. As ϕ^A and ϕ^P are monotone, ϕ^* is monotone and is therefore the unique solution to Problem 34.

We next prove the remaining parts of the statement by explicitly constructing the claimed supporting price schedules and purchases. From Lemma 2, we can construct the claimed formula for the price schedule directly. Because ϕ^* is invertible over $(\phi^*(0), \phi^*(1))$ and using its extension on the boundaries (see Footnote 14), we have that for all $x \in X^* = [\phi^*(0), \phi^*(1)]$:

$$T^*(x) = t({\phi^*}^{-1}(x)) = u(x, {\phi^*}^{-1}(x)) - \int_0^{{\phi^*}^{-1}(x)} u_\theta(\phi^*(s), s)\,\mathrm{d}s \tag{37}$$

As T^* is monotone, it is almost everywhere differentiable. Moreover, whenever it is differentiable, by differentiating Equation 37 we obtain ${T^*}'(x) = u_x(x, {\phi^*}^{-1}(x))$. Integrating, we obtain the price schedule in Equation 6 on X^*.

Finally, we show that the optimal level of consumption is supported by any selection from Ξ_{ϕ^*}, and only by selections from Ξ_{ϕ^*}. To this end, consider the selection $\overline{\xi} \in \Xi_{\phi^*}$ defined as $\overline{\xi} = \max \Xi_{\phi^*}$. We want to show that the triple $(\phi^*, \overline{\xi}, T^*)$ satisfies Obedience, Incentive Compatiblility, and Individual Rationality. We now define $t = T^* \circ \overline{\xi}$.

Consider first the Obedience constraint that $\phi^*(\theta) \in \arg\max_{x\in[0,\overline{\xi}(\theta)]} u(x,\theta)$, for all $\theta \in \Theta$. Observe that $\phi^* \leq \overline{\xi}$ by construction of $\overline{\xi}$. Moreover, toward a contradiction, suppose that there exists $\theta \in \Theta$ and $x \leq \overline{\xi}(\theta)$ such that $u(\phi^*(\theta),\theta) < u(x,\theta)$. There are two cases:

1. If $\phi^*(\theta) < \phi^A(\theta)$, then by construction $x \leq \overline{\xi}(\theta) = \phi^*(\theta) < \phi^A(\theta)$ implying that $u(\phi^*(\theta),\theta) \geq u(x,\theta)$ by strict quasiconcavity of $u(\cdot,\theta)$, hence yielding a contradiction.
2. If $\phi^*(\theta) = \phi^A(\theta)$, then by construction $u(\phi^*(\theta),\theta) \geq u(x,\theta)$ yielding a contradiction.

Consider now the Incentive Compatibility constraint that for all $\theta \in \Theta$:

$$\overline{\xi}(\theta) \in \arg\max_{y\in X} \left\{ \max_{x\in[0,y]} u(x,\theta) - T^*(y) \right\} \tag{38}$$

and define $g(y,\theta) = \max_{x\in[0,y]} u(x,\theta)$. Toward a contradiction, suppose that there exist $\theta, \theta' \in \Theta$ such that $g(\overline{\xi}(\theta),\theta) - t(\theta) < g(\overline{\xi}(\theta'),\theta) - t(\theta')$. There are two cases to consider:

1. If $\phi^*(\theta') < \phi^A(\theta')$, then $\Xi_{\phi^*}(\theta') = \{\phi^*(\theta')\}$. Thus, $\overline{\xi}(\theta') = \phi^*(\theta')$. Hence:

$$g(\phi^*(\theta), \theta) - t(\theta) = g(\overline{\xi}(\theta), \theta) - t(\theta) < g(\overline{\xi}(\theta'), \theta) - t(\theta') = g(\phi^*(\theta'), \theta) - t(\theta') \tag{39}$$

   where the first equality follows by Obedience, the inequality follows by hypothesis, and the last equality follows as $\overline{\xi}(\theta') = \phi^*(\theta')$.

2. If $\phi^*(\theta') = \phi^A(\theta')$, then define $\theta'' = \inf\left\{\hat{\theta} \in \Theta : \hat{\theta} \geq \theta', \phi^*(\hat{\theta}) < \phi^A(\hat{\theta})\right\}$. Note that, by monotonicity of ϕ^* and by construction of $\overline{\xi}$, we have $\phi^*(\theta'') = \overline{\xi}(\theta'') = \overline{\xi}(\theta')$. Moreover, by construction we necessarily have that $[\theta', \theta''] \subseteq \left\{\hat{\theta} \in \Theta : \phi^*(\hat{\theta}) = \phi^A(\hat{\theta})\right\}$. Therefore, by Equation 26 in Lemma 1, we have that:

$$t(\theta'') - t(\theta') = \int_{\theta'}^{\theta''} u_x(\phi^A(s), s) \left(\phi^A\right)'(s) \, \mathrm{d}s = 0 \tag{40}$$

   by optimality of $\phi^A(s)$ for all $s \in [0, 1]$. Thus, $t(\theta') = t(\theta'')$ and we have that:

$$g(\phi^*(\theta), \theta) - t(\theta) = g(\overline{\xi}(\theta), \theta) - t(\theta) < g(\overline{\xi}(\theta'), \theta) - t(\theta') = g(\phi^*(\theta''), \theta) - t(\theta'') \tag{41}$$

   where the first equality is by Obedience, the inequality is by hypothesis and the second equality follows as $\phi^*(\theta'') = \overline{\xi}(\theta')$ and $t(\theta') = t(\theta'')$.

In both cases, there exists $\theta'' \in \Theta$ such that $g(\phi^*(\theta), \theta) - t(\theta) < g(\phi^*(\theta''), \theta) - t(\theta'')$ (in case 1, $\theta'' = \theta'$). This contradicts the fact that (ϕ^*, ϕ^*, T^*) is implementable, which we established in Lemma 2. Thus, Incentive Compatibility is satisfied.

Finally, consider the Individual Rationality constraint that $u(\phi^*(\theta), \theta) - T^*(\overline{\xi}(\theta)) \geq 0$ for all $\theta \in \Theta$. Observe that $T^*(\overline{\xi}(\theta)) = T^*(\phi^*(\theta))$ for all θ such that $\phi^*(\theta) < \phi^A(\theta)$. When $\phi^*(\theta) = \phi^A(\theta)$, we have that $T^*(\overline{\xi}(\theta)) - T^*(\phi^*(\theta)) = \int_{\phi^*(\theta)}^{\overline{\xi}(\theta)} u_x(z, {\phi^*}^{-1}(z)) \, \mathrm{d}z = 0$ as all types that consume between $\phi^*(\theta) = \phi^A(\theta)$ and $\overline{\xi}(\theta)$ consume their bliss point, by construction. Thus, $T^* \circ \overline{\xi} = T^* \circ \phi^*$ and by implementability of (ϕ^*, ϕ^*, T^*) (see Lemma 2), Individual Rationality holds.

This proves that $(\phi^*, \overline{\xi}, T^*)$ is implementable and therefore optimal. We now argue that for any other selection $\xi \in \Xi_{\phi^*}$, the triple (ϕ^*, ξ, T^*) is necessarily implementable and therefore optimal. Indeed, by way of contradiction, suppose that the latter is not implementable. It follows that $(\phi^*, \overline{\xi}, T^*)$ is not implementable either as all feasible deviations under purchase function ξ are still feasible under $\overline{\xi}$ and $T^* \circ \overline{\xi} = T^* \circ \xi$. However, this contradicts our demonstration that $(\phi^*, \overline{\xi}, T^*)$ is implementable.

We finally show that if $\xi \notin \Xi_{\phi^*}$, then it is not part of an optimal contract. We will use the observation that all agents' payments to the seller are pinned down by the envelope formula for t. There are two cases to consider. First, suppose that there exists a $\theta \in \Theta$ such that $\xi(\theta) \neq \phi^*(\theta)$ and $\phi^*(\theta) < \phi^A(\theta)$. If $\xi(\theta) < \phi^*(\theta)$, then $\phi^*(\theta) \notin [0, \xi(\theta)]$, which makes ϕ^* infeasible. If $\xi(\theta) > \phi^*(\theta)$, then, as $\phi^*(\theta) < \phi^A(\theta)$, $t(\theta)$ is strictly increasing at θ. Thus $T^*(\xi(\theta)) > t(\theta)$, which is a contradiction. Second, suppose that there exists a $\theta \in \Theta$ such that $\xi(\theta) \notin [\phi^A(\theta), \inf_{\theta' \in [\theta,1]} \{\phi^*(\theta') : \phi^*(\theta') < \phi^A(\theta')\}]$ and $\phi^*(\theta) = \phi^A(\theta)$. Once again if $\xi(\theta) < \phi^*(\theta)$, then $\phi^*(\theta) \notin [0, \xi(\theta)]$, which makes ϕ^* infeasible. If $\xi(\theta) > \inf_{\theta' \in [\theta,1]} \{\phi^*(\theta') : \phi^*(\theta') < \phi^A(\theta')\}$, then as before $T^*(\xi(\theta)) > t(\theta)$, which is a contradiction. $\square$

A.2 Proof of Proposition 2, Corollary 1, and Corollary 2

Proof of Proposition 2. We first prove that $H(x) > 0$ implies that $T^*(x)$ is flat at x, for any $x \in X^*$. By the definition of a multi-part tariff, this will also prove that T^* is a multi-part tariff. Consider first any $x \in \text{Int}(X^*)$, where $\text{Int}(\cdot)$ denotes the interior of a set. Observe that $\phi^* = \min\{\phi^P, \phi^A\}$ is invertible over $\text{Int}(X^*)$. Suppose now that $H(x) = J_x(x, (\phi^A)^{-1}(x)) > 0$ and define $\theta(x) = (\phi^*)^{-1}(x)$. It is either the case that $x = \overline{x}$ (which is not in $\text{Int}(X^*)$), or $\phi^A(\theta(x)) < \phi^P(\theta(x))$. Thus, when $H(x) > 0$, $\phi^A(\theta(x)) < \phi^P(\theta(x))$, so $\phi^*(\theta(x)) = \phi^A(\theta(x))$. As ϕ^A and ϕ^P are continuous functions by the Theorem of the Maximum and invertible at x, we can find a neighborhood $O(x)$ of x, such that for all $x' \in O(x)$, and corresponding $\theta' = (\phi^*)^{-1}(x')$, we have that $\phi^*(\theta') = \phi^A(\theta')$. To see that prices are constant on $O(x)$, take any two points $x_1, x_2 \in O(x)$, and observe that (by Equation 6 of Proposition 1):

$$T(x_1) - T(x_2) = \int_{x_2}^{x_1} u_x(z, \phi^{A^{-1}}(z)) \, \mathrm{d}z = 0 \tag{42}$$

by optimality of z for type $\phi^{A^{-1}}(z)$, which implies the necessary optimality condition, for all $z \in [x_2, x_1]$, $u_x(z, \phi^{A^{-1}}(z)) = 0$. It remains to consider all $x \notin \text{Int}(X^*)$. Continuity of H implies the result for the boundary points of X^*.[28] Thus, we have shown that, if $H(x) > 0$, then there exists a neighborhood of x such that prices are constant, proving the claim.

We now prove that, for every $x \in X^*$, if T^* is a multi-part tariff that is flat at x, then $H(x) \geq 0$. First, consider $x \in \text{Int}(X^*)$. If T is flat at x, then there exists a neighborhood $O(x)$ such that for all $x_1, x_2 \in O(x)$, we have that $T(x_1) - T(x_2) = 0$. Thus, by Equation 6 of Proposition 1, we have that $\int_{x_2}^{x_1} u_x(z, \phi^{*^{-1}}(z)) \, \mathrm{d}z = 0$ for all $x_1, x_2 \in O(x)$. Thus, we have that $u_x(z, \phi^{*^{-1}}(z)) = 0$ (as $u_x(z, \phi^{*^{-1}}(z)) \geq 0$ by Obedience) for almost all $z \in O(x)$. By strict quasiconcavity of u, this implies that $\phi^{*^{-1}}(z) = \phi^{A^{-1}}(z)$ for almost all

[28] A neighborhood at $\max X^*$ is of the form $(\max X^* - \varepsilon, \max X^*]$ for some $\varepsilon > 0$, and at $\min X^*$ of the form $[\min X^*, \min X^* + \varepsilon)$.

$z \in O(x)$. Toward a contradiction, suppose that $H(x) < 0$. By continuity of H, there exists a neighborhood $O'(x) \subseteq O(x)$ such that ${\phi^*}^{-1}(z) = {\phi^P}^{-1}(z) < {\phi^A}^{-1}(z)$ for all $z \in O'(x)$. But we have already shown that ${\phi^*}^{-1}(z) = {\phi^A}^{-1}(z)$ for almost all $z \in O(x)$. This is a contradiction, and so $H(x) \geq 0$. It remains to consider all $x \notin \text{Int}(X^*)$. As before, continuity of H implies the result for the boundary points of X^*.[29] $\square$

Proof of Corollary 1. Immediate from Proposition 2 and the pricing-scheme definitions. $\square$

Proof of Corollary 2. By Proposition 2, if $H(x) > 0$ at $\phi^A(1)$, then T^* is flat at $\phi^*(1) = \phi^A(1)$. Moreover, at $x = \phi^A(1)$, we have $H(\phi^A(1)) = \pi_x(\phi^A(1), 1)$. It follows that, when $\pi_x(\phi^A(1), 1) > 0$, $H(\phi^A(1)) > 0$ and T^* features an unlimited subscription. Likewise, if $H(x) > 0$ at $\phi^A(0)$, then T^* is flat at $\phi^*(0) = \phi^A(0)$. Moreover, at $x = \phi^A(0)$, we have $H(\phi^A(0)) = \pi_x(\phi^A(0), 0) - \frac{1}{f(0)} u_{x\theta}(\phi^A(0), 0)$. It follows that, when $f(0)\pi_x(\phi^A(0), 0) - u_{x\theta}(\phi^A(0), 0) > 0$, $H(\phi^A(0)) > 0$ and T^* features a trial. $\square$

A.3 Proofs of Propositions 3 and 4

Proof of Proposition 3. We first prove that $V(\theta; T) \geq V_N(\theta; T)$, for all $\theta \in \Theta$. We compare the values with and without free disposal to each type $\theta \in \Theta$ under any T:

$$V(\theta; T) = \sup_{y \in X, x \in [0, y]} \{u(x, \theta) - T(y)\} \geq \sup_{y \in X} \{u(y, \theta) - T(y)\} = V_N(\theta; T) \tag{43}$$

because any payoff in the problem on the right of the inequality is attainable in the problem on the left of the inequality.

We now show that $V^*(\theta) \leq V_N^*(\theta)$, for all $\theta \in \Theta$. Without free disposal, the optimal allocation is $\phi_N^*(\theta) = \phi^P(\theta)$. With free disposal, the optimal allocation is $\phi^*(\theta) = \min\{\phi^A(\theta), \phi^P(\theta)\}$. It follows that $\phi^*(\theta) \leq \phi_N^*(\theta)$ for all $\theta \in \Theta$. Using the formula for agent welfare under the optimal mechanism (see Equation 29), we can then see that:

$$V^*(\theta) = \int_0^\theta u_\theta(\phi^*(s), s)\, \mathrm{d}s \leq \int_0^\theta u_\theta(\phi_N^*(s), s)\, \mathrm{d}s = V_N^*(\theta) \tag{44}$$

for all $\theta \in \Theta$, where the inequality follows as u is strictly single-crossing in (x, θ) and $\phi^* \leq \phi_N^*$.

For the seller, by Proposition 1, we have that for all $\theta \in \Theta$:

$$\Pi^*(\theta) = \max_{x \in [0, \phi^A(\theta)]} J(x, \theta) \leq \max_{x \in X} J(x, \theta) = \Pi_N^*(\theta) \tag{45}$$

[29] A careful reader may ask why it is not true that T^* being flat at $x \in X^*$ implies $H(x) > 0$. Toward a counter-example to this claim, suppose that $\phi^P \equiv \phi^A$. We have that T^* is flat everywhere but $H(x) \equiv 0$.

The inequality follows because the problem without free disposal allows for more choices of $x \in X$. $\square$

Proof of Proposition 4. We first show how J and ϕ^P change when (i) π changes to $\tilde{\pi}$ such that $\tilde{\pi}_x \geq \pi_x$ and (ii) F changes to $\tilde{F}$ such that F dominates $\tilde{F}$ in the hazard-rate order. Observe that $J(\cdot, \theta)$ increases pointwise for all $\theta \in \Theta$ as we may write (noting that $J(0, \theta) = 0$ by the properties that $u(0, \theta) \equiv \pi(0, \theta) \equiv 0$):

$$J(x, \theta) = \int_0^x \left[\pi_x(z, \theta) + u_x(z, \theta) - \frac{1 - F(\theta)}{f(\theta)} u_{x\theta}(z, \theta) \right] \mathrm{d}z \tag{46}$$

and see that the integrand, $J_x(z, \theta)$, increases pointwise under (i) and (ii). As J_x increases pointwise and J is strictly quasiconcave, we moreover have that ϕ^P increases pointwise while ϕ^A remains unchanged. Let ϕ^P, J, V^*, and Π^* be evaluated at the original π and/or F, and $\tilde{\phi}^P$, $\tilde{J}$, $\tilde{V}^*$, and $\tilde{\Pi}^*$ be the same objects evaluated at the new $\tilde{\pi}$ and/or $\tilde{F}$.

We first study consumer welfare and establish that $\tilde{V}^* \geq V^*$. See that (by Equation 29):

$$\tilde{V}^*(\theta) = \int_0^\theta u_\theta(\tilde{\phi}^*(s), s) \, \mathrm{d}s \geq \int_0^\theta u_\theta(\phi^*(s), s) \, \mathrm{d}s = V^*(\theta) \tag{47}$$

for all $\theta \in \Theta$, where the inequality follows as u is strictly single-crossing in (x, θ) and $\tilde{\phi}^* = \min\{\tilde{\phi}^P, \phi^A\} \geq \min\{\phi^P, \phi^A\} = \phi^*$. Showing that $\tilde{V}_N^* - V_N^* \geq \tilde{V}^* - V^*$ is equivalent to showing that $\tilde{V}_N^* - \tilde{V}^* \geq V_N^* - V^*$, or (by Equation 29):

$$\int_0^\theta \left[\left(u_\theta(\tilde{\phi}_N^*(s), s) - u_\theta(\tilde{\phi}^*(s), s) \right) - \left(u_\theta(\phi_N^*(s), s) - u_\theta(\phi^*(s), s) \right) \right] \mathrm{d}s \geq 0 \, , \forall \theta \in \Theta \tag{48}$$

There are three possible cases for each $s \in \Theta$ to compute the integrand:

i $\phi^P(s) < \phi^A(s)$ and $\tilde{\phi}^P(s) < \phi^A(s)$. Hence: $\phi^*(s) = \phi^P(s) = \phi_N^*(s)$ and $\tilde{\phi}^*(s) = \tilde{\phi}^P(s) = \tilde{\phi}_N^*(s)$. In this case, the value of the integrand is zero.

ii $\phi^P(s) < \phi^A(s)$ and $\tilde{\phi}^P(s) \geq \phi^A(s)$. Hence: $\phi^*(s) = \phi^P(s) = \phi_N^*(s)$ and $\tilde{\phi}^*(s) = \phi^A(s)$. Thus, the integrand is $u_\theta(\tilde{\phi}^P(s), s) - u_\theta(\phi^A(s), s) \geq 0$ by strict single-crossing of u.

iii $\phi^P(s) \geq \phi^A(s)$ and $\tilde{\phi}^P(s) \geq \phi^A(s)$, so $\phi^*(s) = \phi^A(s)$ and $\tilde{\phi}^*(s) = \phi^A(s)$. Thus, the value of the integrand is $u_\theta(\tilde{\phi}^P(s), s) - u_\theta(\phi^P(s), s) \geq 0$ by strict single-crossing of u.

Thus, the integrand is positive for all $s \in \Theta$ and the claimed inequality holds.

We now study producer welfare and establish that $\tilde{\Pi}^* \geq \Pi^*$. By Proposition 1, we have:

$$\tilde{\Pi}^*(\theta) = \tilde{J}(\tilde{\phi}^*(\theta), \theta) \geq \tilde{J}(\phi^*(\theta), \theta) \geq J(\phi^*(\theta), \theta) = \Pi^*(\theta) \tag{49}$$

where the first inequality is by feasibility of ϕ^* before and after the change (as ϕ^A is unchanged), and the second inequality follows as $\tilde{J} \geq J$. Showing that $\tilde{\Pi}^*_N(\theta) - \Pi^*_N(\theta) \geq \tilde{\Pi}^*(\theta) - \Pi^*(\theta)$ is equivalent to showing that $\tilde{\Pi}^*_N(\theta) - \tilde{\Pi}^*(\theta) \geq \Pi^*_N(\theta) - \Pi^*(\theta)$, or:

$$\left(\tilde{J}(\tilde{\phi}^*_N(\theta),\theta) - \tilde{J}(\tilde{\phi}^*(\theta),\theta)\right) - \left(J(\phi^*_N(\theta),\theta) - J(\phi^*(\theta),\theta)\right) \geq 0\,, \forall\theta \in \Theta \tag{50}$$

We have that $\phi^*_N(\theta) = \phi^P(\theta)$ and $\tilde{\phi}^*_N(\theta) = \tilde{\phi}^P(\theta)$, and there are three cases for each $\theta \in \Theta$:

i $\phi^P(\theta) < \phi^A(\theta)$ and $\tilde{\phi}^P(\theta) < \phi^A(\theta)$, so $\phi^*(\theta) = \phi^P(\theta) = \phi^*_N(\theta)$ and $\tilde{\phi}^*(\theta) = \tilde{\phi}^P(\theta) = \tilde{\phi}^*_N(\theta)$. In this case, the value of the expression is zero.

ii $\phi^P(\theta) < \phi^A(\theta)$ and $\tilde{\phi}^P(\theta) \geq \phi^A(\theta)$, so $\phi^*(\theta) = \phi^P(\theta) = \phi^*_N(\theta)$ and $\tilde{\phi}^*(\theta) = \phi^A(\theta)$. In this case, the value of the expression is $\tilde{J}(\tilde{\phi}^P(\theta),\theta) - \tilde{J}(\phi^A(\theta),\theta) \geq 0$ as $\tilde{\phi}^P$ is maximal for $\tilde{J}$.

iii $\phi^P(\theta) \geq \phi^A(\theta)$ and $\tilde{\phi}^P(\theta) \geq \phi^A(\theta)$, so $\phi^*(\theta) = \phi^A(\theta)$ and $\tilde{\phi}^*(\theta) = \phi^A(\theta)$. In this case, the value of the expression is $\left(\tilde{J}(\tilde{\phi}^P(\theta),\theta) - \tilde{J}(\phi^A(\theta),\theta)\right) - \left(J(\phi^P(\theta),\theta) - J(\phi^A(\theta),\theta)\right)$, and we wish to show that this is positive. Now observe that we can write this inequality as:

$$\int_{\phi^P(\theta)}^{\tilde{\phi}^P(\theta)} \tilde{J}_x(z,\theta)\,\mathrm{d}z + \int_{\phi^A(\theta)}^{\phi^P(\theta)} \left(\tilde{J}_x(z,\theta) - J_x(z,\theta)\right)\mathrm{d}z \geq 0 \tag{51}$$

As $\tilde{J}$ is strictly quasiconcave and $\tilde{\phi}^P$ is $\tilde{J}$ maximal, we know that $\int_{\phi^P(\theta)}^{\tilde{\phi}^P(\theta)} \tilde{J}_x(z,\theta)\,\mathrm{d}z \geq 0$. Moreover, as $\tilde{J}_x \geq J_x$, we have that $\int_{\phi^A(\theta)}^{\phi^P(\theta)} \left(\tilde{J}_x(z,\theta) - J_x(z,\theta)\right)\mathrm{d}z \geq 0$. The claimed inequality follows.

Thus, the expression in (50) is positive for all $\theta \in \Theta$ and the claimed inequality follows. □

B Additional Results

B.1 Optimal Bunching and Free Disposal

This appendix extends our main analysis to cover cases in which the virtual surplus function J does not satisfy single-crossing and thereby allows for the possibility that multiple buyer types bunch on the same level of consumption. We apply techniques from Nöldeke and Samuelson (2007) to study the inverse problem of assigning types to consumption. For this reason, we make make the additional assumptions that J is concave and that both π_{xx} and $u_{xx\theta}$ exist and are continuous.

Denote an inverse consumption function by $\psi : X \to \Theta$. This corresponds to an inverse of the standard consumption function ϕ. For any monotone ψ, define the correspondence:

$$\Psi(x) = \left[\lim_{y \to^- x} \psi(y), \lim_{y \to^+ x} \psi(y) \right] \tag{52}$$

which "fills in" discontinuities in the inverse consumption function.[30] Moreover, define the generalized inverse of ϕ^A as $\left(\phi^A\right)^{[-1]}(x) = \min\left\{\theta \in [0,1] : \phi^A(\theta) = x\right\}$. Our first result concerns implementation in this setting.

Lemma 4. *A consumption function ϕ is implementable and supported by (ϕ, T) if and only if there exists a monotone inverse consumption $\psi : X \to \Theta$ such that $\psi(x) \geq \left(\phi^A\right)^{[-1]}(x)$ for all $x \in X$, $\theta \in \Psi(\phi(\theta))$ for all $\theta \in \Theta$, and $T(x) = C + \int_0^x u_x(z, \psi(z))\,\mathrm{d}z$ with $C \leq 0$.*

Proof. By construction, (T, ψ) are consistent as defined in Equation 5 in Nöldeke and Samuelson (2007). Moreover, $\phi \leq \phi^A$ if and only if $\psi \geq (\phi^A)^{[-1]}$. Therefore, the statement immediately follows from Lemma 1 and Lemma 2 in Nöldeke and Samuelson (2007) and the proof of Lemma 2 in this paper. □

We now provide the solution to the seller's screening problem. Toward simplifying the seller's problem, we define the following function:

$$\hat{J}(x, \theta) = u_x(x, \theta)(1 - F(\theta)) + \int_\theta^1 \pi_x(x, s)\,\mathrm{d}F(s) \tag{54}$$

Using this function as well as Lemma 4 in this paper and Remark 1 and Lemma 5 in Nöldeke and Samuelson (2007), we can re-express the seller's problem as:

$$\begin{aligned} \max_{\psi} \quad & \int_0^{\overline{x}} \hat{J}(x, \psi(x))\,\mathrm{d}x \\ \text{s.t.} \quad & \psi(x') \geq \psi(x),\ \psi(x) \geq \left(\phi^A\right)^{[-1]}(x),\ \forall x', x \in X : x' \geq x \end{aligned} \tag{55}$$

The following result solves this problem and uses the solution to solve Problem 1.

Proposition 5. *Problem 55 is solved by the inverse consumption $\psi^* : X \to \Theta$ given by*

$$\psi^*(x) = \max\left\{\arg\max_{\theta \in [(\phi^A)^{[-1]}(x), 1]} \hat{J}(x, \theta)\right\} \tag{56}$$

[30] Where we follow the convention from Nöldeke and Samuelson (2007) that:

$$\lim_{y \to^- 0} \psi(y) = 0, \quad \lim_{y \to^+ \overline{x}} \psi(y) = 1 \tag{53}$$

Moreover, Problem 1 is solved by $\phi^*(\theta) = \inf\{x \in X : \psi^*(x) \geq \theta\}$ *for all* $\theta \in \Theta$.

Proof. We first show that $\hat{J}$ is supermodular. We follow Lemma 6 in Nöldeke and Samuelson (2007) and observe that the cross partial derivative of $\hat{J}$ is:

$$\hat{J}_{x\theta}(x,\theta) = -\left[u_{xx}(x,\theta) + \pi_{xx}(x,\theta)\right] f(\theta) + \left[1 - F(\theta)\right] u_{xx\theta}(x,\theta) = -J_{xx}(x,\theta) f(\theta) \geq 0 \tag{57}$$

where the last inequality uses the concavity of J and $f > 0$. Next, we argue that the correspondence $x \mapsto [(\phi^A)^{[-1]}(x), 1]$ is monotone in the strong set order (SSO). This immediately follows from the fact that $(\phi^A)^{[-1]}$ is increasing. We then apply Theorem 4' in Milgrom and Shannon (1994) to argue that ψ^* is monotone. Since $\psi^* \geq (\phi^A)^{[-1]}$, the inverse consumption function ψ^* is implementable and therefore optimal in Problem 55.

We now prove the optimality of ϕ^* in Problem 1. Given that ψ^* is monotone and such that $\psi^* \geq (\phi^A)^{[-1]}$, it follows that ϕ^* is also monotone and such that $\phi^* \leq \phi^A$. Hence, by Lemma 2, it is implementable. Next, suppose there exists an implementable consumption function ϕ such that $\int_0^1 J(\phi(\theta),\theta)\,\mathrm{d}F(\theta) > \int_0^1 J(\phi^*(\theta),\theta)\,\mathrm{d}F(\theta)$. Given that ϕ is implementable, there exist (ξ, T) that support it. By the proof of Lemma 1 in Nöldeke and Samuelson (2007) it follows that there exists an inverse consumption function ψ such that $T(x) = T(0) + \int_0^x u(z,\psi(z))\,\mathrm{d}z$. In turn, Lemma 3 in Nöldeke and Samuelson (2007) implies that

$$\int_0^{\overline{x}} \hat{J}(x,\psi^*(x))\,\mathrm{d}x = \int_0^1 J(\phi^*(\theta),\theta)\,\mathrm{d}F(\theta) < \int_0^1 J(\phi(\theta),\theta)\,\mathrm{d}F(\theta) = \int_0^{\overline{x}} \hat{J}(x,\psi(x))\,\mathrm{d}x \tag{58}$$

contradicting the optimality of ψ^* in Problem 55. Therefore, ϕ^* solves Problem 1. □

As in Nöldeke and Samuelson (2007), bunching manifests in the solution to this problem as a discontinuity in the resulting inverse consumption function ψ. In particular, whenever ψ is discontinuous the outcome at the discontinuity is assigned to a positive measure of types.

As explained in Remark 2, these bunching regions in the type space do not generate flat regions of the optimal price schedule. Similarly to Proposition 2, we can fully characterize the regions where T^* is flat. These are the quantities x where the local constraint $\theta \in [(\phi^A)^{[-1]}(x), 1]$ in (56) binds at the optimum. However, here we do not assume strict concavity of $\hat{J}$ since this would be equivalent to assuming strict supermodularity of J. Therefore, we cannot rely on first-order conditions to replicate the local characterization of Proposition 2. We conclude with an example in which the optimal contract features both bunching and a multi-part tariff:

Example 4. Consumer preferences, the outcome space, and the external revenue function

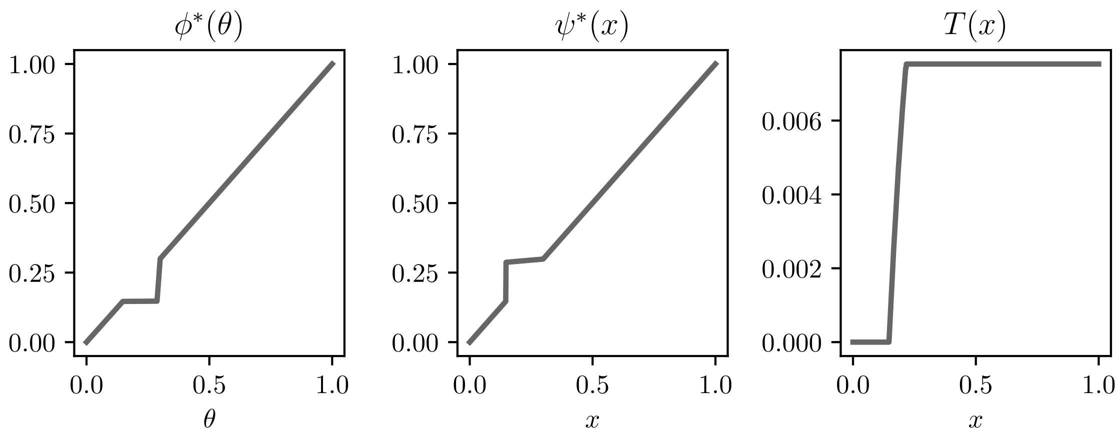

Figure 6: Multi-part tariff with bunching in Example 4.

are identical to those in Example 1. The type distribution has density

$$f(\theta) = 1 + \frac{k}{2\pi\omega}(\cos(2\pi\omega) - 1) + k\sin(2\pi\omega\theta) \tag{59}$$

for some $k > 0$ and $\omega > 0$. We solve the example for $\alpha = 1$, $\beta = 0$, $k = \frac{1}{2}$, and $\omega = 3$. In Figure 6, we plot $\phi^*(\theta)$, $\psi^*(x)$, and $T(x)$ in the optimal contract. In the price schedule, there is both an unlimited subscription and a free trial. A mass of types, approximately between 0.15 and 0.29, is bunched at the allocation $\phi^* = 0.15$. These types all consume the maximum amount possible in the free trial. Anecdotally, this is a common occurrence for free trials in practice (e.g., the free allotment of online *Wall Street Journal* articles).

△

B.2 Competition and Free Disposal

In this appendix, we study the relationship between competition and optimal pricing under free disposal. We do this by comparing our monopoly screening benchmark with one model of perfect competition. We show that our results are robust to this extension by demonstrating that zero marginal pricing is in fact more prevalent under perfect competition.

The nature of perfect competition we consider is that our monopolist faces a perfectly competitive fringe of firms that can enter and displace them to serve the entire market. In this case (as in, e.g., Grubb, 2009), the equilibrium contract maximizes expected consumer surplus subject to our usual implementation constraints and a new constraint that the monopolist actually wishes to serve the market. That is, the screening problem becomes:

$$\begin{aligned} \sup_{\phi,\xi,T} \quad & \int_\Theta (u(\phi(\theta),\theta) - T(\xi(\theta)))\,\mathrm{d}F(\theta) \\ \text{s.t.} \quad & \text{(O),(IC),(IR)} \\ & \int_\Theta (\pi(\phi(\theta),\theta) + T(\xi(\theta)))\,\mathrm{d}F(\theta) \geq 0 \end{aligned} \tag{60}$$

The last constraint, which we call "Monopolist's IR," encodes the requirement that the monopolist wishes to serve the market compared to an outside option of earning nothing.

Toward characterizing the solution of this problem, define the total surplus function as $S(x,\theta) = \pi(x,\theta) + u(x,\theta)$. In analogy to our assumptions that J is strictly single-crossing and strictly quasiconcave, we assume that S is strictly single-crossing in (x,θ) and strictly quasiconcave in x. We further define the total surplus maximizing consumption level as $\phi^O(\theta) = \arg\max_{x\in X} S(x,\theta)$.

Proposition 6. *The equilibrium consumption under perfect competition is* $\phi^{PC} = \min\{\phi^A, \phi^O\}$.

Proof. As in the proof of Lemma 3, we have that agents' transfers under any locally incentive compatible menu are given by Equation 23 for some $C \in \mathbb{R}$. We can therefore rewrite the objective (using the same integration-by-parts argument as Lemma 3) as:

$$-C + \int_\Theta \frac{1-F(\theta)}{f(\theta)} u_\theta(\phi(\theta),\theta)\,\mathrm{d}F(\theta) \tag{61}$$

By integrating over types, we can then express the monopolist's IR constraint as:

$$\int_\Theta \left(\pi(\phi(\theta),\theta) + u(\phi(\theta),\theta) - \frac{1-F(\theta)}{f(\theta)} u_\theta(\phi(\theta),\theta)\right) \mathrm{d}F(\theta) + C \geq 0 \tag{62}$$

Thus, the optimal C sets this inequality tight. Substituting, we obtain that the objective function becomes $\int_\Theta S(\phi(\theta),\theta)\,\mathrm{d}F(\theta)$. Moreover, by the same arguments as in Lemma 2, the remaining implementation constraints are that $\phi(\theta) \leq \phi^A(\theta)$ for all $\theta \in \Theta$, ϕ is monotone increasing and $u(\phi(0),0) - t(0) \geq 0$. By identical arguments to Proposition 1 (as S is strictly single-crossing and quasiconcave), it follows that the optimal consumption levels satisfy $\phi^{PC}(\theta) = \min\{\phi^A(\theta), \phi^O(\theta)\}$, which is monotone. Moreover, $t(0) = C + u(\phi^{PC}(0),0) \leq 0$ as C is negative and $u(\phi^{PC}(0),0) \geq 0$ as $\phi^{PC}(0) \in [0, \phi^A(0)]$. □

We now show that zero marginal pricing is more prevalent under perfect competition:

Corollary 3. *The set of outcomes at which there is flat pricing under perfect competition includes the set of outcomes at which there is flat pricing under monopoly pricing.*

Proof. Define $H^{PC}(x) = S_x\left(x, \left(\phi^A\right)^{-1}(x)\right)$ for all $x \in X^*$ and observe that:

$$
\begin{aligned}
H^{PC}(x) &= S_x\left(x, \left(\phi^A\right)^{-1}(x)\right) = u_x\left(x, \left(\phi^A\right)^{-1}(x)\right) + \pi_x\left(x, \left(\phi^A\right)^{-1}(x)\right) \\
&\geq u_x\left(x, \left(\phi^A\right)^{-1}(x)\right) + \pi_x\left(x, \left(\phi^A\right)^{-1}(x)\right) - \frac{1 - F\left(\left(\phi^A\right)^{-1}(x)\right)}{f\left(\left(\phi^A\right)^{-1}(x)\right)} u_{x\theta}\left(x, \left(\phi^A\right)^{-1}(x)\right) \\
&= J_x\left(x, \left(\phi^A\right)^{-1}(x)\right) = H(x)
\end{aligned}
\tag{63}
$$

Thus, $H(x) \geq 0 \implies H^{PC}(x) \geq 0$. Hence, by an identical argument to Proposition 2, whenever T^* is flat, so is T^{PC}. □

The intuition for this result is that there are no quantity distortions from information rents under the competitive solution. Thus, total-surplus-maximizing consumption is greater than virtual-surplus-maximizing consumption, and the constraint $\phi \leq \phi^A$ binds more often.

C Microfoundations of Revenue from Usage

C.1 Network Effects from Platform Externalities

Sellers may value usage because it makes the platform more valuable for other end users. That is, usage generates network effects. Examples include networking services (e.g., LinkedIn), matching services (e.g., Tinder, Match.com, or OK Cupid), online games (e.g., Fortnite, Candy Crush Saga, or World of Warcraft), and content-streaming platforms with social rating systems (e.g., Hulu or Netflix).

The function $W : X \times \Theta \to \mathbb{R}_+$ maps each agent's consumption to a positive externality for every agent. Agents' payoffs if they participate, given a consumption function ϕ, are:

$$
v\left(x, \theta, (\phi(s))_{s\in[0,1]}\right) = u(x, \theta) + \int_0^1 W(\phi(s), s)\,\mathrm{d}F(s)
\tag{64}
$$

with the maintained assumption of a zero outside option otherwise. The rest of the model is as in Section I. The externality of others' usage is obtained by an agent whenever they use the platform at the extensive margin. This makes the model amenable to settings where an agent may gain from participating, even if they do not regularly use the platform. For example, having a LinkedIn profile may generate the "passive" benefit of being findable by job recruiters, even if the user spends essentially zero time using the website. In analogy to the main analysis, we assume that the modified virtual surplus function

$$J^{\dagger}(x,\theta) = \pi(x,\theta) + u(x,\theta) + W(x,\theta) - \frac{1-F(\theta)}{f(\theta)} u_\theta(x,\theta) \tag{65}$$

is strictly quasiconcave in x and strictly single-crossing in (x,θ). We now show how this setting maps to our baseline setting of Section I.

Lemma 5. *Optimal consumption is given by* $\phi^*(\theta) = \min\{\phi^A(\theta), \phi^P(\theta)\}$, *where* $\phi^A(\theta) = \arg\max_{x\in X} u(x,\theta)$ *and* $\phi^P(\theta) = \arg\max_{x\in X} J^{\dagger}(x,\theta)$.

Proof. Observe first that the externality cannot affect the Obedience or Incentive Compatibility constraints since it has no dependence on consumer choice. The Individual Rationality constraint becomes $v\left(\phi(\theta), \theta, (\phi(s))_{s\in[0,1]}\right) \geq 0$. The same arguments from the proof of Lemma 3 imply that, without loss of optimality, we can restrict attention to allocations in which all agents participate (as $W \geq 0$), but now where $C = \int_\Theta W(\phi(\theta),\theta)\,\mathrm{d}F(\theta)$. Thus, by Equation 36, the objective of the monopolist is now $\int_\Theta J^{\dagger}(x,\theta)\,\mathrm{d}F(\theta)$ and the constraints are the same as those in Equation 34. The result then follows by application of the arguments in the proof of Proposition 1. □

Intuitively, since the externality is excludable, or not available to agents that do not participate in the mechanism, the seller can extract the full value of the externality as part of a "participation fee." Thus, each agent's marginal contribution to the externality, $W(x,\theta)$, is "as if" additional usage-derived revenue.

C.2 Irrational Addiction

Addicted users are commonly cited as a major source of revenue for digital goods (see, e.g., Allcott, Gentzkow and Song, 2022). In this appendix, we describe a simple microfoundation of how external revenue could be derived from irrational addiction of consumers.

Suppose that agents live for two periods but are myopic. Let $x \in X$ be the agent's consumption today ($t = 0$) and $\tilde{x} \in X$ their consumption tomorrow ($t = 1$). An agent of type $\theta \in \Theta$ believes they have lifetime payoff from consumption x given by $u(x,\theta)$, where u satisfies our running assumptions. In reality, however, the agent also values consumption tomorrow. Moreover, the more (or less) that they consumed today the more (or less) they value consumption tomorrow. Thus, at $t = 1$, the agent has utility function $\tilde{u} : X^2 \times \Theta \to \mathbb{R}$, where $u(x,\tilde{x},\theta)$ is their payoff. This *complete myopia* can be thought of as an extreme form of the inattention toward habit formation that Allcott, Gentzkow and Song (2022) find is necessary to empirically rationalize the total demand for six ubiquitous mobile apps (Facebook, Instagram, Twitter, Snapchat, web browsers, and YouTube).

Observe that given a full-revelation mechanism (or equivalently under observation of agent consumption under an implementable mechanism), the seller knows the agent's type tomorrow. Thus, when agents consume x today and their type is θ, tomorrow the monopolist sells them $\tilde{x}^*(x,\theta) \in \arg\max_{\tilde{x}\in X} \tilde{u}(x,\tilde{x},\theta)$ and charges a transfer of $\pi(x,\theta) = \tilde{u}(x,\tilde{x}^*(x,\theta),\theta)$ to extract full surplus. Thus, from the perspective of today, the monopolist faces the non-linear pricing problem we study in the main text, with an external revenue function π that captures the gains from addicting a user through contemporaneous consumption and extracting this surplus from them in the future.

C.3 Overconfidence

A natural reason why a seller may allocate more of a good than an agent wants *ex post* is that the agent expected to want something different *ex ante*. This story is at the heart of Grubb (2009)'s analysis of selling to overconfident consumers and his leading example of pricing cell phone plans, a context in which individuals regularly (based on anecdotes and empirical exploration) underestimate the variance of their future demand (see also Grubb and Osborne, 2015). We now illustrate how over-confidence at the participation stage can be mapped to our framework as a particular external revenue function.

The Grubb (2009) model is a monopoly pricing model, with continuous, increasing, and convex production costs $K(x)$ and no additional revenue from usage. The twist relative to the standard model is that agents decide whether to participate *ex ante* without knowing their type θ, but with a prior belief $\theta \sim \check{F}$ which may differ from the objective truth $\theta \sim F$ (see Grubb (2009) for the full details of the model). The common individual rationality constraint for all consumers is that the expected payoff at the allocation $(\phi(\theta), \xi(\theta), t(\theta))_{\theta\in\Theta}$ exceeds the outside option 0, or

$$(66) \qquad \int_0^1 (u(\phi(\theta),\theta) - t(\theta))\,\mathrm{d}\check{F}(\theta) \geq 0$$

We derive the following mapping of the Grubb (2009) model into ours:

Lemma 6. *The optimal consumption in the monopoly problem of Grubb (2009) is equal to the consumption that solves Problem 1, with* $\pi(x,\theta) = \frac{1-\check{F}(\theta)}{f(\theta)} u_\theta(x,\theta) - K(x)$

Proof. This follows immediately from our Lemma 3 and Proposition 1 in Grubb (2009). □

Observe first that, in a classical model with correctly specified expectations $\check{F} = F$, the first term in π cancels with the information rents and the Obedience constraint is always slack in the optimum. With mis-specified $\check{F} \neq F$, the first term and information rents do not cancel. When the first term stemming from overconfidence dominates both production costs and information rents on the margin at $x \in X$, the model generates $H(x) > 0$ and multi-part tariffs. Grubb (2009) applies this model to understand the occurrence of trial tiers (in our language) in cell phone pricing.

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