Adaptive Priority Mechanisms

Oğuzhan Çelebi*　　Joel P. Flynn†
Stanford　　Yale

April 10, 2024

Abstract

How should authorities that care about match quality and diversity allocate resources when they are uncertain about the market? We introduce *adaptive priority mechanisms* (APM) that prioritize agents based on their scores and characteristics. We derive an APM that is optimal and show that the ubiquitous priority and quota mechanisms are optimal if and *only if* the authority is risk-neutral or extremely risk-averse over diversity, respectively. Deferred Acceptance implements the unique stable matching when all authorities use the optimal APM. We provide a practical roadmap for implementing APM as a market-design solution and illustrate this using Chicago Public Schools data.

*Stanford University Department of Economics, 579 Jane Stanford Way, Stanford, CA 94305. Email: ocelebi@stanford.edu

†Yale University Department of Economics, 30 Hillhouse Avenue, New Haven, CT, 06511. Email: joel.flynn@yale.edu

We are grateful to Daron Acemoglu, Mohammad Akbarpour, George-Marios Angeletos, Nick Arnosti, Jonathan Cohen, Viola Corradini, Roberto Corrao, Mert Demirer, Glenn Ellison, Aytek Erdil, Arda Gitmez, Marina Halac, Ömer Karaduman, Stephen Morris, Anh Nguyen, Parag Pathak, Charlie Rafkin, Karthik Sastry, Tayfun Sönmez, Bertan Turhan, Alexander Wolitzky, Bumin Yenmez and seminar participants at the 2022 INFORMS Workshop on Market Design, the 2022 Society for the Advancement of Economic Theory Conference, UC Berkeley, Iowa State University and the MIT Theory Lunch for helpful comments. We thank Chicago Public Schools for graciously sharing their data and Eryn Heying, Talia Gerstle and Jim Shen for invaluable administrative support. The views expressed here are those of the authors and do not reflect the views of Chicago Public Schools.

1 Introduction

Authorities that allocate resources such as school seats, university courses, refugee visas, and medical supplies often face conflicting objectives. On the one hand, they want to maximize match quality or fairness by allocating resources to the highest-scoring agents according to various criteria such as academic attainment, qualifications (Kornbluth and Kushnir, 2021), likelihood of gaining employment (Ahani, Andersson, Martinello, Teytelboym, and Trapp, 2021; Delacrétaz, Kominers, and Teytelboym, 2023), mortality risk (Pathak, Sönmez, Unver, and Yenmez, 2021), or distance (Dur, Kominers, Pathak, and Sönmez, 2018). On the other hand, they want to achieve diversity across socioeconomic attributes including race, religion, and gender (Aygün and Turhan, 2017; Dur, Pathak, and Sönmez, 2020; Sönmez and Yenmez, 2022a). Resolving this conflict is complicated, especially in new markets, due to uncertainty about the distribution of individuals' scores, characteristics, and preferences (Doğan and Erdil, 2022).[1]

To balance these trade-offs, when the use of prices is seen as infeasible or unethical, authorities have broadly used two classes of policies: *quotas*,[2] where a certain portion of the resource is set aside for given groups; and *priorities*, where individuals in given groups receive higher scores. These policies have been applied across many different markets in many different countries, for example: the Indian government reserves some government jobs for disadvantaged groups; Chicago Public Schools employs quotas for students from different socioeconomic groups at its competitive exam schools; many countries gave differential priority to healthcare workers in the receipt of Covid-19 vaccines; and the University of Michigan and the University of Texas have used different priority scales for minority students.

But what mechanism *should* such an authority use? Despite the practical importance of this question, we do not know if an authority should use a priority mechanism, a quota mechanism, or something else entirely. In this paper, we formulate and solve the optimal mechanism design problem of an authority that allocates a resource to agents who are heterogeneous in their scores and belong to different groups. The authority cares about the *scores* and *diversity* of the agents who are assigned the resource.[3] Moreover, they are uncertain

[1] Emphasizing the importance of uncertainty in the context of U.K. university admissions, Doğan and Erdil (2022) write: "[V]olatile trends of demand for different courses mean even meeting intake targets can be a difficult juggling act for universities."

[2] We use *quota* as a general term that includes the widely used reserve policies (see Definition 4).

[3] This diversity preference can be interpreted more generally as encoding a preference of the authority over the composition of assigned agents across a range of attributes. Moreover, when scores represent individuals' property rights over objects, we can interpret the preference for higher scores as a preference for procedural fairness. Whenever a lower-scoring agent obtains the resource while a higher-scoring agent does not, the latter agent has *justified envy* towards the former. In the two-sided matching literature, justified envy is often seen as inimical to fairness (see *e.g.,* Balinski and Sönmez, 1999).

about the market, *i.e.,* they have some beliefs over joint distributions of scores and groups in the population.

Summary of Main Results. We propose a new class of *adaptive priority mechanisms* (APM) that adjust agents' scores as a function of the number of assigned agents with the same characteristics and that allocate the resource to the set of agents with the highest adjusted scores. With a single authority, we derive an APM that is optimal, implements a unique outcome, and can be specified solely in terms of the *preferences* of the authority (*i.e.,* it is optimal regardless of their beliefs). By contrast, we show that priorities and quotas are optimal if and only if risk aversion over diversity is extremely low or high, respectively. Moreover, optimally set priority and quota policies depend on both the preferences and beliefs of the authority. Thus, the optimal APM improves outcomes, is robust to uncertainty, and requires less information. With many authorities, we show that the combination of the deferred acceptance algorithm along with each authority using its optimal APM implements the unique stable allocation. Finally, we provide a practical roadmap for implementing APM as a market design solution. To demonstrate the practicality and potential benefits of adopting APM, we follow this roadmap and provide a proof-of-concept implementation using application and admission data from Chicago Public Schools. We find that the gains from adopting APM have the potential to be large.

Single-Authority Model. We begin our analysis by studying a setting with a single authority that has some amount of a homogeneous resource (*e.g.,* seats at a school, medical resources) that it can allocate to a continuum of agents.[4] Agents differ in their scores (*e.g.,* exam score, clinical need) and discrete attributes (*e.g.,* socioeconomic status, whether they are a frontline health worker). The authority cares separably about the distributions of scores (through some index such as the average score) and characteristics (such as gender and race) of the agents who are assigned the resource. Thus, the authority's preferences over agents depend on the joint distribution of agents' scores and groups. We assume that this distribution is potentially unknown and varies arbitrarily across states of the world. The authority's problem is to design a *first-best optimal* mechanism: a mechanism that is optimal regardless of their beliefs and implements an *ex post* optimal allocation in all states.

Adaptive Priority Mechanisms. To this end, we introduce the class of adaptive priority mechanisms (APM), which proceed in two steps. First, each agent is given an *adaptive priority* that is a function of their own score and the number of agents from the same group to whom the resource is assigned. Second, APM allocate the resource to agents in order of adaptive priorities, subject to fully allocating the available amount. This class of mechanisms

[4] In Appendix G, we generalize our analysis and results to a setting with discrete agents.

allows the implicit preference for agents from different groups to depend upon the ultimate allocation. The allocation under an APM is defined as the fixed point of the above operation: an allocation is implemented by APM if the adaptive priority of all agents who are allocated the resource (evaluated at the allocation) is higher than those who are not allocated the resource. When an agent's adaptive priority is increasing in their own score and decreasing in the number of agents with the same attributes that are assigned the resource – a property we call *monotonicity* – the APM implements a unique allocation. Moreover, this allocation can be computed by prioritizing agents according to their adaptive priority, evaluated at the number of higher-scoring agents in their group.

Most importantly, we derive a particular, monotone APM that is first-best optimal. Under this optimal APM, an agent's priority is equal to the contribution of their own score plus their marginal contribution to diversity utility. Intuitively, this mechanism equates the benefits and costs of allocating to the marginal agent, regardless of the ultimate joint distribution of agents' scores and groups. Moreover, this APM can be described and communicated to stakeholders *ex ante* as a function of the authority's preferences, without any reference to its beliefs or hypothetical states of the world.

(Sub)Optimality of Priorities and Quotas. We next establish that the widely used priority and quota mechanisms are generally dominated by APM. We do so by fully characterizing the conditions on the preferences of the authority such that priorities and quotas attain first-best optimality. Concretely, we find that priorities and quotas are first-best optimal if and only if (i) the authority is risk-neutral over diversity, in which case priorities are optimal, or (ii) the authority is extremely risk-averse over diversity, in which case quotas are optimal. Hence, outside of extreme cases, APM deliver strict improvements relative to the *status quo*.

A Price-Theoretic Intuition. To both illustrate and develop the intuition behind these results, we study a detailed example that allows for a closed-form comparison of priorities, quotas, and the optimal adaptive priority mechanism. We do this in the spirit of the seminal analysis of Weitzman (1974), who compares price and quantity regulation in product markets. In the example, the resource corresponds to seats at a school and there are two groups of students (minority and majority students). The authority is uncertain over the relative scores of minority and majority students, and has preferences over the scores of admitted students and the number of minority students admitted to the school.

The preference of the authority between priority and quota mechanisms is governed by its risk aversion over the number of admitted minority students: there is a cutoff value such that quotas are preferred when risk aversion exceeds this threshold and priorities are otherwise preferred. On the one hand, by mandating a minimal level of minority admissions, quotas *guarantee* a level of diversity. On the other hand, as relatively more minority students receive the resource in the states in which they have relatively higher scores, priorities *positively select* minority students. Adaptive priority mechanisms optimally exploit the guarantee effects of quotas and the positive selection effects of priorities, and are always optimal. In a price-theoretic sense, this is analogous to the idea that firms setting either a price or a quantity is typically suboptimal and it is better to set an entire supply function (for a review of the literature on supply functions, see Rostek and Yoon, 2023).

Multiple Authorities and Stable Allocations. While the single-authority model is relevant for studying settings with a single resource, in many markets there are multiple authorities who control heterogeneous resources (*e.g.,* school seats) over which agents have heterogeneous preferences. We generalize our analysis to this setting and show that there is a unique stable allocation. This constitutes a methodological contribution as we establish the uniqueness of stable allocations in continuum economies in which there are multiple socioeconomic groups and authorities have non-linear preferences over the composition of the agents that they admit. Importantly, this means that authorities have *endogenous* preferences over various agents: how an authority ranks one agent relative to another depends on the representation of their groups in the ultimate allocation. Moreover, we characterize this unique stable allocation and show that a mechanism is consistent with stability if and only if it coincides with the single-authority-optimal APM. Furthermore, if all authorities use the optimal APM to determine the set of admitted agents, then the widely used Deferred Acceptance (DA) algorithm implements the unique stable matching.

APM as a Practical Market Design Solution. Based on the fact that APM outperform conventional priority and quota mechanisms, the adoption of APM has the scope to improve resource allocation in many two-sided matching contexts. We conclude the paper by providing a practical roadmap for implementing APM as a market design solution. The roadmap proceeds in three steps. First, understand the preferences of stakeholders over allocations using a combination of standard stated and revealed preference methodologies. A key benefit of designing the optimal APM is that it only requires elicitation of preferences and not beliefs, which the optimal design of priority and quota mechanisms would require. Second, estimate the optimal APM and communicate this to stakeholders. We show that this can be achieved via a simple tabular format, in which rows correspond to groups, columns correspond to the number of people from that group that are allocated the resource, and the entries correspond to score boosts. Third, implement the optimal APM by running the Deferred Acceptance algorithm. As we have shown theoretically, this implements a stable allocation and makes it dominant for agents to report their preferences truthfully.

We finally perform a proof-of-concept exercise to obtain a sense of the benefit of using APM and demonstrate the simplicity of its implementation. Concretely, we apply our practical roadmap and benchmark the improvements from APM using application and admission data from 2013-2017 on the selective exam schools of Chicago Public Schools (CPS), a setting also empirically studied by Angrist, Pathak, and Zárate (2019) and Ellison and Pathak (2021). CPS uses a reserve system to increase the admissions of underrepresented groups. In this system, as we later detail, academic scores and the socioeconomic characteristics of the census tracts in which students live determine the schools that students can attend. Moreover, there is substantial variation in the joint distribution of student characteristics over time. This justifies our focus on the importance of uncertainty and implies that APM *must* generate gains for the authority. Estimating preference parameters to best rationalize the pursued reserve policy, we find that the gains from using the optimal APM are equivalent to eliminating 37.5% of the loss to CPS' payoffs from failing to admit a diverse class of students. This gain is 2.3 times larger than the estimated gain from an actual 2012 policy reform that increased the size of all reserves. This exercise shows both that APM could be practically implemented and that the gains from so doing may be considerable.

Related Literature. The market design literature has largely studied the comparative statics and axiomatic foundations of mechanisms. In this context, our paper relates to the literature on matching with affirmative action concerns initiated by Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005). For example, in the study of quotas, Kojima (2012) shows how affirmative action policies that place an upper bound on the enrollment of non-minority students may hurt all students, Hafalir, Yenmez, and Yildirim (2013) introduce the alternative and more efficient minority reserve policies, Ehlers, Hafalir, Yenmez, and Yildirim (2014) generalize reserves to accommodate policies that have floors and ceilings for minority admissions, and Doğan (2016) shows that stronger affirmative action can (weakly) harm all minority students under reserve policies. The quota policies studied in this paper are a special case of the slot-specific priorities introduced in Kominers and Sönmez (2016).[5] A number of alternative mechanisms have also been studied. For example, Kamada and Kojima (2017, 2018) and Goto, Kojima, Kurata, Tamura, and Yokoo (2017) study stability and efficiency in more general matching-with-constraints models. Relatedly, Echenique and Yenmez (2015) characterize a class of substitutable choice rules under diversity preferences, Erdil and Kumano (2019) study tie-breaking rules under substitutable priorities under stable matching mechanisms and distributional constraints, and Imamura (2020) presents axioms

[5]Quota policies have been used in many countries, including India (Aygün and Turhan, 2020; Sönmez and Yenmez, 2022a,b), Germany (Westkamp, 2013) and Brazil (Aygün and Bó, 2021).

to compare the meritocracy and diversity of different choice rules and characterizes reserves and quotas.

In this paper, we instead pursue the methodological approach of mechanism design and welfare economics by analyzing optimal mechanisms from the perspective of an authority with some given preferences over allocations. Chan and Eyster (2003) share this perspective in their analysis of the costs and benefits of banning affirmative action.[6] In this vein, we have previously analyzed the narrower problem of how to optimally coarsen agents' scores into priorities (Çelebi and Flynn, 2022). This analysis nevertheless restricted authorities to using a priority mechanism that does not consider agents' characteristics and implementing only allocations that are stable with respect to these priorities. Thus, our focus on comparing priorities, quotas, and optimal mechanisms distinguishes our analysis from our prior work and the previous literature, which study the properties of each policy in isolation and without an explicit treatment of uncertainty. By doing so, we arrive at a new and practical market design solution for two-sided matching markets that dominates the existing alternatives, including priority and quota mechanisms.[7]

Finally, by showing that deferred acceptance with adaptive priority mechanisms implements the unique stable allocation in the presence of endogenous preferences, we extend results from Azevedo and Leshno (2016) and Abdulkadiroğlu, Che, and Yasuda (2015) that show that deferred acceptance implements the unique stable allocation with fixed preferences. As a byproduct, our paper contributes to a literature that studies the uniqueness of stable allocations in large markets when authorities have endogenous preferences over agents, complementing Che, Kim, and Kojima (2019), who show that, when authorities' choice functions are generated by submodular quotas, there is a unique stable allocation when the distribution of agents has full support. Our uniqueness result concerns another important class of economies in which authorities have separable and smooth preferences over scores and diversity of a form that is commonly assumed in applied theoretical work (see *e.g.,* Chan and Eyster, 2003; Ellison and Pathak, 2021; Dessein, Frankel, and Kartik, 2023; Passaro, Kojima, and Pakzad-Hurson, 2023).

Outline. Section 2 exemplifies our main results. Section 3 studies optimal mechanisms with a single authority. Section 4 extends the model to include many authorities. Section 5 presents a general roadmap for implementing APM in practice and, as a proof-of-concept, applies this to Chicago Public Schools. Section 6 concludes. The proofs are in Appendix A.

[6] Other analyses of this issue include Epple, Romano, and Sieg (2008) and Temnyalov (2023).

[7] In a similar spirit, Combe, Dur, Tercieux, Terrier, and Ünver (2022) propose new mechanisms to overcome problems associated with unequal distribution of experienced teachers in schools and quantify the improvements compared to benchmarks.

2 Comparing Mechanisms: An Example

The Setting. A single school has capacity $q < 1$. Students are of unit total measure, have scores in $[0,1]$, and are either minority or majority students. The authority has linear-quadratic preferences $\xi : \mathbb{R}^2 \to \mathbb{R}$ over students' total scores $\bar{s}$ and the measure of admitted minority students x:

$$\xi(\bar{s}, x) = \bar{s} + \gamma \left(x - \frac{\beta}{2} x^2 \right) \tag{1}$$

where $\gamma \geq 0$ indexes their general preference for admitting minority students and $\beta \geq 0$ indexes the degree of risk aversion regarding the measure of admitted minority students.

The minority students are of measure κ and have scores that are uniform over $[0,1]$. The majority students are of measure $1-\kappa$ and all have common underlying score $\omega \in [\underline{\omega}, \overline{\omega}] \subseteq [0,1]$ with distribution Λ. The score of the majority students, ω, parameterizes how well they score relative to the minority students. Finally, we assume that the affirmative action preference is neither too small nor too large with the following: $\min\{\kappa, q\} > \frac{1+\gamma-\underline{\omega}}{\frac{1}{\kappa}+\gamma\beta} + \kappa(\overline{\omega} - \underline{\omega})$, $\kappa(1-\underline{\omega}) < \frac{1+\gamma-\overline{\omega}}{\frac{1}{\kappa}+\gamma\beta}$. These conditions ensure that optimal affirmative action policies will neither be so large as to award all slots to minority students in some states nor so small that there is no affirmative action in some states.

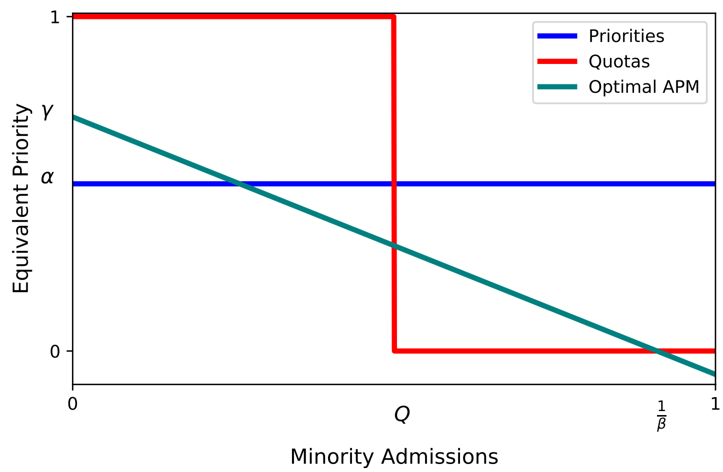
The authority can implement an APM, a priority mechanism, or a quota mechanism. An APM increases the scores of minority students by $A(y)$ when y other minority students are admitted, does not change the scores of majority students, and allocates seats to the students with highest transformed scores.[8] An (additive) priority mechanism $\alpha \in \mathbb{R}_+$ increases uniformly the scores of minority students by α. The authority then admits the highest-scoring measure q students. A quota policy $Q \in [0, \min\{\kappa, q\}]$ sets aside measure Q of the capacity for the minority students. The highest-scoring minority students of measure Q are first allocated to quota slots, and all other agents are then admitted to the residual $q - Q$ places according to the underlying score.[9]

We illustrate how these three policies prioritize minority students in Figure 1. By definition, priority mechanisms award a constant score boost of α. Quota mechanisms give enough points to always ensure admission until measure Q is reached and then give no advantage. APM allow any pattern of prioritization as a function of minority admissions (we plot only the optimal APM, which turns out to be linear in this context).

[8] Formally, this mechanism allocates seats to $x(\omega)$ minority students and $q - x(\omega)$ majority students, where $s(x(\omega)) + A(x(\omega)) = \omega$, and $s(x(\omega))$ denotes the score of the marginal minority student when the highest-scoring $x(\omega)$ minority students are admitted.

[9] This corresponds to a precedence order that processes quota slots first. We discuss the importance of precedence orders in Section 2.1 and in Appendix B.3.

Figure 1: How Priorities, Quotas, and APM Prioritize Minority Students



Note: Illustration of the equivalent priority given to a minority student as a function of the measure of admitted minority students under: the optimal APM (see Proposition 1), a priority mechanism α, and a quota mechanism Q.

Comparing Mechanisms. Let the authority's expected utility be V^* under any optimal (expected utility maximizing) mechanism, V_A under an optimal adaptive priority mechanism, V_P under an optimal priority mechanism, and V_Q under an optimal quota mechanism. The following proposition characterizes the relationships between these mechanisms:

Proposition 1. *The following statements are true:*

1. *The APM* $A(y) = \gamma(1 - \beta y)$ *is optimal,* $V^* = V_A$
2. *The comparative advantage of priorities over quotas is given by:*

$$\Delta \equiv V_P - V_Q = \frac{\kappa}{2}\left(1 - \kappa\gamma\beta\right)\mathrm{Var}[\omega] \tag{2}$$

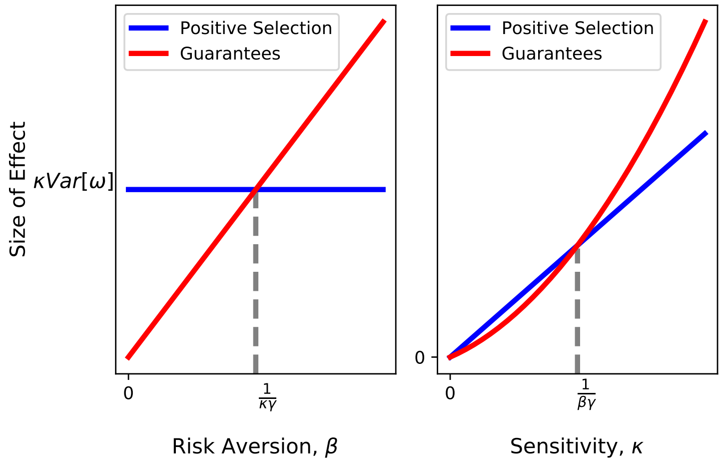
3. *The comparative advantage of APM over priorities and quotas is given by:*

$$\Delta^* \equiv \min\{V^* - V_P, V^* - V_Q\} = \begin{cases} \frac{1}{2}\left(\kappa\gamma\beta\right)^2 \frac{\kappa\mathrm{Var}[\omega]}{1+\kappa\gamma\beta}, & \kappa\gamma\beta \leq 1, \\ \frac{1}{2}\frac{\kappa\mathrm{Var}[\omega]}{1+\kappa\gamma\beta}, & \kappa\gamma\beta > 1. \end{cases} \tag{3}$$

We now develop intuition for the comparative advantage of priorities over quotas. First, observe that a quota of Q admits measure Q minority students in all states of the world under our assumptions. However, a priority policy induces variability in the measure of admitted minority students across states of the world. We call the gain to quota policies in eliminating this variation the *guarantee effect* and find mathematically that it is equal to $\frac{\kappa}{2}\left(1 + \kappa\gamma\beta\right)\mathrm{Var}[\omega]$ in payoff terms.

Second, the optimal priority policy admits more minority students when minority students score relatively well and fewer when minority students score relatively poorly. To

Figure 2: Comparative Statics for the Positive Selection and Guarantee Effects

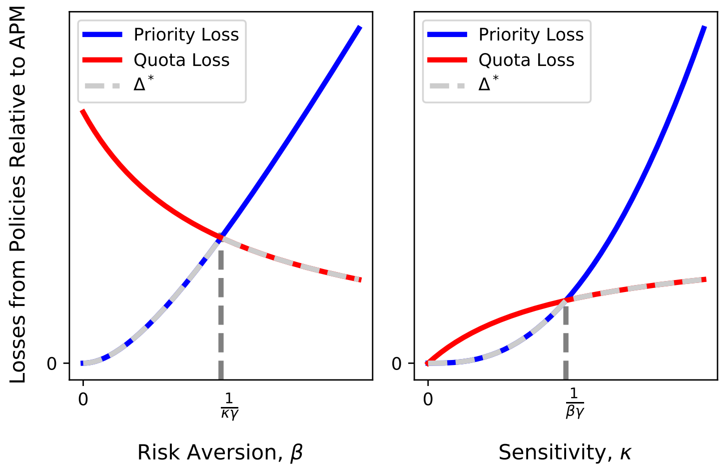


Note: Illustration of the comparative statics for the trade-offs between priority and quota mechanisms. Positive Selection plots the positive selection effect, $\kappa \text{Var}[\omega]$, and Guarantee plots the guarantee effect, $\frac{\kappa}{2}(1+\kappa\gamma\beta)\,\text{Var}[\omega]$. As per Equation 2 in Proposition 1, priorities dominate quotas if and only if $1 \geq \kappa\gamma\beta$, where the point of indifference is denoted by the dashed grey line.

demonstrate this, we show that minority admissions in state ω under the optimal priority policy are $x(\alpha,\omega) = \bar{x}(\alpha) + \varepsilon(\omega)$ where $\bar{x}(\alpha) = \kappa(1+\alpha-\mathbb{E}[\omega])$ and $\varepsilon(\omega) = \kappa\,(\mathbb{E}[\omega]-\omega)$. Thus, in the states where minority students score relatively better ($\omega < \mathbb{E}[\omega]$), we have that $\varepsilon(\omega) > 0$ and $x(\alpha,\omega) > \bar{x}(\alpha)$. We call this effect the *positive selection* effect and find that this benefits a priority policy by $-\text{Cov}[\omega, \varepsilon(\omega)] = \kappa\text{Var}[\omega]$ in payoff terms.

The ultimate preference between priority and quota mechanisms is determined by which of the guarantee and positive selection effects dominates. This is itself determined by the extent to which the authority values diversity γ, the risk preferences of the authority β, and the measure of minority students κ. We illustrate how risk aversion and the measure of minority students affect the sizes of the positive selection and guarantee effects in Figure 2. If the authority is close enough to risk-neutral (*i.e.*, $\frac{1}{\kappa\gamma} > \beta$), then priorities are strictly preferred as positive selection dominates guarantees. If the authority is sufficiently risk-averse (*i.e.*, $\frac{1}{\kappa\gamma} < \beta$), then quotas are strictly preferred as the guarantee effects dominate positive selection. The threshold for risk aversion scales inversely with the measure of minority students κ. Because minority students' scores are uniform, κ corresponds to the density of minority students' scores. Hence, the change in minority admissions from a small change in their priority equals κ. Thus, κ indexes the *sensitivity* of minority admissions to the state under priority policies. As a result, higher κ favors quota policies by increasing the magnitude of the guarantee effect relative to the positive selection effect. Finally, the extent of uncertainty $\text{Var}[\omega]$ may intensify an underlying preference but never determines which regime is preferred.

Figure 3: Comparative Statics for the Losses from Priorities and Quotas



Note: Illustration of the comparative statics for the losses from optimal priority and quota policies relative to the optimal APM (as presented in Equation 3 in Proposition 1). The lower envelope of the losses, Δ^*, corresponds to the comparative advantage of the optimal APM over priorities and quotas. The point of indifference between priorities and quotas is denoted by the dashed grey line.

An APM is optimal and overcomes the limitations posed by both priorities and quotas. In this case, the optimal APM is linear in the measure of admitted minority students, with slope given by the authority's risk aversion over minority admissions, awarding each minority student a subsidy equivalent to their marginal contribution to the diversity preferences of the authority. This allows the adaptive priorities to optimally balance the positive selection and guarantee effects, and implement the first-best allocation in every state. In Figure 3, we illustrate how the losses from priority mechanisms and quota mechanisms vary with risk aversion and sensitivity. As risk aversion moves, the loss from priority and quota policies relative to the optimum is greatest when the authority is indifferent between the two regimes. The loss from restricting to priority or quota policies is zero when the authority is risk-neutral or there is no uncertainty regarding relative scores, and decreases as the authority becomes extremely risk-averse. As sensitivity increases, the scope for affirmative action increases and so the gains from APM also increase. Thus, we should expect there to be large gains from switching to APM precisely when authorities have intermediate levels of risk aversion and/or the scope for implementing affirmative action is significant.

Finally, optimal APM have a further advantage that we have not yet highlighted: they depend only on the authority's preferences, γ and β, and not their beliefs, Λ. This contrasts with the optimal priority and quota policies, which depend on Λ.[10] As a result, APM improve outcomes while using *less* information and are robust to changes in beliefs.

[10]The optimal quota policy is given by $Q^* = \frac{1+\gamma-\mathbb{E}[\omega]}{\frac{1}{\kappa}+\gamma\beta}$, while the optimal priority policy sets the expected measure of minorities to Q^*. In this simple setting, the policies depend on Λ through $\mathbb{E}[\omega]$.

2.1 Discussion

Before moving to the general analysis, we discuss three additional findings that emphasize the broader economics and scope of these results.

A Price-Theoretic Intuition. This comparison of *priorities vs. quotas* echoes the comparison of *prices vs. quantitities* by Weitzman (1974). We show in Appendix B.1 that there is a formal mapping between the two. Intuitively, the positive selection effect is equivalent to the effect that price regulation gives rise to the greatest production in states where the firm's marginal cost is lowest. Moreover, the guarantee effect is equivalent to the ability of quantity regulation to stabilize the level of production. An APM corresponds in the Weitzman (1974) setting to a regulator setting neither a price nor a quantity, but completely specifying the optimal demand curve (as in the literature on demand functions reviewed by Rostek and Yoon, 2023). Thus, the comparison of mechanisms for allocating goods without prices boils down to similar trade-offs between well-understood price-based mechanisms for goods allocation.

Medical Resource Allocation. In Appendix B.2, we apply this model to understand the trade-offs between priority and quotas in the context of medical resource allocation. This topic received enormous attention during the Covid-19 pandemic (see *e.g.,* Pathak, Sönmez, Unver, and Yenmez, 2021). Our analysis provides a formal justification for the idea that priorities may lose out relative to quotas from ignoring some groups or ethical values in the allocation of scarce resources (the guarantee effect). However, we also uncover a benefit of priorities that was not previously understood: they induce positive selection. Thus, if we care mostly about treating the neediest (β is low), priorities may yet be optimal.

Optimal Precedence Orders. We have modelled quotas by first allocating minority students to quota slots and then allocating all remaining students according to the underlying score. However, we could have done the opposite. The orders in which quotas are processed are called *precedence orders* and their importance has been the subject of a growing literature (see *e.g.,* Dur, Kominers, Pathak, and Sönmez, 2018; Dur, Pathak, and Sönmez, 2020; Pathak, Rees-Jones, and Sönmez, 2022). In Corollary 1 in Appendix B.3, we show that processing quotas second is equivalent to using a priority policy in this setting. Thus, processing quotas first is better than processing them second if and only if $1 \leq \kappa\gamma\beta$. The main aspect of this conclusion is robust in the general theory: in Theorem 2, we show that for any quota policy to be optimal in the presence of uncertainty, it must process quotas first.

3 Optimal Mechanisms with a Single Authority

We begin our general analysis by studying the resource allocation problem of a single authority. In this context, we define APM and derive an optimal APM that attains the first-best. We moreover provide necessary and sufficient conditions for the optimality of the ubiquitous priority and quota mechanisms and find that they are extremely restrictive, implying that there are likely gains from switching to APM.

3.1 Model

An authority allocates a single resource of measure $q \in (0,1)$. Agents differ in their type $\theta \in \Theta = [0,1] \times \mathcal{M}$ comprising their scores $s \in [0,1]$ and the group to which they belong, $m \in \mathcal{M}$, where their score denotes their suitability for the resource and $\mathcal{M}$ is a finite set comprising potential attributes such as race, gender, or socioeconomic status. We denote the score and group of any type θ by $s(\theta)$ and $m(\theta)$, respectively. The true distribution of types is unknown to the authority. The authority's uncertainty is parameterized by $\omega \in \Omega$, where Ω is the set of all distributions over Θ that admit a density. The authority believes that ω has distribution $\Lambda \in \Delta(\Omega)$. In state of the world ω, we denote the measure of types by F_ω with density f_ω.[11] In Appendix G, we translate our analysis and results to the discrete context.[12]

An allocation $\mu : \Theta \to \{0,1\}$ specifies for any type $\theta \in \Theta$ whether they are assigned to the resource.[13] Two allocations μ and μ' are *essentially the same* if they coincide up to a measure zero set. The set of possible allocations is $\mathcal{U}$. An allocation is feasible if it allocates no more than measure q of the resource. A mechanism is a function $\phi : \Omega \to \mathcal{U}$ that returns a feasible allocation for any possible measure of types.

As motivated, authorities often have preferences over scores and diversity. To model this, we define the aggregate score index of any allocation as:

$$\bar{s}_h(\mu, \omega) = \int_\Theta \mu(s,m)h(s)\mathrm{d}F_\omega(s,m) \tag{4}$$

for some continuous, strictly increasing function $h : [0,1] \to \mathbb{R}_+$, which determines the extent to which the authority values agents with higher scores. To capture diversity, we compute

[11]Formally, we mean that $f_\omega(s,m) = \frac{\partial}{\partial s}F_\omega(s,m)$ exists for all $s \in [0,1]$ and $m \in \mathcal{M}$.

[12]Concretely, we establish the optimality of APM (Theorem 7), characterize the (sub)-optimality of priorities and quotas (Theorem 8), and show that agent-proposing deferred acceptance, when combined with the optimal APM, implements the agent-optimal stable allocation (Theorem 9).

[13]Formally, μ is a measurable function with respect to the Borel $\sigma-$algebra of the product topology in Θ.

the measure of agents of each group allocated the resource $x(\mu, \omega) = \{x_m(\mu, \omega)\}_{m \in \mathcal{M}}$ as:

$$x_m(\mu, \omega) = \int_{[0,1]} \mu(s, m) f_\omega(s, m) \mathrm{d}s \tag{5}$$

To separate the roles of scores and diversity, we impose that their utility function over these dimensions $\xi : \mathbb{R}^{|\mathcal{M}|+1} \to \mathbb{R}$ satisfies the following assumption:

Assumption 1. *The authority's utility function can be represented as:*

$$\xi(\bar{s}_h, x) \equiv g\left(\bar{s}_h + \sum_{m \in \mathcal{M}} u_m(x_m)\right) \tag{6}$$

for some continuous, strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ and differentiable and concave functions $u_m : \mathbb{R} \to \mathbb{R}$ for all $m \in \mathcal{M}$.

We also assume that the authority always prefers to allocate the entire resource.[14] The preference of the authority is a monotone transformation of a quasi-linear utility index comprised of scores and a diversity preference. Intuitively, u_m determines the preference for assigned agents of group m, with its concavity following from a preference for diversity.[15] The function g determines their risk preferences as well as the complementarity/substitutability of scores and diversity (if g is convex (concave) at a point, then scores and diversity are complements (substitutes) at that point). More fundamentally, Assumption 1 can be justified by the axiomatization developed by Çelebi (2023): if preferences satisfy appropriate adaptations of responsiveness (in the sense of Roth, 1985), substitutes (in the sense of Roth, 1984), and acyclicity (which is a strengthening of transitivity, proposed by Tversky, 1964), then Assumption 1 is satisfied.

This notwithstanding, we explore the robustness of our results to relaxing the separability, differentiability, and concavity embedded in Assumption 1 in Appendix D. Most importantly, we show that our results are essentially unchanged when preferences are non-separable over diversity, *i.e.,* when $\sum_{m \in \mathcal{M}} u_m(x_m)$ is replaced with $u(x)$. Among other things, this allows our model to capture preferences in situations with overlapping group membership (Aygün and Bó, 2021; Sönmez and Yenmez, 2022a), *e.g,* when people have different genders and belong to different socioeconomic groups.[16] The essential assumption for our results is the

[14] A necessary and sufficient condition for this is: $h(0) + u'_m(q) \geq 0$ for all $m \in \mathcal{M}$. This condition is sufficient as the lowest utility the authority can get from allocating the resource is always positive. It is also necessary: if $h(0) + u'_m(q) < 0$ for some m, in the state of the world where there are only measure q of group m agents with uniform score distribution, the authority would prefer not to allocate a portion of the resource to the lowest-scoring agents.

[15] Note that u_m depends on m, so our specification allows the designer to have different preferences for allocating the resource to agents from different groups. For example, this allows for a designer with affirmative action motives who prefers to assign the resource to some particular group m: $u'_m(x) > u'_{m'}(x)$ for all x or a designer who prefers a balanced composition of allocated agents: $u'_m(x) = u'_{m'}(x)$ for all $m \in \mathcal{M}$ and x.

[16] As a simple example, if people can be men m or women w and rich r or poor p (so the groups are

weak separability of diversity and score preferences.[17] When this fails, it is no longer possible to specify optimal mechanisms without explicitly conditioning the allocation on the realized distribution of agents. While separability does impose structure, we re-emphasize that the separability embodied in Assumption 1 does not rule out complementarity or substitutability between scores and diversity and does not even rule out that scores and diversity can be substitutes local to some allocations and complements local to other allocations.

We define the value of a mechanism ϕ under distribution Λ as the authority's expected utility of the allocations induced by that mechanism:

$$\Xi(\phi, \Lambda) = \int_{\Omega} \xi(\bar{s}_h(\phi(\omega), \omega), x(\phi(\omega), \omega)) \mathrm{d}\Lambda(\omega) \tag{7}$$

We say that a mechanism is first-best optimal if it maximizes the authority's expected utility for all possible *distributions of* measures of agents' characteristics.

Definition 1 (First-Best Optimality)**.** *A mechanism ϕ^* is first-best optimal if:*

$$\Xi(\phi^*, \Lambda) = \sup_{\phi} \Xi(\phi, \Lambda) \tag{8}$$

for all $\Lambda \in \Delta(\Omega)$.

This is a demanding property for a mechanism to possess as it requires a mechanism to implement an *ex post* optimal allocation in all states of the world. Moreover, as the example from Section 2 shows, priority and quota mechanisms can fail to be first-best optimal while APM can attain first-best optimality. This is despite the fact that the optimal APM can be described without reference to the state of the world. This allows the optimal APM to be defined *ex ante* and communicated to stakeholders (for example, students' families) in a simple way, without any reference to the beliefs of the authority or the states of the world. In the remainder of this section, we formally define APM, show that (when suitably designed) they are first-best optimal, and characterize the conditions under which priorities and quotas are first-best optimal.

3.2 Adaptive Priority Mechanisms

Toward deriving a first-best optimal mechanism, we introduce APMs. To this end, we first introduce an *adaptive priority policy* $A = \{A_m\}_{m \in \mathcal{M}}$, where $A_m : \mathbb{R} \times [0, 1] \to \mathbb{R}$. The

$\mathcal{M} = \{wp, wr, mp, mr\}$) and the authority cares about increasing the representation of women and poor people, then the utility function could be given by $u(x) = \hat{u}(x_{wp} + x_{wr}, x_{mp} + x_{wp})$ and our analysis would apply so long as u is concave.

[17] The assumption of separable preferences over scores and diversity is common in the literature on affirmative action concerns (see e.g., Athey, Avery, and Zemsky, 2000; Chan and Eyster, 2003; Ellison and Pathak, 2021).

adaptive priority policy assigns priority $A_m(y_m, s)$ to an agent with score s in group m when measure y_m of agents of the same group is allocated the object. Given an adaptive priority policy, an APM implements allocations in the following way:

Definition 2 (Adaptive Priority Mechanism)**.** *An adaptive priority mechanism, induced by an adaptive priority A, implements an allocation μ in state ω if the following are satisfied:*

1. *Allocations are in order of priorities: $\mu(\theta) = 1$ if and only if for all θ' with $\mu(\theta') = 0$, we have that:*

$$A_{m(\theta)}(x_{m(\theta)}(\mu, \omega), s(\theta)) > A_{m(\theta')}(x_{m(\theta')}(\mu, \omega), s(\theta')) \tag{9}$$

2. *The resource is fully allocated:*

$$\sum_{m \in \mathcal{M}} x_m(\mu, \omega) = q \tag{10}$$

With some abuse of terminology, we will often refer to an APM as the adaptive priority A that induces it. By way of illustration, we provide a simple example of the flexibility of APM to act like a hybrid of priority and quota policies.

Example 1. Let $\mathcal{M} = \{m, n\}$ and the capacity be $q = 0.5$. We consider the adaptive priority policy $A = \{A_m, A_n\}$ given by:

$$A_m(x, s) = s, \qquad A_n(x, s) = \begin{cases} s + 1 & \text{if } x \leq 0.1 \\ s + 0.1 & \text{if } x \in (0.1, 0.25) \\ s & \text{if } x \geq 0.25 \end{cases} \tag{11}$$

This leaves the score of group m agents unchanged and gives agents of group n a score boost of: 1 if less than measure 0.1 group n agents is assigned, 0.1 if between measure 0.1 and 0.25 group n agents is assigned, and no score boost at all if measure greater than 0.25 group n agents is assigned.

To understand the properties of this adaptive priority policy, observe that the highest-scoring measure 0.1 group n agents is guaranteed the resource, even in states where they score badly. Therefore, A_n practically embeds a quota of 0.1. For admissions levels between 0.1 and 0.25, the APM acts like a priority policy and boosts the scores of group n agents by 0.1, increasing the admissions of group n when group n agents score moderately well. For admissions levels beyond 0.25, group n agents are given no further advantage. Thus, when diversity is attained, this APM "phases out" and no longer advantages any group. △

At this point, we have not established that a given APM implements any allocation at all, or that it implements a unique allocation (*i.e.,* it may not even be a mechanism).

However, there is a natural subclass of APM that do implement a unique allocation: those that are monotone. An APM A is *monotone* when (i) $A_m(\cdot, s)$ is a decreasing function for all $m \in \mathcal{M}, s \in [0,1]$ and (ii) $A_m(y_m, \cdot)$ is a strictly increasing function for all $m \in \mathcal{M}, y_m \in \mathbb{R}$.[18]

Proposition 2. *Any Monotone APM A implements an essentially unique allocation.*

Moreover, the unique outcome of a monotone APM can be implemented by a simple algorithm:[19]

Algorithm 1 (Algorithm for Implementation of APM)**.** *The APM algorithm proceeds in the following four steps:*

1. *For each θ, define*

$$\overline{x}(\theta) = \int_{s(\theta)}^{1} f_\omega(s, m(\theta))\mathrm{d}s \tag{12}$$

as the measure of agents who have higher scores than θ and belong to the same group.

2. *Construct a ranking of the agents as*

$$R(\theta) = A_{m(\theta)}(\overline{x}(\theta), s(\theta)) \tag{13}$$

3. *Define the cutoff ranking for the agents as $\overline{R}_\omega$ by*

$$\int_\Theta \mathbb{I}\{R(\theta) \geq \overline{R}_\omega\}\mathrm{d}F_\omega(\theta) = q \tag{14}$$

4. *Allocate the resource to all θ with $R(\theta) \geq \overline{R}_\omega$.*

Intuitively, this algorithm works by ranking all agents by their score within each group m and assigning agents in order of their transformed scores evaluated at the measure of *already assigned* agents of the same group, conditional on their admission. Informally, the algorithm moves down the ranking of agents until the resource is exhausted.

3.3 Adaptive Priority Mechanisms Achieve the First-Best

Having shown that monotone APM implement a unique allocation and provided an algorithm to compute this allocation, we now show that a certain, monotone APM is first-best optimal:

[18]Observe that monotone adaptive priority mechanisms are fair in the sense that they preserve the ranking of agents within any group and assign higher priority to an agent whenever there are fewer agents from her group who are allocated the resource.

[19]Formally, when we consider an Adaptive Priority *Mechanism*, we are studying any selection from the set of allocations that the APM implements. As monotone APMs implement an essentially unique allocation, this is without loss of optimality. When we refer to the "unique" allocation, we refer to the cutoff allocation defined in the proof of Proposition 2 and which is implemented by Algorithm 1.

Theorem 1. *The following APM is monotone and first-best-optimal:*

$$A_m^*(y_m, s) \equiv h^{-1}(h(s) + u_m'(y_m)) \tag{15}$$

Moreover, if a mechanism is first-best-optimal, then it implements essentially the same allocations as A^*.

To gain intuition for the form of this mechanism, suppose that the authority has linear utility over scores $h(s) \equiv s$. In this case, $A_m^*(y_m, s) = s + u_m'(y_m)$, so an agent in group m is awarded a boost of $u_m'(y_m)$ when there are y_m higher-scoring agents of the same group, their direct marginal contribution to the diversity preferences of the authority. This is optimal because this boost precisely trades off the marginal benefit of additional diversity with the marginal costs of reduced scores. Moreover, failing to award this precise level of boost would result in a suboptimal allocation. Thus, any optimal mechanism must be essentially identical to the optimal APM we have characterized. To generalize this beyond linear utility of scores, consider the following observation: we can map agents' scores from s to $h(s)$, and consider the optimal boost in this space. As h is monotone, this preserves the ordinal structure of the optimal allocation, and the authority has linear preferences over $h(s)$. Thus, in this transformed space, the optimal boost remains additive and given by $u_m'(y_m)$. To find the optimal transformed score in the original space, we simply invert the transformation h and apply it to the optimal score in the transformed space, yielding the formula for the optimal mechanism in Theorem 1.

This result is fundamentally simple and echoes standard consumer preference theory: just as it is best to equate the marginal rates of substitution between different goods for a consumer, an authority should set allocations to equate the marginal rates of substitution between merit and diversity across all groups. Theorem 1 shows that this can be achieved through a simple generalization of priority mechanisms that allows the priority boost to depend on the representation of each group.

Moreover, Theorem 1 has important practical implications and, for these practical applications, its simplicity is its strength. Critically, we now show that this mechanism outperforms the alternatives that are used in practically all real-world allocation settings that don't use prices: priority and quota mechanisms. Moreover, this mechanism can outperform while using *less* information. To design this mechanism, a designer need only know the preferences of the authority and does not need any information about the designer's beliefs about what states of the world are likely and even what these states of the world are. This contrasts with optimally set priority and quota mechanisms, which require detailed information on the authority's beliefs, *i.e.,* one must elicit the authority's subjective distribution over joint distributions of groups' scores and measures.

In addition to its practical relevance, Theorem 1 is important as an intermediate step in our more involved theoretical results that characterize stable allocations with multiple authorities in Section 4. In Section 5, we will provide a practical roadmap for implementing this mechanism as a market design solution and demonstrate that it can be designed and communicated to stakeholders straightforwardly.

3.4 (Sub)Optimality of Priorities and Quotas

We have shown that APM are optimal. However, the primary classes of mechanisms that have been used in practice are priority and quota mechanisms. Therefore, it is important to understand whether (and when) these mechanisms are also optimal. We now establish that APM generally provide a strict improvement over priority and quota mechanisms and characterize when priority and quota mechanisms attain optimality.

We first formally define priority and quota mechanisms. A *priority policy* $P : \Theta \to [0, 1]$ awards an agent of type $(s, m) \in \Theta$ a priority $P(s, m)$, that depends on both their score and group.

Definition 3 (Priority Mechanisms)**.** *A priority mechanism, induced by a priority policy P, allocates the resource in order of priorities until measure q has been allocated, with ties broken uniformly and at random.*

We define a *quota policy* as (Q, D), where $Q = \{Q_m\}_{m\in\mathcal{M}}$ and $D : \mathcal{M} \cup \{R\} \to \{1, 2, \ldots, |\mathcal{M}| + 1\}$ is a bijection. The vector Q reserves measure of the capacity Q_m for agents in group m, with residual capacity $Q_R = q - \sum_{m\in\mathcal{M}} Q_m$ open to agents of all types. The bijection D (the precedence order) gives the order in which the groups are processed.

Definition 4 (Quota Mechanisms)**.** *A quota mechanism, induced by a quota policy (Q, D), proceeds by allocating the measure $Q_{D^{-1}(k)}$ agents from group $D^{-1}(k)$ (if there are sufficient agents from this group) to the resource in ascending order of k, and in descending order of score within each k. If there are insufficiently many agents of any group to fill the quota, the residual capacity is allocated to a final round in which all agents are eligible.*

We now characterize when priority and quota mechanisms are (sub)optimal. To do this, we first provide some definitions. Authority preferences are *non-trivial* if for all $m, n \in \mathcal{M}$, we have that:

$$h(1) + u_n'(0) > h(0) + u_m'(q) \tag{16}$$

Intuitively, the authority's preferences are non-trivial when their concerns for representation of certain groups do not always outweigh the consideration of scores.[20] The authority is *risk-neutral* over diversity if for all $m \in \mathcal{M}$, $u'_m : [0, q] \to \mathbb{R}$ is constant, *i.e.,* there are constant marginal returns to admitting more agents from all groups. If there are decreasing marginal returns, then the authority's preferences feature risk aversion. We define extremely risk-averse preferences as follows. Let $\tilde{u}$ and $\tilde{h}$ be functions describing diversity and score preferences, and let $\{x_m^{\text{tar}}\}_{m\in\mathcal{M}}$ be a vector of target allocation levels. Moreover, assume that these satisfy: (i) $\tilde{u}'_m(x_m) = 0$ for all $x_m > x_m^{\text{tar}}$ (ii) $\tilde{u}'_m(x_m) \geq \tilde{h}(1) - \tilde{h}(0)$ for $x_m \leq x_m^{\text{tar}}$ and (iii) $\sum_{m\in\mathcal{M}} x_m^{\text{tar}} \leq q$. Intuitively, an authority whose preferences are represented by $\tilde{u}$ and $\tilde{h}$ is very risk-averse as the condition that $\tilde{u}'_m(x_m) \geq \tilde{h}(1) - \tilde{h}(0)$ implies that the loss from being below the target level for a group x_m^{tar} dominates any benefit from increased scores. Thus, they are infinitely risk-averse to failing to meet this target. We say that the authority is *extremely risk-averse* if the authority's preferences over the optimal allocations can be represented by $(\tilde{u}, \tilde{h})$.[21]

Theorem 2. *Suppose that the authority has non-trivial preferences. The following statements are true:*

1. *There exists a first-best optimal priority mechanism if and only if the authority is risk-neutral. Moreover, this mechanism is given by* $P(s, m) = h^{-1}(h(s) + u'_m)$.
2. *There exists a first-best optimal quota mechanism if and only if the authority is extremely risk-averse. Moreover, this mechanism is given by* $Q_m = x_m^{tar}$ *and* $D(R) = |\mathcal{M}| + 1$.

Theorem 2 provides precise conditions on preferences such that the inability of priorities and quotas to adapt to the state is not problematic. That risk-neutrality and high risk aversion are sufficient for the optimality of priority and quota mechanisms is intuitive. On the one hand, if the authority is risk-neutral over the measure of agents from different groups, then they can perfectly balance their score and diversity goals without regard for the state of the world. This is because, under risk-neutrality, there is a constant "exchange rate" between the two: how the authority compares any two agents does not depend on the final allocation and thus can be specified *ex ante* by a priority policy. On the other hand, if the authority is extremely risk-averse as to the prospect of failing to assign x_m^{tar} agents from group m, then a quota allows them to always achieve this target level of allocation in all states of the world while minimally sacrificing scores. It is less obvious that risk-neutrality and high risk

[20] Note that failure of non-triviality means there exists m and n such that $h(1) + u'_n(0) \leq +h(0) + u'_m(q)$, *i.e.,* a group n agent with the maximum score is less preferred than a group m agent with the minimum score even when all of the entire capacity is allocated to group m agents.

[21] More formally, this means that there exists $(\tilde{u}, \tilde{h})$ such that the optimal allocation under (u, h) is also optimal under $(\tilde{u}, \tilde{h})$ for all $\omega \in \Omega$.

aversion are necessary. We prove this result by constructing certain adversarial measures of agents that render any priority or quota mechanism suboptimal unless the authority is risk-neutral or extremely risk-averse, respectively. Importantly, this result also shows that the only optimal quota mechanisms are those that process open slots last.

This result highlights the fragility of priority mechanisms to uncertainty absent the strong assumption of risk-neutrality over diversity. Intuitively, this is because they feature no guarantees as to how many agents of different groups will be assigned. Indeed, the unfortunate interaction between priority mechanisms and unforeseen market realizations has led to public backlash against priority mechanisms. For example, in the Vietnamese university admissions system, which combines exam scores with priority boosts for students from disadvantaged groups, a year of unexpectedly easy exams led to "top-scoring students missing out on the opportunity to attend their university of choice" and generated backlash against the system (Tuoi Tre News, 2017).

Moreover, our result highlights that quota mechanisms similarly fail to achieve the first-best away from high levels of risk aversion as they do not take advantage of the potential for positive selection. Our quantitative analysis in Section 5 in the context of quota mechanisms in Chicago Public Schools suggests that the variation in the distribution of characteristics across years generates meaningful welfare gains from switching to APM.

To formalize the connection between uncertainty and the importance of the adaptability of APM, we consider a setting with *no uncertainty*, where Λ is a Dirac measure. In this context, we say that a mechanism is optimal without uncertainty if it is a utility maximizer.

Proposition 3. *If there is no uncertainty, then there exist optimal priority and quota mechanisms.*

This result shows that if an authority is certain about the market, then appropriately constructed priority and quota mechanisms would be optimal. This formalizes the idea that the suboptimality of priority and quota mechanisms stems from their inability to adapt to the state. Of course, in practice, an authority is always somewhat uncertain of the market they face. We will later show empirically that this is the case for CPS as we find substantial variation in the joint distribution of student scores and groups across years. Thus, absent the strong conditions on authority preferences that we have characterized in Theorem 2, APM dominate priority and quota mechanisms. In Section 5, over and above its superior theoretical performance, we describe several practical benefits of APM and estimate its potential benefits using Chicago Public Schools data.

4 Stable Mechanisms with Multiple Authorities

The single-authority model is relevant for many resource allocation contexts, such as the medical resource allocation problem of a hospital and the allocation of (homogeneous) government jobs to candidates. However, in other settings such as school or university admissions, multiple authorities must decide upon their admissions policies and rules. In this section, we generalize our single authority model to a setting with multiple authorities. We define *stability* in this setting and show that there is a unique stable allocation. Moreover, we show that a mechanism is consistent with stability if and only if it coincides with the *single-authority-optimal* APM from Theorem 1. Finally, we show that when authorities use the optimal APM, the widely used deferred acceptance algorithm implements the unique stable matching, which makes the use of APM practical in a multiple-authority setting.

4.1 The Multi-Authority Model

There are authorities $c \in \mathcal{C} = c_0 \cup \bar{\mathcal{C}} = \{c_0, c_1, \ldots, c_{|\mathcal{C}|-1}\}$ with capacities q_c, where c_0 is a dummy authority that corresponds to an agent going unmatched. The agents differ in their authority-specific scores, the group to which they belong, and their preferences over the authorities, $\succ$. We index agents by their type $\theta = (s, m, \succ) \in [0,1]^{|\mathcal{C}|} \times \mathcal{M} \times \mathcal{R} = \Theta$, where $\mathcal{R}$ is set of all complete, transitive, and strict preference relations over $\mathcal{C}$. For each type θ, $s_c(\theta)$ denotes the score of θ at authority c and $m(\theta)$ denotes the group of θ. From now, to economize on notation, we suppress indexing states by $\omega \in \Omega$ and let the measure of types be F, with density f.[22] We assume that f has full support over Θ (*i.e.,* $f > 0$) and that $F(\Theta)$ is less than the capacity of c_0 and greater than the capacity of $\bar{\mathcal{C}}$.

Each authority has preferences over the agents they are assigned of the form introduced in the previous section:

$$\xi_c(\bar{s}_{h_c}, x_c) = g_c\left(\bar{s}_{h_c} + \sum_{m\in\mathcal{M}} u_{m,c}(x_{m,c})\right) \tag{17}$$

where the extent to which they care about risk g_c, scores h_c, and diversity $\{u_{m,c}\}_{m\in\mathcal{M}}$ are potentially specific to each authority.

A matching is a function $\mu : \mathcal{C} \cup \Theta \to 2^\Theta \cup \mathcal{C}$ where $\mu(\theta) \in \mathcal{C}$ is the authority that any type θ is assigned and $\mu(c) \subseteq \Theta$ is the set of agents that is assigned to authority c.[23] Given

[22]Formally, this density is given by $f(s, m, \succ) = \frac{\partial}{\partial s} F(s, m, \succ)$.

[23]The mathematical definition of a matching for the continuum economy we study follows Azevedo and Leshno (2016) and requires that μ satisfies the following four properties: (i) for all $\theta \in \Theta$, $\mu(\theta) \in \mathcal{C}$; (ii) for all $c \in \mathcal{C}$, $\mu(c) \subseteq \Theta$ is measurable and $F(\mu(c)) \leq q_c$; (iii) $c = \mu(\theta)$ iff $\theta \in \mu(c)$; (iv) (open on the right) for any $c \in \mathcal{C}$, the set $\{\theta \in \Theta : c \succ_\theta \mu(\theta)\}$ is open.

a matching μ, $\bar{s}_{h_c,c}(\mu)$ and $x_c(\mu) = \{x_{m,c}(\mu)\}_{m\in\mathcal{M}}$ denote the score indices and measures of agents from different groups matched to c at μ. We say that c *prefers* μ to μ', which we denote by $\mu \succ_c \mu'$, if $\xi_c(\bar{s}_{h_c,c}(\mu), x_c(\mu)) > \xi_c(\bar{s}_{h_c,c}(\mu'), x_c(\mu'))$. Toward representing a matching as a lower-dimensional object, we moreover define a cutoff matching as one in which agents are assigned to the authority that they most prefer among the set of authorities in which their score clears a group-specific threshold:

Definition 5. *A matching μ is a cutoff matching if there exist cutoffs $S = \{S_{m,c}\}_{m\in\mathcal{M},c\in\mathcal{C}}$ such that $\mu(\theta) = c$ if (i) $s_c(\theta) \geq S_{m(\theta),c}$ and (ii) for all c' with $c' \succ_\theta c$, $s_{c'}(\theta) < S_{m(\theta),c'}$.*

Given S, the *demand* of an agent θ is their favorite authority among those for which they clear the cutoff:

$$D^\theta(S) = \{c : s_c(\theta) \geq S_{m(\theta),c} \text{ and } c \succeq_\theta c' \text{ for all } c' \text{ with } s_{c'}(\theta) \geq S_{m(\theta),c')}\} \tag{18}$$

The *aggregate demand* for authority c is the set of agents who demand it $D_c(S) = \{\theta : D^\theta(S) = c\}$, while $\tilde{D}_c(S_{-c}) = D_c((0,\ldots,0), S_{-c})$ returns the set of all agents who would demand c if offered admission when other authorities' cutoffs are S_{-c}.

4.2 Characterization of Stable Allocations

We first characterize the set of stable allocations. Our context presents two challenges in this regard. First, the priorities that are typically used to define stability are not primitives of our model. Therefore, to define stability, we will use the preferences of the authorities induced by Equation 17. Second, unlike discrete models, a single agent does not affect the preferences of an authority. Therefore, we need to consider a positive mass of agents to define blocking.

For each matching μ, authority $c \neq c_0$, and two sets of agents $\tilde{\Theta}$ and $\hat{\Theta}$, we let $\hat{\mu}_{(\hat{\Theta},\tilde{\Theta},c,\mu)}$ denote the matching that maps $\hat{\Theta}$ to c and $\tilde{\Theta}$ to c_0 and otherwise coincides with μ.[24] A set of agents $\hat{\Theta}$ *blocks* matching μ at authority c by $\tilde{\Theta}$ if (i) for all $\theta \in \hat{\Theta}$, $c \succ_\theta \mu(\theta)$, (ii) $\tilde{\Theta} \subseteq \mu(c)$, (iii) $F(\tilde{\Theta}) = F(\hat{\Theta})$, and (iv) $\hat{\mu}_{(\hat{\Theta},\tilde{\Theta},c,\mu)} \succ_c \mu$. A matching μ is *not blocked* if there does not exist such a $(\hat{\Theta}, \tilde{\Theta}, c)$. A matching μ satisfies *within-group fairness* if for all $\theta, \theta' \in \Theta$ such that $m(\theta') = m(\theta)$ and $s_{\mu(\theta)}(\theta') > s_{\mu(\theta)}(\theta)$, it holds that $\mu(\theta') \succeq_{\theta'} \mu(\theta)$.[25] A matching μ is

[24]Formally,

$$\hat{\mu}_{(\hat{\Theta},\tilde{\Theta},c,\mu)}(\theta) = \begin{cases} c_0 & \text{if } \theta \in \tilde{\Theta} \\ c & \text{if } \theta \in \hat{\Theta} \\ \mu(\theta) & \text{otherwise} \end{cases} \tag{19}$$

[25]Within-group fairness requires an authority to not reject an agent if it is admitting an agent from the same group with a lower score. Under our assumption that authorities prefer higher scores (h_c is strictly increasing), within-group fairness is satisfied if there is no blocking in discrete models.

stable if it satisfies within-group fairness, is not blocked, and all non-dummy authorities fill their capacity. The following result establishes that there exists a unique stable matching and that this is a cutoff matching.

Theorem 3. *There is a unique stable matching. This matching is a cutoff matching.*

This result complements existing results on the uniqueness of stable allocations in continuum economies under full support (Azevedo and Leshno (2016) for responsive preferences and Che, Kim, and Kojima (2019) for submodular quotas). Theorem 3 extends the scope of known economies in which there is a unique stable allocation by considering economies in which agents belong to different socioeconomic groups and the preferences of authorities depend non-linearly on their admissions of various groups. Importantly, our setting results in the authorities having preferences over various sets of agents that are endogenous to the composition of the admitted agents.

We now formally describe these complications and how we resolve them by using our characterization of the single authority optimal APM from Theorem 1. First, imagine that there is only one group of agents $|\mathcal{M}| = 1$, so that authorities' preferences are determined by the scores of the agents, as in Azevedo and Leshno (2016). Given a set of cutoffs S_{-c}, a cutoff t_c clears the market for c if $F(D_c(t_c, S_{-c})) = q_c$. When $|\mathcal{M}| = 1$, for a given S_{-c}, there is a unique t_c that clears the market since a smaller cutoff will exceed the capacity while a larger one will leave a positive measure of the capacity empty. Define $T = \{T_c\}_{c \in \mathcal{C}}$, where $T_c(S)$ is the function that maps each S to the market-clearing cutoff t_c under S_{-c}. The result then follows from (i) showing the fixed points of T correspond to market-clearing cutoffs of stable matchings, (ii) establishing that T is monotone, (iii) applying Tarski's fixed point theorem to show that the set of market-clearing cutoffs is a lattice, and (iv) showing that there can only be one market-clearing cutoff as, if there were two, one would strictly exceed the capacities of at least one authority.

Suppose now that there are multiple groups, $|\mathcal{M}| > 1$. A significant complication arises: there is a potential continuum of cutoffs that would clear the market for authority c. A selection from this set is provided by the cutoffs induced by the optimal APM characterized by Theorem 1, $A^*_{m,c}(y_m, s) \equiv h_c^{-1}(h_c(s) + u'_{m,c}(y_m))$. We show that the APM cutoffs are unique among the market-clearing cutoffs in being compatible with stability. This is because, for any other t'_c, there is a set $\hat{\Theta}$ of agents (with positive measure) who have scores lower than the cutoff for their group and a set $\tilde{\Theta}$ of agents (with positive measure) who have scores higher than the cutoff for their group, but the authority is strictly better off by admitting $\hat{\Theta}$ and rejecting $\tilde{\Theta}$. We define $T_c(S)$ as the market-clearing cutoffs induced by the optimal APM, show that the fixed points of T_c correspond to market-clearing cutoffs of stable matchings, and then use this to establish the existence and uniqueness of the stable allocation.

This hints at a connection between the stable allocation and the allocation induced by all authorities pursuing the optimal APM, which we now make explicit. The demand set of c at μ, $D_c(\mu)$, is the set of agents who prefer c to their allocation under μ. A mechanism is *consistent with stability* if for all F with stable matching μ_F, it chooses $\mu_F(c)$ from $D_c(\mu_F)$. In other words, evaluated at the set of agents who demand an authority, this mechanism chooses the set of agents with which the authority is already matched. Moreover, we say that a mechanism ϕ is *equivalent* to ϕ' if it chooses the same agents under all full support measures. We now establish that single-authority-optimal APM, as characterized by Theorem 1, comprise the full set of mechanisms that are consistent with stability (up to equivalent mechanisms).

Proposition 4. *A mechanism ϕ is consistent with stability if and only if it is equivalent to A^*_c.*

Thus, not only is the optimal APM A^* inherent to the structure of stable allocations, but it also characterizes stability in this setting in the sense that any deviation from A^* would result in a violation of stability.

4.3 APM and DA Implement the Unique Stable Matching

We have now shown there is a unique stable matching and that any mechanism that is consistent with stability is equivalent to the optimal APM. These results suggest that authorities using their optimal APM can implement the unique stable matching. We now show that this is indeed the case: the standard deferred acceptance (DA) algorithm implements the unique stable matching when the authorities' choice rules follow the optimal APM.

First, we define the Deferred Acceptance (DA) algorithm, following Abdulkadiroğlu, Che, and Yasuda (2015), which we describe below under our notational conventions for completeness. Each authority $c \in \mathcal{C}$ submits a choice function $Ch_c : 2^\Theta \to 2^\Theta$ such that $Ch_c(\Theta') \subseteq \Theta'$ and $F(Ch_c(\Theta')) \leq q_c$. To define the DA mapping DA, first, consider a mapping $Q : \Theta \to \mathcal{R}$, where $Q(\theta)$ is an ordered list of authorities. The DA mapping $Q' = DA(Q)$ is determined as follows. Informally, every agent θ applies to her most preferred authority in $Q(\theta)$. Every authority c tentatively accepts its applicants according to Ch_c. If all seats of c are assigned, it rejects the remaining agents. If an agent θ is rejected by c, $Q'(\theta)$ is obtained from $Q(\theta)$ by deleting c in $Q(\theta)$. If an agent θ is not rejected, then $Q'(\theta) = Q(\theta)$.

More formally, let $\mathcal{T}_c(Q) = \{\theta \in \Theta : \text{c is ranked first in } Q(\theta)\}$ be the set of agents that rank c as first choice. Each authority admits agents in $Ch_c(\mathcal{T}_c(Q))$ and rejects agents in $\mathcal{T}_c(Q) \setminus Ch_c(\mathcal{T}_c(Q))$. If $\theta \in \mathcal{T}_c(Q) \setminus Ch_c(\mathcal{T}_c(Q))$ for some c, then $Q'(\theta)$ is obtained from $Q(\theta)$ by deleting c from the top of $Q(\theta)$; otherwise $Q'(\theta) = Q(\theta)$. Repeated application of the DA

mapping gives us the DA algorithm. That is, given the set of authorities C, the distribution of agents F and capacities q, let $Q^0(\theta) = \succ_\theta$ for all θ and $Q^t = DA(Q^{t-1})$ for $t \in \mathbb{N}$. By generalizing Theorem 0 of Abdulkadiroğlu, Che, and Yasuda (2015), we obtain that this DA algorithm converges for every agent:

Lemma 1. *$DA^t(Q^0)$ converges pointwise.*

We let $Q^* : \Theta \to \mathcal{R}$ denote the limit of $DA^t(Q_0)$ as $t \to \infty$. Given $Q^*(\theta)$, we define $\mu^{DA}(\theta)$ as the $\succ_\theta$-maximal element of $Q^*(\theta)$. We say that μ^{DA} is the matching that is implemented by DA. We now show that if Ch_c is generated by the optimal APM A_c^* for all c, then μ^{DA} is the unique stable matching of this economy.

Theorem 4. *If all authorities use the optimal APM, then DA implements the unique stable matching.*

This result demonstrates that deferred acceptance, a widely used and standard matching algorithm, can readily be combined with APM to compute and implement the unique stable matching in practice in a multi-authority setting.[26] Moreover, as is well known (see Theorem 5.16 in Roth and Sotomayor, 1990), this establishes that the combination of DA with APM is strategy-proof for agents: truthfully reporting preferences is a dominant strategy.[27]

4.4 Extensions: Efficient Mechanisms and Dominance Under Decentralized Sequential Admissions

Decentralized Admissions. In our main analysis of multiple authorities, we have considered a centralized allocation mechanism (deferred acceptance) and shown that this implements the unique stable matching. It is also interesting to consider what might happen as an equilibrium outcome between authorities in a decentralized setting. In Appendix E, we study such a decentralized setting in which the agents apply sequentially to the authorities, who then decide which agents to admit. We show that a mechanism implements a dominant strategy for an authority c if and only if it is the optimal APM A_c^* characterized by Theorem 1 (Theorem 5). Moreover, in any equilibrium in which authorities use the optimal APM, the equilibrium allocation is the unique stable matching of the economy (Proposition 9). Moreover, this is true whatever the order in which the authorities admit agents. These results

[26] We extend this result to discrete markets in Theorem 9 in Appendix G.4, replacing the unique stable matching with the agent-optimal stable matching.

[27] Of course, it is not a dominant strategy for authorities to report truthfully (*i.e.*, use the optimal APM) under deferred acceptance. This has nothing to do with their using APM or restricting attention to DA: it is not possible to find a stable matching mechanism that makes truth-telling dominant for authorities (see Theorem 5.14 in Roth and Sotomayor, 1990).

imply that one could advise authorities to use APMs with confidence that outcomes will be stable and that they could do no better under any alternative mechanism in decentralized allocation contexts.

Efficient Mechanisms. In our main analysis, as is standard in market design, we considered the problem of implementing stable allocations. It is also natural to ask what kind of mechanisms might implement allocations that are efficient for the *authorities*. In Appendix F, we study authority-efficient mechanisms. Considering utilitarian efficiency over authorities as our welfare criterion, we show that the unique stable allocation is generally inefficient (Proposition 10). We propose an augmentation of an APM to solve this problem, an adaptive priority mechanism with quotas (APM-Q). When authorities evaluate the scores (but not necessarily diversity) in the same way, we show that an APM-Q implements efficient allocations (Theorem 6). The idea behind this hybrid mechanism is to construct a fictitious aggregate authority in our single object setting and use aggregate, market-level priorities with authority-specific quotas to implement efficient allocations.

5 A Practical Method to Implement APM as a Market Design Solution

We have so far shown theoretically that APM outperform conventional priority and quota mechanisms and we have shown in multi-authority contexts that the combination of APM with deferred acceptance implements stable allocations. Given these desirable theoretical properties, we argue that adopting APM has the scope to improve the allocation of resources in many two-sided matching contexts.

In this section, we demonstrate how to practically implement APM as a market design solution. We do this in two steps. First, we describe a general three-step roadmap for how APM can be implemented in practice. Second, as a proof-of-concept exercise, we use data from Chicago Public Schools to show that: (i) APM can be easily designed, (ii) APM can be simply communicated to stakeholders, and (iii) the gains from switching to APM are *necessarily* positive and may also be quantitatively significant.

5.1 A General Methodology to Implement APM in Practice

Implementation of APM in practice is simple and can be achieved via a three-step approach.

Step I: Understand the Preferences of the Authorities. As in any other market design solution, the market designer must understand the preferences of the authorities.

This can be achieved in at least three ways. The first and simplest approach is simply to ask authorities to report their preferences. This is possible if authorities can determine and communicate their preferences. A second, and likely more practical strategy, would be to adopt a discrete-choice revealed-preference approach: ask the authorities to compare pairs of allocations for a large set of allocations and use the corresponding revealed preferences to set identify the utility functions that can represent these preferences. This second approach, which dates back to at least McFadden (1986), can be implemented by using standard econometric methods and is widely used in marketing and policy analysis (see Ben-Akiva, McFadden, and Train, 2019, for a review).[28] A third approach would be to estimate their preferences based on historical data, *e.g.,* by using the admissions rules adopted in previous years to structurally estimate preferences that are consistent with these rules. This approach is less desirable than the discrete-choice approach but is likely to be feasible in a larger set of circumstances and/or as an initial proof-of-concept exercise that can demonstrate the utility of the market design approach to the stakeholders. We will shortly adopt this third approach and use observational data from Chicago Public Schools to estimate preferences.

An important feature of this step is that we only require the preferences of the authorities and *not* their beliefs about the market they face. This is because the optimal APM from Theorem 1 can be computed using *only* information about the authorities' preferences. This is an important property of the optimal APM that is not shared by priority and quota mechanisms: outside of the specific and knife-edge conditions on preferences described by Theorem 2 (risk-neutrality and extreme risk-aversion), the optimal design of priority and quota mechanisms requires not only the elicitation of preferences but also the elicitation of beliefs about the distribution of agent characteristics in the population. We provided an explicit example of this fact in Section 2.[29] While the elicitation of preferences may be demanding, it is always necessary to design optimal (or improved) mechanisms and evaluate their effectiveness. Adopting the APM approach allows the market designer to sidestep entirely the even more demanding task of jointly estimating both preferences and beliefs.

Step II: Compute and Communicate the Optimal APM. Once preferences have been recovered, Theorem 1 provides an explicit formula for the optimal APM and its computation is immediate. The communication of the optimal APM to stakeholders is also simple and, in addition to reporting the function that describes the optimal APM, can be achieved via a tabular format in which the score boost that agents from various groups receive is described for different levels of admissions of that group to a school.

We provide a simple example of how that could be done in Table 1. Suppose that there are

[28] In the context of psychology, the approach dates much further back to Thurstone (1931).

[29] See Footnote 10.

Table 1: A Simple Example of How to Communicate APM to Stakeholders

School: Great School								
Score Boost		Number Admitted						
		0	5	10	15	20	25	...
Group	Blue	+10	+8	+6	+4	+2	+0	...
	Red	+5	+3	+1	+0	+0	+0	...
	Green	+0	+0	+0	+0	+0	+0	...
	Orange	+10	+8	+6	+4	+2	+0	...

Note: The table reports the score boost that an individual in each group receives at Great School depending on the number of people that have been admitted from their group. For example, a Blue individual will receive 6 bonus points for admission when 10 Blue individuals are admitted to Great School.

four groups $\mathcal{M} = \{\text{Red}, \text{Green}, \text{Blue}, \text{Orange}\}$ and consider an authority $c = $ Great School. After computing the optimal APM A^*_c, we can print the additive priority boost that the APM awards $h(A^*_{m,c}(x_{x,c})) - h(s) = u'_{m,c}(x_{m,c})$ as a function of $x_{m,c}$. Communication strategies like Table 1 provide a simple way to communicate how APM functions to stakeholders.

There are three further practical advantages of APM that Table 1 illustrates. First, to achieve diversity, it is not necessary to treat different groups differently *ex ante*. This is because even if score boosts are *ex ante* identical, this does not mean that individuals of different groups will receive the same realized score boost *ex post*. In this example, Blue and Orange individuals receive the same score boosts as a function of the number of individuals that are admitted from their groups. However, if Orange individuals were to score worse *ex post* and fewer were to be admitted, then Orange individuals would receive higher score boosts. This constitutes a significant benefit of APM over priority mechanisms, which must treat different groups differently *ex ante* to achieve a more equitable allocation of resources *ex post*. Indeed, the optimal APM we will compute for the CPS in Section 5.2.3 treats all groups symmetrically *ex ante* while yielding more equitable allocations *ex post*. We believe this feature has the potential to make affirmative action more acceptable to the general public. This is because a symmetric APM ensures that effective affirmative action is due to inherent inequality in the distribution of scores and *not* the preferences of the authority. Moreover, while a general APM is slightly more complex to communicate than a priority mechanism, which would require only one column of Table 1, a symmetric APM is simple to communicate as it requires only a single row of Table 1.

Second, while it is of course ideal to communicate the optimal APM which will be smooth, it is possible to "coarsen" the APM into a smaller set of tiers—as illustrated in the tabular format—if this better facilitates transparent communication to stakeholders.

Third, it may be simpler to communicate an APM than a quota mechanism. This is because recent work has shown that precedence orders, an essential component of describing a quota mechanism, are hard for even administrators to understand and have great effect on the final allocation.[30] Moreover, precedence orders increase considerably the dimensionality of quota mechanisms. For example, a quota mechanism with 4 groups does not only need to specify the quota levels for each group, but also the order in which the 5 different types of seats (including the *merit* seats that are not reserved for any group) will be processed, which can take $5! = 120$ different values.

Step III: Implement the Optimal APM Using the DA Algorithm. With the optimal APM computed and communicated, all that remains is to implement it. As Theorem 4 shows, running the agent-proposing deferred acceptance algorithm, in which all authorities tentatively accept and reject agents by running the optimal APM on the set of agents that apply, implements stable allocations. Moreover, as Theorem 9 in Appendix G.4 shows, the combination of agent-proposing DA and the optimal APM in a discrete matching environment always implements the agent-optimal stable allocation. Thus, standard DA implementations need only be updated with the optimal APM to practically implement APM as a market design solution. A benefit of this DA implementation is that it is strategy-proof for the agents, *i.e.,* it is dominant for agents to report their preferences to the mechanics, truthfully.

5.2 A Proof-Of-Concept: Implementing This Roadmap for Chicago Public Schools

To show how this roadmap could be practically implemented, we now use Chicago Public Schools (CPS) data in a proof-of-concept exercise. We first provide some institutional background on the current CPS policies. We then implement each of the three steps of our roadmap to estimate the preferences of CPS, compute the optimal APM, and benchmark the gains from switching to the optimal APM from the current policy.

5.2.1 Institutional Context on Chicago Public Schools

We first describe the institutional context of CPS. Under current policy, CPS admits students to its selective exam schools based on two criteria. First, CPS ranks students according to a

[30] For example, Dur, Kominers, Pathak, and Sönmez (2018) show that the use of an inappropriate precedence order lead to the failure of walk-zone priority in Boston, Dur, Pathak, and Sönmez (2020) show that by changing the precedence order, CPS affected the allocation of fifteen percent of students without changing the quota levels and Pathak, Rees-Jones, and Sönmez (2023) demonstrate that reserve systems face widespread misunderstanding by the public.

score that combines the results of a specialized entrance exam, prior standardized test scores, and grades in prior coursework. This composite score ranges from 0 to 900 and higher-scoring students are admitted before lower-scoring ones, reflecting CPS's desire to allocate seats in exam schools to the students with the best academic standing. In our model, these are the students' scores. Second, CPS divides the census tracts in the city into four *tiers* based on socioeconomic characteristics.[31] Tier 1 tracts are the most disadvantaged, while Tier 4 tracts are the most advantaged. This is reflected in the composite scores of students from Tier 1, who represent 25% of the city's population but comprise relatively few of the high-scoring students (Ellison and Pathak, 2021). As a result, Tier 1 students would have a very small share in the city's top exam schools without affirmative action. To ensure more equal representation across socioeconomic status in these schools, between 2013 and 2017 CPS implemented a quota policy that reserves 17.5% of the seats for each tier, yielding a total of 70% reserve seats and 30% merit slots that are open to students from all tiers. CPS allocates the seats by first assigning the highest-scoring students (regardless of their tier) to the merit slots and then the highest-scoring students from each tier to the 17.5% reserve seats.

We focus on the most selective CPS school, Walter Payton College Preparatory High School (Payton), which has the highest cutoffs for each tier in each year in our data and would have very few tier 1 students without affirmative action.[32] Table 2 presents the cutoff scores of each tier (*i.e.,* the composite score of the last admitted student from each tier).

We make two observations. First, the cutoff students from less advantageous tiers face is lower than the cutoff for students from more advantageous tiers. Therefore, CPS has a *revealed* preference (and not merely a stated preference) for a diverse student body.

Second, cutoff scores vary across years. This implies that the distribution of applicant characteristics varies from year to year. Given this uncertainty and the fact that CPS uses a policy that processes quotas after open slots, we know by Theorem 2 that CPS' baseline policy cannot be rationalized as optimal (even if they are extremely risk-averse). Thus, without even estimating their preferences, we know that the gains from the optimal APM will be strictly positive. Nevertheless, it is always possible that the gains from APM could

[31] Concretely, 800 census tracts are divided into four tiers based on six characteristics of each census tract: (i) median family income, (ii) percentage of single-parent households, (iii) percentage of households where English is not the first language, (iv) percentage of homes occupied by the homeowner, (v) adult educational attainment, and (vi) average Illinois Standards Achievement Test scores for attendance-area schools. These characteristics are then combined to construct the socioeconomic score for the tract. Finally, the tracts are ranked according to socioeconomic scores and partitioned into 4 tiers with approximately the same number of school-age children. See Ellison and Pathak (2021) for a more detailed account of the CPS system.

[32] This approach follows the analysis in Ellison and Pathak (2021), who focus on the two most competitive schools, Northside College Preparatory High School (Northside) alongside Payton. In the years that we study, the cutoff scores for Northside are below some other schools frequently, which is why we restrict attention to Payton.

Table 2: Admissions Cutoff Scores for Payton

Cutoff Score	2013	2014	2015	2016	2017
Tier 1	801	838	784	769	771
Tier 2	845	840	831	826	846
Tier 3	871	883	877	853	875
Tier 4	892	896	891	890	894

Note: The table reports the score of the lowest-scoring student that was admitted to Payton in each of the four tiers and five years.

be small. To quantify the size of these gains and demonstrate what the optimal APM might look like, we now follow our three-step approach.

5.2.2 Step I: Understanding Preferences

We perform our preference estimation in two sub-steps: establishing a parametric class of preferences for the authority and then estimating the parameters.

Preferences. We assume a parsimonious, parametric form for CPS's preferences. In particular, we impose that the preferences of CPS over the scores and diversity of the student body are represented by the following parametric utility function:

$$\xi(\bar{s}, x; \beta, \gamma) = \bar{s} + \sum_{t=1}^{4} \beta |x_t - 0.25|^{\gamma} \tag{20}$$

where $\bar{s}$ is the average score of admitted students and $x_t \in [0, 1]$ is the proportion of tier t students. Motivated by CPS' desire to allocate the highest-scoring students, ξ is increasing in $\bar{s}$. To model the diversity preferences of CPS, we assume that CPS loses as the gap from equal representation (which corresponds to $x_t = 0.25$ of each group) in each tier increases. We do this through the functional form $\beta|x_t - 0.25|^{\gamma}$. The parameters β and γ index the slope and curvature of utility in losses from unequal representation and are the two free parameters of our framework. The preferences that we assume follow Ellison and Pathak (2021), but also allow for a score-diversity trade-off that depends on the level of diversity. Our goal in assuming these preferences is to arrive at some sense of the gains from APM while acknowledging that it is, as in all contexts, impossible to know the parametric class in which the authority's preferences lie. To probe robustness to this functional form assumption, in Appendix C.2 we consider two other parametric utility functions that: i) allow for asymmetric effects of under-representation and over-representation, and ii) only consider losses from under-represented tiers.

Estimation. We estimate β and γ to best rationalize the pursued policy choice of 17.5% reserves for each tier as optimal within the class of all reserve policies. We believe this to be a reasonable approach as the size of the reserves is an important issue that is decided only after much deliberation.[33] Moreover, CPS has used the size of the reserves as a policy tool, increasing them from 15% to 17.5% in 2012 and is currently deliberating another change that would further boost the representation of tier 1 and tier 2 students (Chicago Public Schools, 2022). Given our functional form assumptions, the optimality of the chosen reserve sizes yields moment conditions that we use to estimate the parameters β and γ.

Formally, we index reserve mechanisms by the reserve sizes of the four socioeconomic tiers $r = (r_1, r_2, r_3, r_4)$. We let $\bar{s}(r, y)$ and $x(r, y)$ denote the average scores and tier percentages that would be obtained in year y, with distribution F_y, under reserve policy r. The payoff of the policymaker under reserve policy r is given by $\Xi(r, \Lambda; \beta, \gamma)$, as per Equation 7:

$$\Xi(r, \Lambda; \beta, \gamma) = \mathbb{E}_\Lambda[\xi(\bar{s}(r, y), x(r, y); \beta, \gamma)] \tag{21}$$

where the expectation is taken over distributions of agents' characteristics F_y under the subjective probability measure Λ over such distributions. Define the expected marginal benefit of increasing reserve i and decreasing reserve j as:

$$G_{ij}(r, \Lambda; \beta, \gamma) = \frac{\partial}{\partial r_i}\Xi(r, \Lambda; \beta, \gamma) - \frac{\partial}{\partial r_j}\Xi(r, \Lambda; \beta, \gamma) \tag{22}$$

Any (interior) optimal reserve policy r^* must equate the expected marginal benefit of increasing reserve i and decreasing reserve j at r^* to zero for all (i, j) pairs, *i.e.,* $G_{ij}(r^*, \Lambda; \beta, \gamma) = 0$ for all $\{i, j\} \subset \{1, 2, 3, 4\}$ such that $j > i$. These six first-order conditions yield six moments.

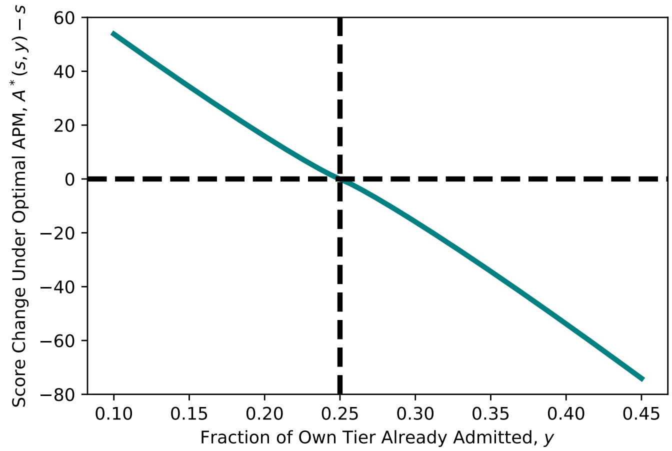
We take empirical analogs of the theoretical moments and estimate preference parameters by minimizing the sum of squared deviations of these moments from zero. We take CPS' pursued reserve policy from 2012 to 2017 as optimal, $\hat{r}^* = (0.175, 0.175, 0.175, 0.175)$. We estimate the empirical joint distribution of students' scores and tiers in CPS in each year $\hat{F}_y$ for $y \in \{2013, 2014, 2015, 2016, 2017\}$. We then estimate the distribution over distributions by setting $\hat{\Lambda}$ as a distribution that places equal probability on each of these five measured distributions, *i.e.,*

$$\mathbb{P}_{\hat{\Lambda}}[F = \tilde{F}] = \begin{cases} \frac{1}{5}, & \text{if } \tilde{F} = \hat{F}_y \text{ for } y \in \{2013, 2014, 2015, 2016, 2017\} \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

We plug these sample estimates into the theoretical moment functions. This yields six

[33] These points are emphasized in Dur, Pathak, and Sönmez (2020): "This change was made at the urging of a Blue Ribbon Commission (BRC, 2011), which examined the racial makeup of schools under the 60% reservation compared to the old Chicago's old system of racial quotas. They advocated for the increase in tier reservations on the basis it would be "improving the chances for students in neighborhoods with low performing schools, increasing diversity, and complementing the other variables."

Figure 4: The Estimated Optimal APM



Note: This figure plots the change in a student's score when fraction y of students in their own tier has already been admitted under the estimated optimal APM, A^*. At $y = 0.25$ (the vertical dashed black line), the score is unchanged. For $y < 0.25$, students receive a score boost. For $y > 0.25$, students receive a score penalty. The range of the x-axis, $[0.1, 0.45]$, is chosen to cover the full range of fractions of admitted students under both the optimal APM and the CPS reserve policy from all tiers in all years of our sample (see Figure 5).

empirical moment functions, $G_{ij}(\hat{r}^*, \hat{\Lambda}; \beta, \gamma)$, that depend only on the preference parameters. Motivated by the theoretical necessity of $G_{ij}(r^*, \Lambda; \beta, \gamma) = 0$, we estimate the preference parameters by minimizing the sum of squared deviations of the empirical moments from zero:

$$(\beta^*, \gamma^*) \in \arg\min_{\beta,\gamma} \sum_{i=1}^{4} \sum_{j>i} G_{ij}(\hat{r}^*, \hat{\Lambda}; \beta, \gamma)^2 \tag{24}$$

Performing this estimation yields estimated parameter values of $\beta^* = -209.5$ and $\gamma^* = 2.11$.

5.2.3 Step II: Computing and Communicating the Optimal APM

In Figure 4, we illustrate how the estimated optimal APM changes students' scores to arrive at their ultimate priorities. In accordance with the preferences we have assumed, students receive a score boost when their tier is underrepresented and a score penalty when their tier is overrepresented. As we found $\gamma^* = 2.11$, the estimated diversity preference is very close to quadratic. Thus, the optimal APM is very close to linear. From Theorems 1 and 2, we know that this APM achieves the first-best allocation in each year while the implemented quota policy does not.

In Table 3, we demonstrate how the estimated optimal APM for CPS can be communicated to stakeholders using the tabular approach presented in Table 1. We restrict attention to the observed proportions of students from each tier $[0.10, 0.45]$ and recalibrate the score

Table 3: The Estimated Optimal APM in a Simple Tabular Format

Walter Payton College Preparatory High School									
Score Boost		Percentage Admitted							
		10	15	20	25	30	35	40	45
Group	Tier 1	+128	+108	+90	+74	+58	+39	+20	+0
	Tier 2	+128	+108	+90	+74	+58	+39	+20	+0
	Tier 3	+128	+108	+90	+74	+58	+39	+20	+0
	Tier 4	+128	+108	+90	+74	+58	+39	+20	+0

Note: The table reports the score boost that an individual in each group receives depending on the number of people that have been admitted from their group, rounded to the nearest integer. As the proportion of admitted students from each group under the Optimal APM has a range in $[0.10, 0.45]$ (see Figure 5), the score boost is calibrated to be zero at the highest level of representation and set to be the maximum value of +128 for representation below 10% and zero above 45%.

boost so that it is equal to zero at the highest level of representation. We believe this recalibration will be useful in communication as it is only required to keep track of bonuses instead of bonuses and penalties, which may look controversial to some stakeholders. The optimal APM corresponds to giving students of each tier a score boost of 128 points out of a possible 900, equivalent to 14.2% of the total available points. This boost decreases to 0 almost linearly as the representation of a group approaches 45% of the student body.

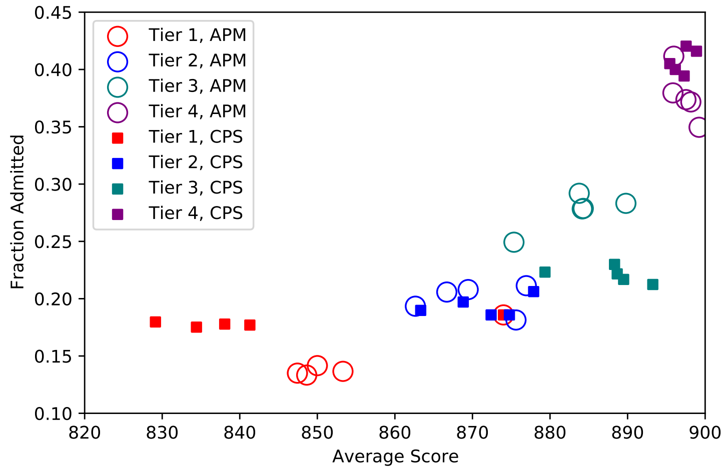
As the assumed preferences treat all groups in the same way, so does the optimal APM, and so the rows of Table 3 are identical. Therefore, the optimal APM treats groups in a symmetric manner and a group receives score boosts relative to another if and only if its members have systematically lower scores. Any realized affirmative action is therefore due to differences in the score distributions of groups of students. As we have highlighted, this contrasts with priority policies that give explicit score boosts to certain groups (that only depend on the group, and not their representation) at the expense of others to affect the allocation.

5.2.4 Step III: Implementing the Optimal APM and Computing its Gains

We now use our estimated model to implement the optimal APM instead of the pursued reserve policy between 2012 and 2017 to quantify the welfare gains from using APM.[34] To do this, we compare the empirical payoff $\Xi(\phi, \hat{\Lambda}, \beta^*, \gamma^*)$ under two mechanisms: the pursued quota policy, r^*, and the optimal APM from Theorem 1, A^*. In words, the empirical payoff from Mechanism ϕ is the average estimated utility (*i.e.,* given β^* and γ^*) of the authority of

[34]As we are studying one authority for simplicity, the implementation is simple and does not require the DA algorithm.

Figure 5: Comparing Admissions under the Optimal APM and the CPS Policy



Note: Each point corresponds to one of the four tiers of students in one of the five years under either the optimal APM or the CPS policy. The x-axis corresponds to the average score of those admitted from that tier in that year under that policy. The y-axis corresponds to the fraction of admitted students from that tier in that year under that policy.

the allocations that ϕ implements under the empirical distribution over distributions (*i.e.*, under $\hat{\Lambda}$).

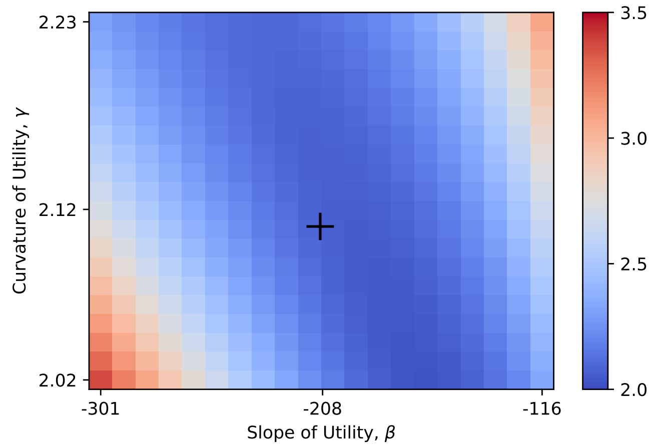
The empirical payoff under APM is 876.9, while it is 874.8 under the CPS reserve policy. Thus, the gains from APM are equivalent to increasing average scores by 2.1, holding diversity fixed. To benchmark the size of the gains, we require units in which they can be meaningfully expressed. To this end, we define the *loss from underrepresentation* as the payoff lost by CPS under its baseline policy from not admitting a fully balanced class, while holding fixed the average score of the class. This is equal to 5.6 points under our estimated parameters. Thus, the gains from APM are equal to 37.5% of the loss from underrepresentation incurred under the CPS policy.[35]

To contextualize the magnitude of this improvement, we can compare the gains from the optimal APM to the gains from the 2012 CPS reform that gave rise to the CPS policy from 2012-2017 and increased the size of all reserves from 15% to 17.5%. Under the estimated preferences, the empirical payoff under the 15% reserve rule is 873.9, and so the gains from the reform are equivalent to increasing average scores by 0.9. Thus, the gains from switching to the optimal APM are 2.3 times larger than the gains from this recent reform.

These estimates suggest that the gains from APM are economically meaningful. These gains stem from the variation across years in the joint distribution of student scores and

[35] This is equivalent to increasing the percentage of students from tier 1 from 0.179 to 0.21 and decreasing the percentage of students from tier 4 from 0.407 to 0.378. This corresponds to swapping 8.7 students from the most overrepresented group (tier 4) for the most underrepresented group (tier 1) each year.

Figure 6: Robustness of the Gains from APM



Note: This chart plots the difference in empirical payoffs from the optimal APM and CPS reserve policy under alternative parameter values, with the shaded colors corresponding to the numerical value of the gains from APM, ranging from 2.0 to 3.5. The black '+' indicates our baseline parameter values. The ranges for the axes are obtained by estimating β and γ separately for each year of our data and separately taking the minimum and maximum estimated values of each set of estimated parameters.

tiers. This can be seen in Table 2, which shows the variability in the scores of the marginally admitted students from tiers 1, 2, and 3. More systematically, we visualize the difference in outcomes under CPS' reserves and the optimal APM by plotting the average scores and fraction admitted for each tier for each year under both mechanisms in Figure 5. There are two main differences between the allocations. First, the APM allocates systematically fewer tier 1 and tier 4 students and more tier 3 students. These level effects are a consequence of the second difference: the APM admits a greater fraction of students from each tier (especially tiers 1 and 3) in the years in which that tier scores well. The fact that tier 3 students score well relative to their admissions level is what leads the authority to admit more tier 3 students and fewer tier 1 students. These positive selection and level effects generate the welfare gains.

5.2.5 Discussion of Robustness and Limitations

Robustness. Finally, we explore the robustness of our estimates to the three core assumptions of our analysis: (i) that CPS has the correct beliefs about the distribution of distributions of students, (ii) that CPS has preferences that lie in the assumed parametric family, and (iii) that CPS optimizes the sizes of all four tiers.

Our baseline analysis took the beliefs of CPS to be the true empirical distribution of student distributions over years. To test robustness to this assumption, we take $\hat{\Lambda}$ as a Dirac distribution on the realized distribution for each of the five years of our data and re-estimate the preference parameters. In Figure 6, we plot the difference in welfare under the optimal APM and CPS reserve policies over the full range of these re-estimated parameters (*i.e.,* we take the minimum and maximum of the estimated parameters across years as the ranges for the axes). We find that the gains from APM range from 2.0 to 3.5, while our baseline estimate was 2.1. Thus, the point estimate of our welfare gains from APM appears to be conservative by this metric.

To gauge robustness to the functional form we have assumed for CPS' preferences, in Appendix C.2, we estimate two different parametric specifications of utility. First, we consider a utility function that includes a loss term only for underrepresented tiers (and does not penalize overrepresentation of any tier). Second, we allow for CPS to care differentially about underrepresentation and overrepresentation by considering a utility function with separate coefficients for underrepresented and overrepresented tiers. We find that, under these specifications, the improvement from APM corresponds to 9.7% and 8.7% of the loss from underrepresentation, which is attenuated relative to our baseline, but remains considerable.

To study the robustness of our findings to the assumption that CPS optimizes the size of all four tiers, in Appendix C.1 we consider a setting where CPS sets a *single* reserve size for all tiers. As we now have only one moment condition, we vary γ over the interval [1,10], estimate $\beta^*(\gamma)$ as the exact solution to the moment condition, and compute the gains from APM as a function of γ. The *minimum* gain from APM over the estimated range is 1.98 points, which corresponds to 26.2% of the loss from underrepresentation under that parameterization. This is slightly smaller than our baseline estimate but still considerable.

Limitations. We conclude our quantitative analysis by stating some limitations. First, even though we argue that our functional form assumptions are reasonable and parsimonious in this setting, there are possibly many other parametric utility functions that might represent the preferences of CPS and these alternative preference structures may give different estimates of the gains from APM. This notwithstanding, we have documented the robustness of our conclusion that APM deliver considerable gains relative to the *status quo* under several alternative estimation methods and preference assumptions. Moreover, if a researcher prefers an alternative structure for preferences, they could estimate the optimal APM and the gains from the optimal APM using precisely the same approach that we developed. Furthermore, given that CPS does face uncertainty, we have shown theoretically that the pursued quota policy (with merit slots processed first) cannot be rationalized as optimal *even* if CPS' preferences are extremely risk averse (by Theorem 2). Thus, while the numerical gains from APM will be sensitive to parametric assumptions on utility and should be treated as a benchmarking exercise, the fundamental conclusion that APM deliver strict gains is not sensitive to such assumptions.

Second, one of the main aims of the tier system employed by CPS is to increase racial diversity in the prestigious exam schools. Indeed, the pursued tier system is a race-neutral alternative that replaced the previous race-based system following two Supreme Court Rulings in 2003 and 2007 (see Ellison and Pathak, 2021, for a summary). Because of this, CPS uses tiers based on socioeconomic status instead of race and so we estimate their preferences over tiers. Of course, if admission rules could depend on race, then one could perform a similar analysis in which APM simply prioritize based on race rather than tier.

Finally, we also note that one key benefit of APM is to yield optimal allocations even under rare circumstances. By assuming that the designer is risk-neutral (rather than risk-averse) over their score-diversity index (*i.e,* g is taken to be linear rather than concave), we are likely to understate the gains from APM in dealing with such situations.

6 Conclusion

Motivated by the use of priority and quota policies in resource allocation settings with diversity concerns, we consider an authority that has preferences over scores and diversity. We introduce Adaptive Priority Mechanisms (APM) and characterize an APM that is both optimal and can be specified solely in terms of the preferences of the authority. We study the priority and quota policies that are used in practice and show that they are optimal if and only if the authority is either risk-neutral or extremely risk-averse over diversity. Moreover, all authorities following the optimal APM implements the unique stable allocation when combined with the standard deferred acceptance algorithm.

Our analysis therefore proposes a new and practical allocation mechanism that provably improves upon existing alternatives and, we argue, can be transparently communicated to stakeholders and implemented using a leading existing approach (the deferred acceptance algorithm). Thus, we argue that APM could be a valuable tool for improving the design of real-world allocation mechanisms in many settings, including the allocation of seats at schools, places at universities, and medical resources to patients. Moreover, we have provided a concrete roadmap for how APM could be designed, communicated, and implemented in practice. Implementing this roadmap, our proof-of-concept analysis using Chicago Public Schools data suggests that the use of APM has the potential to yield considerable welfare gains over the *status quo*.

References

Abdulkadiroğlu, A., Y.-K. Che, and Y. Yasuda (2015): "Expanding 'choice' in school choice," *American Economic Journal: Microeconomics*, 7(1), 1–42.

Abdulkadiroğlu, A. (2005): "College Admission with Affirmative Action," *International Journal of Game Theory*, 33, 535–549.

Abdulkadiroğlu, A., and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Ahani, N., T. Andersson, A. Martinello, A. Teytelboym, and A. C. Trapp (2021): "Placement optimization in refugee resettlement," *Operations Research*, 69(5), 1468–1486.

Angrist, J. D., P. A. Pathak, and R. A. Zárate (2019): "Choice and consequence: Assessing mismatch at chicago exam schools," Discussion paper, NBER.

Athey, S., C. Avery, and P. Zemsky (2000): "Mentoring and diversity," *American Economic Review*, 90(4), 765–786.

Aygün, O., and I. Bó (2021): "College admission with multidimensional privileges: The Brazilian affirmative action case," *American Economic Journal: Microeconomics*, 13(3), 1–28.

Aygün, O., and B. Turhan (2017): "Large-scale affirmative action in school choice: Admissions to IITs in India," *American Economic Review*, 107(5), 210–13.

——— (2020): "Dynamic reserves in matching markets," *Journal of Economic Theory*, 188, 105069.

Azevedo, E. M., and J. D. Leshno (2016): "A supply and demand framework for two-sided matching markets," *Journal of Political Economy*, 124(5), 1235–1268.

Balinski, M., and T. Sönmez (1999): "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, 84, 73–94.

Ben-Akiva, M., D. McFadden, and K. Train (2019): "Foundations of stated preference elicitation: Consumer behavior and choice-based conjoint analysis," *Foundations and Trends in Econometrics*, 10(1-2), 1–144.

Çelebi, O. (2023): "Diversity Preferences, Affirmative Action and Choice Rules," *arXiv preprint arXiv:2310.14442*.

Çelebi, O., and J. P. Flynn (2022): "Priority design in centralized matching markets," *The Review of Economic Studies*, 89(3), 1245–1277.

Chan, J., and E. Eyster (2003): "Does banning affirmative action lower college student quality?," *American Economic Review*, 93(3), 858–872.

Che, Y.-K., J. Kim, and F. Kojima (2019): "Stable matching in large economies," *Econometrica*, 87(1), 65–110.

Chicago Public Schools (2022): "Selective Enrollment Admissions Process: Community Feedback," *Chicago Public Schools*, March 9, 2022.

Combe, J., U. M. Dur, O. Tercieux, C. Terrier, and M. U. Ünver (2022): *Market design for distributional objectives in (re) assignment: An application to improve the distribution of teachers in schools*. MIT Department of Economics.

Delacrétaz, D., S. D. Kominers, and A. Teytelboym (2023): "Matching mechanisms for refugee resettlement," *American Economic Review*, 113(10), 2689–2717.

Dessein, W., A. Frankel, and N. Kartik (2023): "Test-Optional Admissions," *arXiv preprint arXiv:2304.07551*.

Doğan, B. (2016): "Responsive affirmative action in school choice," *Journal of Economic Theory*, 165, 69–105.

Doğan, B., and A. Erdil (2022): "Widening Access in University Admissions," *Available at SSRN 4309391*.

Dur, U., S. D. Kominers, P. A. Pathak, and T. Sönmez (2018): "Reserve design: Unintended consequences and the demise of Boston's walk zones," *Journal of Political Economy*, 126(6), 2457–2479.

Dur, U., P. A. Pathak, and T. Sönmez (2020): "Explicit vs. statistical targeting in affirmative action: theory and evidence from Chicago's exam schools," *Journal of Economic Theory*, p. 104996.

Echenique, F., and M. B. Yenmez (2015): "How to control controlled school choice," *American Economic Review*, 105(8), 2679–94.

Ehlers, L., I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim (2014): "School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds," *Journal of Economic Theory*, 153, 648–683.

Ellison, G., and P. A. Pathak (2021): "The Efficiency of Race-Neutral Alternatives to Race-Based Affirmative Action: Evidence from Chicago's Exam Schools," *American Economic Review*, 111(3), 943–75.

Epple, D., R. Romano, and H. Sieg (2008): "Diversity and affirmative action in higher education," *Journal of Public Economic Theory*, 10(4), 475–501.

Erdil, A., and T. Kumano (2019): "Efficiency and stability under substitutable priorities with ties," *Journal of Economic Theory*, 184, 104950.

Goto, M., F. Kojima, R. Kurata, A. Tamura, and M. Yokoo (2017): "Designing matching mechanisms under general distributional constraints," *American Economic Journal: Microeconomics*, 9(2), 226–62.

Hafalir, I. E., M. B. Yenmez, and M. A. Yildirim (2013): "Effective Affirmative Action in School Choice," *Theoretical Economics*, 8, 325–363.

Imamura, K. (2020): "Meritocracy versus diversity," Discussion paper, Boston College.

Kamada, Y., and F. Kojima (2017): "Stability concepts in matching under distributional constraints," *Journal of Economic Theory*, 168, 107–142.

——— (2018): "Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition," *Theoretical Economics*, 13(2), 761–793.

Kojima, F. (2012): "School Choice: Impossibilities for Affirmative Action," *Games and Economic Behavior*, 75, 685–693.

Kominers, S. D., and T. Sönmez (2016): "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 11(2), 683–710.

Kornbluth, D., and A. Kushnir (2021): "Undergraduate Course Allocation through Competitive Markets," *Available at SSRN 3901146*.

McFadden, D. (1986): "The choice theory approach to market research," *Marketing science*, 5(4), 275–297.

Passaro, D. G., F. Kojima, and B. Pakzad-Hurson (2023): "Equal Pay for Similar Work," *arXiv preprint arXiv:2306.17111*.

Pathak, P. A., A. Rees-Jones, and T. Sönmez (2022): "Immigration lottery design: engineered and coincidental consequences of H-1B reforms," *Review of Economics and Statistics*, pp. 1–43.

——— (2023): "Reversing reserves," *Management Science*.

Pathak, P. A., T. Sönmez, M. U. Unver, and M. B. Yenmez (2021): "Leaving No Ethical Value Behind: Triage Protocol Design for Pandemic Rationing," Discussion paper.

Rostek, M. J., and J. H. Yoon (2023): "Equilibrium theory of financial markets: Recent developments," Working Paper 3710206, SSRN.

Roth, A. E. (1984): "Stability and polarization of interests in job matching," *Econometrica: Journal of the Econometric Society*, pp. 47–57.

——— (1985): "The college admissions problem is not equivalent to the marriage problem," *Journal of economic Theory*, 36(2), 277–288.

Roth, A. E., and M. A. O. Sotomayor (1990): *Two-sided matching: a study in game-theoretic modeling and analysis*. Econometric Society monographs, Cambridge.

Sönmez, T., and M. B. Yenmez (2022a): "Affirmative action in India via vertical, horizontal, and overlapping reservations," *Econometrica*, 90(3), 1143–1176.

——— (2022b): "Constitutional Implementation of Affirmative Action Policies in India," *arXiv preprint arXiv:2203.01483*.

Temnyalov, E. (2023): "An information theory of efficient differential treatment," *American Economic Journal: Microeconomics*, 15(1), 323–358.

Thurstone, L. L. (1931): "The indifference function," *The journal of social psychology*, 2(2), 139–167.

Tuoi Tre News (2017): "Vietnamese college hopefuls miss dream school despite acing national exam," *Tuoi Tre News*, August 4, 2017.

Tversky, A. (1964): "Finite additive structures," *Michigan Mathematical Psychology Program*.

Weitzman, M. L. (1974): "Prices vs. quantities," *The Review of Economic Studies*, 41(4), 477–491.

Westkamp, A. (2013): "An analysis of the German university admissions system," *Economic Theory*, 53(3), 561–589.

For Online Publication

Appendices

A Omitted Proofs

A.1 Proof of Proposition 1

Proof. Part (i): In state ω the payoff from admitting the highest-scoring minority students of measure $x(\omega)$ is:

$$q\omega + (1 + \gamma - \omega)x(\omega) - \frac{1}{2}\left(\frac{1}{\kappa} + \gamma\beta\right)x(\omega)^2 \tag{25}$$

Thus, the $x(\omega)$ that solves the FOC is given by:

$$x(\omega) = \frac{\kappa(1 + \gamma - \omega)}{1 + \kappa\gamma\beta} \tag{26}$$

Under our maintained assumptions, we have that:

$$x(\omega) = \frac{\kappa(1 + \gamma - \omega)}{1 + \kappa\gamma\beta} \leq \frac{1 + \gamma - \underline{\omega}}{\frac{1}{\kappa} + \gamma\beta} + \kappa(\overline{\omega} - \underline{\omega}) < \min\{\kappa, q\} \tag{27}$$

and:

$$x(\omega) = \frac{\kappa(1 + \gamma - \omega)}{1 + \kappa\gamma\beta} \geq \frac{1 + \gamma - \overline{\omega}}{\frac{1}{\kappa} + \gamma\beta} > \kappa(1 - \underline{\omega}) \geq 0 \tag{28}$$

Thus, this level of minority admissions is feasible. Substituting, we have that:

$$V^* = q\mathbb{E}[\omega] + \frac{1}{2}\frac{\mathbb{E}[\kappa(1 + \gamma - \omega)^2]}{1 + \kappa\gamma\beta} \tag{29}$$

Consider now the APM $A(y) = \gamma(1 - \beta y)$. Agents are allocated the resource if their modified scores exceed ω, with a uniform lottery over students with score exactly ω. Thus, in state ω, this policy admits measure $y(\omega)$ minorities that solve the fixed point equation:

$$y(\omega) = \min\left\{\kappa\int_0^1 \mathbb{I}[s + A(y(\omega)) \geq \omega]\mathrm{d}s, q\right\} = \min\{\kappa\left(1 - \max\{\omega - A(y(\omega)), 0\}\right), q\} \tag{30}$$

Denote the RHS of this fixed point equation by the function RHS(y, ω), which is continuous and decreasing in y. Moreover, RHS$(0, \omega) = \min\{\kappa(1 - \max\{\omega - \gamma, 0\}), q\} > 0$ and RHS$(\min\{\kappa, q\}, \omega) < \min\{\kappa, q\}$. The second of these inequalities is true because the condition $\underline{\omega} > \gamma(1 - \beta\min\{q, \kappa\})$ follows from our assumption that $\min\{\kappa, q\} > \frac{1+\gamma-\omega}{\frac{1}{\kappa}+\gamma\beta} + \kappa(\overline{\omega} - \underline{\omega})$. Thus, there exists a unique $y(\omega)$ implemented by the APM. Moreover, let $y_A(\omega)$ denote the unique solution to the equation 30, which gives the measure of admitted minority students under APM A at state ω.

$$\begin{aligned} y_A(\omega) &= \kappa(1-(\omega-\gamma(1-\beta y_A(\omega)))) \\ &= \kappa(1-\omega+\gamma)-\kappa\gamma\beta y_A(\omega) \\ &= \frac{\kappa(1-\omega+\gamma)}{1+\kappa\gamma\beta} \end{aligned} \tag{31}$$

Thus, A implements the optimal level of minority admissions characterized in equation 26 and $V_A = V^*$.

Part (ii): First, if we admit all minority students over some threshold $\hat{s}$, the total score of admitted minority students is $\kappa\int_{\hat{s}}^1 s\mathrm{d}s$. Moreover, when we admit measure x minority students where $x \leq \min\{\kappa, q\}$, this admissions threshold is defined by $x = \kappa\int_{\hat{s}}^1 \mathrm{d}s = \kappa(1-\hat{s})$. Thus, we have that $\hat{s} = 1-\frac{x}{\kappa}$. Finally, the residual measure $q-x$ admitted majority students all score ω. Thus, the total score is given by $\bar{s} = q\omega+(1-\omega)x-\frac{1}{2\kappa}x^2$ for $0 \leq x \leq \min\{\kappa, q\}$. As both quota and priority policies always admit the highest-scoring minority students, the authority's utility is given by:

$$\mathcal{U} = q\mathbb{E}[\omega]+\mathbb{E}[(1+\gamma-\omega)x]-\frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\mathbb{E}[x^2] \tag{32}$$

We now derive the admitted measure of minority students. In the absence of a priority or quota policy, $\alpha = 0$ or $Q = 0$, we have that $x = \kappa(1-\omega)$ measure minority students is admitted. Thus, under a quota policy Q, measure $x = \max\{Q, \kappa(1-\omega)\}$ minority students are admitted. Under a priority policy, the measure of admitted minority students is $x = \kappa\int_{\omega-\alpha}^1 \mathrm{d}x = \kappa(1+\alpha-\omega)$. In each case x is capped by $\min\{\kappa, q\}$ and floored by 0.

The expected utility function over quotas is given by one of four cases. First, $Q > \min\{\kappa, q\}$ and:

$$\mathcal{U}_Q(Q) = q\mathbb{E}[\omega]+(1+\gamma-\mathbb{E}[\omega])\min\{\kappa, q\}-\frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\min\{\kappa, q\}^2 \tag{33}$$

Second, $Q \in [\kappa(1-\underline{\omega}), \min\{\kappa, q\})$ and:[36]

$$\mathcal{U}_Q(Q) = q\mathbb{E}[\omega]+(1+\gamma-\mathbb{E}[\omega])Q-\frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)Q^2 \tag{34}$$

Third, $Q \in (\kappa(1-\overline{\omega}), \kappa(1-\underline{\omega}))$ and:

$$\begin{aligned} \mathcal{U}_Q(Q) = q\mathbb{E}[\omega] &+ \int_{1-\frac{Q}{\kappa}}^{\overline{\omega}}\left((1+\gamma-\omega)Q-\frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)Q^2\right)\mathrm{d}\Lambda(\omega) \\ &+ \int_{\underline{\omega}}^{1-\frac{Q}{\kappa}}\left((1+\gamma-\omega)\kappa(1-\omega)-\frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)(\kappa(1-\omega))^2\right)\mathrm{d}\Lambda(\omega) \end{aligned} \tag{35}$$

[36]By our maintained assumptions we have that this interval has non-empty interior.

Finally, $Q \leq \kappa(1-\overline{\omega})$ and:

$$\mathcal{U}_Q(Q) = q\mathbb{E}[\omega] + \mathbb{E}\left[(1+\gamma-\omega)\kappa(1-\omega)\right] - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\mathbb{E}\left[(\kappa(1-\omega))^2\right] \tag{36}$$

We claim that the optimum lies in the second case. See that in case two the strict maximum is attained at $Q^* = \frac{1+\gamma-\mathbb{E}[\omega]}{\frac{1}{\kappa}+\gamma\beta} \in (\kappa(1-\underline{\omega}), \min\{\kappa, q\})$, by our assumptions that $\min\{\kappa, q\} > \frac{1+\gamma-\omega}{\frac{1}{\kappa}+\gamma\beta} + \kappa(\overline{\omega}-\underline{\omega})$ and $\kappa(1-\underline{\omega}) < \frac{1+\gamma-\overline{\omega}}{\frac{1}{\kappa}+\gamma\beta}$. Moreover, in case three, the first derivative of the payoff is given by:

$$\mathcal{U}_Q'(Q) = \int_{1-\frac{Q}{\kappa}}^{\overline{\omega}} \left((1+\gamma-\omega) - \left(\frac{1}{\kappa}+\gamma\beta\right)Q\right) \mathrm{d}\Lambda(\omega) \tag{37}$$

Thus, checking that the sign of this is positive amounts to verifying that for all $Q \in (\kappa(1-\overline{\omega}), \kappa(1-\underline{\omega}))$, we have that:

$$Q < \frac{1+\gamma-\mathbb{E}[\omega|\omega \geq 1-\frac{Q}{\kappa}]}{\frac{1}{\kappa}+\gamma\beta} \tag{38}$$

As the RHS is an increasing function of Q, it suffices to show that:

$$\kappa(1-\underline{\omega}) < \frac{1+\gamma-\overline{\omega}}{\frac{1}{\kappa}+\gamma\beta} \tag{39}$$

which we have assumed. Moreover, the expected utility in the first case equals $\mathcal{U}_Q(\kappa(1-\underline{\omega}))$, thus is lower than the optimum of the second case. The expected utility in the fourth case equals $\mathcal{U}_Q(\kappa(1-\overline{\omega}))$, thus is lower than the optimum of the third case. We therefore have that:

$$V_Q = q\mathbb{E}[\omega] + (1+\gamma-\mathbb{E}[\omega])Q^* - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)Q^{*2} \tag{40}$$

We now turn to characterizing the value of priorities. There are three cases to consider. First, when $\kappa(1+\alpha-\overline{\omega}) \geq \min\{\kappa, q\}$ we have that $x = \min\{\kappa, q\}$ and:

$$\mathcal{U}_P(\alpha) = q\mathbb{E}[\omega] + (1+\gamma-\mathbb{E}[\omega])\min\{\kappa, q\} - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\min\{\kappa, q\}^2 \tag{41}$$

Second, when $\kappa(1+\alpha-\underline{\omega}) \geq \min\{\kappa, q\} \geq \kappa(1+\alpha-\overline{\omega})$ we have that:

$$\begin{aligned}\mathcal{U}_P(\alpha) = q\mathbb{E}[\omega] &+ \int_{\underline{\omega}}^{1+\alpha-\min\{\frac{q}{\kappa},1\}} \left((1+\gamma-\omega)\min\{\kappa, q\} - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\min\{\kappa, q\}^2\right)\mathrm{d}\Lambda(\omega) \\ &+ \int_{1+\alpha-\min\{\frac{q}{\kappa},1\}}^{\overline{\omega}} \left((1+\gamma-\omega)\kappa(1+\alpha-\omega) - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)[\kappa(1+\alpha-\omega)]^2\right)\mathrm{d}\Lambda(\omega)\end{aligned} \tag{42}$$

Finally, when $\min\{\kappa, q\} \geq \kappa(1+\alpha-\underline{\omega})$, we have that:

$$\mathcal{U}_P(\alpha) = q\mathbb{E}[\omega] + \mathbb{E}[(1+\gamma-\omega)\kappa(1+\alpha-\omega)] - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\mathbb{E}[(\kappa(1+\alpha-\omega))^2] \tag{43}$$

We claim that the optimum under our assumptions lies only in the third case. First, we argue that there is a unique local maximum in the third case. Second, we show the value in the second case is decreasing in α. By continuity, the unique optimum then lies in the third case.

First, it is helpful to write $\bar{x}(\alpha) = \kappa(1 + \alpha - \mathbb{E}[\omega])$ and $\varepsilon = \kappa\,(\mathbb{E}[\omega] - \omega)$. The value in the third case can then be re-expressed as:

$$\begin{aligned}\mathcal{U}_P(\alpha) &= q\mathbb{E}[\omega] + \mathbb{E}[(1+\gamma-\omega)\,(\bar{x}(\alpha)+\varepsilon)] - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\mathbb{E}[(\bar{x}(\alpha)+\varepsilon)^2] \\ &= q\mathbb{E}[\omega] + (1+\gamma-\mathbb{E}[\omega])\bar{x}(\alpha) - \mathbb{E}[\omega\varepsilon] - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\bar{x}(\alpha)^2 - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\mathbb{E}[\varepsilon^2]\end{aligned} \tag{44}$$

Finally, we have that $\mathbb{E}[\varepsilon^2] = \kappa^2\text{Var}[\omega]$ and $\mathbb{E}[\omega\varepsilon] = \text{Cov}[\omega,\varepsilon] = -\kappa\text{Var}[\omega]$. Thus:

$$\mathcal{U}_P(\alpha) = q\mathbb{E}[\omega] + (1+\gamma-\mathbb{E}[\omega])\bar{x}(\alpha) - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)\bar{x}(\alpha)^2 + \frac{\kappa}{2}\,(1-\kappa\gamma\beta)\,\text{Var}[\omega] \tag{45}$$

We then see that the optimal α^* in this range sets $\bar{x}(\alpha^*) = Q^* < \min\{\kappa, q\}$. It remains only to check that this optimal α^* indeed lies within this case, or equivalently that $\kappa(1+\alpha^*-\underline{\omega}) \leq \min\{\kappa, q\}$. To this end, see that $\kappa(1+\alpha^*-\mathbb{E}[\omega]) = Q^*$, and:

$$\begin{aligned}\kappa(1+\alpha^*-\underline{\omega}) &= Q^* + \kappa(\mathbb{E}[\omega]-\underline{\omega}) \leq Q^* + \kappa(\overline{\omega}-\underline{\omega}) \\ &\leq \frac{1+\gamma-\underline{\omega}}{\frac{1}{\kappa}+\gamma\beta} + \kappa(\overline{\omega}-\underline{\omega}) < \min\{\kappa, q\}\end{aligned} \tag{46}$$

where the final inequality follows by our assumption that $\min\{\kappa, q\} > \frac{1+\gamma-\underline{\omega}}{\frac{1}{\kappa}+\gamma\beta} + \kappa(\overline{\omega}-\underline{\omega})$.

Second, in the second case we have that the first derivative of the payoff in α is given by:

$$\begin{aligned}\mathcal{U}_P'(\alpha) &= \int_{1+\alpha-\min\{\frac{q}{\kappa},1\}}^{\overline{\omega}} \frac{d}{d\alpha}\left((1+\gamma-\omega)\kappa(1+\alpha-\omega) - \frac{1}{2}\left(\frac{1}{\kappa}+\gamma\beta\right)[\kappa(1+\alpha-\omega)]^2\right)\mathrm{d}\Lambda(\omega) \\ &= \kappa\int_{1+\alpha-\min\{\frac{q}{\kappa},1\}}^{\overline{\omega}} \left((1+\gamma-\omega) - \left(\frac{1}{\kappa}+\gamma\beta\right)(\bar{x}(\alpha)+\varepsilon(\omega))\right)\mathrm{d}\Lambda(\omega)\end{aligned} \tag{47}$$

Checking that the sign of this is negative for all α such that $\kappa(1+\alpha-\underline{\omega}) \geq \min\{\kappa, q\} \geq \kappa(1+\alpha-\overline{\omega})$ then amounts to checking that:

$$\bar{x}(\alpha) > \frac{1+\gamma-\mathbb{E}[\omega|\omega \geq 1+\alpha-\min\{\frac{q}{\kappa},1\}]}{\frac{1}{\kappa}+\gamma\beta} - \mathbb{E}\left[\varepsilon(\omega)|\omega \geq 1+\alpha-\min\{\frac{q}{\kappa},1\}\right] \tag{48}$$

for all $\bar{x}(\alpha) \in [\min\{\kappa,q\} - \kappa(\mathbb{E}[\omega]-\underline{\omega}), \min\{\kappa,q\} - \kappa(\mathbb{E}[\omega]-\overline{\omega})]$. So it suffices to check that the minimal possible value of the LHS exceeds the maximal possible value of the RHS. A sufficient condition for this is that:

$$\min\{\kappa, q\} - \kappa(\mathbb{E}[\omega] - \underline{\omega}) > \frac{1 + \gamma - \underline{\omega}}{\frac{1}{\kappa} + \gamma\beta} - \kappa(\mathbb{E}[\omega] - \overline{\omega}) \tag{49}$$

Which holds as we assumed that $\min\{\kappa, q\} > \frac{1+\gamma-\underline{\omega}}{\frac{1}{\kappa}+\gamma\beta} + \kappa(\overline{\omega} - \underline{\omega})$. Substituting the optimal priority policy $\bar{x}(\alpha) = Q^*$ in equation 43, we obtain

$$V_P = q\mathbb{E}[\omega] + (1 + \gamma - \mathbb{E}[\omega])Q^* - \frac{1}{2}\left(\frac{1}{\kappa} + \gamma\beta\right) Q^{*2} + \frac{\kappa}{2}(1 - \kappa\gamma\beta)\,\mathrm{Var}[\omega] \tag{50}$$

We have now established that:

$$\Delta = V_P - V_Q = \frac{\kappa}{2}(1 - \kappa\gamma\beta)\,\mathrm{Var}[\omega] \tag{51}$$

Part (iii): We have V^*, V_Q, V_P. Thus, we can compute the loss from restricting to quota policies:

$$\mathcal{L}_Q = \frac{1}{2}\frac{\kappa\mathrm{Var}[\omega]}{1 + \kappa\gamma\beta} \tag{52}$$

To find the loss from restricting to priority policies, we compute:

$$\mathcal{L}_P = \mathcal{L}_Q - \Delta = \frac{1}{2}(\kappa\gamma\beta)^2 \frac{\kappa\mathrm{Var}[\omega]}{1 + \kappa\gamma\beta} \tag{53}$$

Enveloping over these losses yields the claimed formula. □

A.2 Proof of Proposition 2

Proof. Adapting Definition 5 to single object setting, we say that a matching μ admits a cutoff structure if there exists $S(\omega) = \{S_m(\omega)\}_{m\in\mathcal{M}}$ such that $\mu(s, m; \omega) = 1$ if and only if $s \geq S_m(\omega)$. A mechanism admits a cutoff structure if it admits a cutoff structure at every ω. We will first prove that any monotone APM admits a cutoff structure.

Lemma 2. *A monotone APM admits a cutoff structure.*

Proof. For a contradiction, assume it does not. Then there exists ω and matching μ implemented by the monotone APM such that for some $m \in \mathcal{M}$, $s > s'$, $\mu(s, m; \omega) = 0$ but $\mu(s', m; \omega) = 1$. Let x_m denote the measure of group m agents allocated the resource at μ. Since A is a monotone APM and $s > s'$, we have that $A(x_m, s) > A(x_m, s')$, which contradicts that A implements μ. □

We now use Lemma 2 to show that a monotone APM implements a unique allocation. Assume for a contradiction that $A_m(y_m, s)$ is monotone and implements two different allocations, μ and μ'. Let x_m and x'_m denote the measure of type m agents assigned the resource at μ and μ'. First, we prove that if μ and μ' admit the same measure of agents from each group, *i.e.*, $x_m = x'_m$ for all m, then the average score of admitted agents are the same. Let

s_m and s'_m denote the score of the lowest-scoring type m agents assigned the resource at μ and μ'.

Claim 1. *If $x_m = x'_m$ for all $m \in \mathcal{M}$, then $\bar{s}_h(\mu, \omega) = \bar{s}_h(\mu', \omega)$*

Proof. Fix an m. Without loss of generality, let $s_m \geq s'_m$. First, since APM has cutoff structure and $x_m = x'_m$, we have that

$$\int_\Theta \mathbb{I}\{s(\theta) \in [s'_m, s_m], m(\theta) = m\} \mathrm{d}F_\omega(s, m) = 0 \tag{54}$$

Note that this holds regardless of $m' \in \mathcal{M}$ and whether $s_m \geq s'_m$ or $s'_m \geq s_m$. Therefore,

$$\begin{aligned} \bar{s}_h(\mu, \omega) &= \int_\Theta \mu(s, m) h(s) \mathrm{d}F_\omega(s, m) \\ &= \sum_{m \in \mathcal{M}} \int_\Theta \mathbb{I}\{s(\theta) \geq s_m, m(\theta) = m\} h(s(\theta)) \mathrm{d}F_\omega(s, m) \\ &= \sum_{m \in \mathcal{M}} \int_\Theta \mathbb{I}\{s(\theta) \geq s'_m, m(\theta) = m\} h(s(\theta)) \mathrm{d}F_\omega(s, m) \\ &= \int_\Theta \mu(s, m) h(s) \mathrm{d}F_\omega(s, m) \\ &= \bar{s}_h(\mu', \omega) \end{aligned} \tag{55}$$

where line equation holds from Equation 54 and all others are by definition. This finishes the proof of the claim. □

Therefore, if μ and μ' do not yield identical measures, then there are m and n such that $x_m > x'_m$ and $x'_n > x_n$. Since $x_m > x'_m$, it follows that $s'_m > s_m$. Likewise $x'_n > x_n$ implies that $s_n > s'_n$. Note that these imply: (i) $\mu'(s'_n, n) = 1$ while $\mu'(s'_m, n) = 0$ and (ii) $\mu(s_m, m) = 1$ while $\mu(s'_n, n) = 0$. Thus, the following inequalities hold:

$$A_n(s'_n, x'_n) > A_m(s'_m, x'_m) \geq A_m(s_m, x_m) > A_n(s'_n, x_n) \geq A_n(s'_n, x'_n) \tag{56}$$

where the first inequality follows from (i), the second inequality follows from the fact that $x'_m < x_m$ and A is monotone, the third inequality follows from (ii) and the fourth inequality follows from the fact that $x_n < x'_n$ and A is monotone. This equation yields $A_n(s'_n, x'_n) > A_n(s'_n, x'_n)$, which is a contradiction. Therefore, all allocations implemented by A yield the same x. Thus, from Lemma 2, if a monotone APM A implements μ and μ', both allocations admit the highest-scoring measure x_m agents from group m and can differ (at most) on a measure 0 set, proving that all allocations implemented by A are essentially the same. □

A.3 Proof of Theorem 1

Proof. We characterize the optimal allocation for each $\omega \in \Omega$ and show that the claimed adaptive priority mechanism implements the same allocation. Fix an $\omega \in \Omega$ and suppress the dependence of F_ω and f_ω thereon, and define the utility index of a score as $\tilde{s} = h(s)$ with induced densities over $\tilde{s}$ given by $\tilde{f}_m$ for all $m \in \mathcal{M}$. Let the measure of agents from any group $m \in \mathcal{M}$ that is allocated the resource be $x_m \in [0, \bar{x}_m]$ where $\bar{x}_m = \int_{h(0)}^{h(1)} \tilde{f}_m(\tilde{s})\mathrm{d}\tilde{s}$. Observe that, conditional on fixing the measures of agents from each group that are allocated the resource $x = \{x_m\}_{m\in\mathcal{M}}$, there is a unique optimal allocation (*i.e.*, ξ-maximal μ up to measure zero transformations). In particular, as g and h are continuous and strictly increasing, the optimal allocation conditional on x satisfies $\mu^*(\tilde{s}, m; x) = 1 \iff \tilde{s} \geq \underline{\tilde{s}}_m(x_m)$ for some thresholds $\{\underline{\tilde{s}}_m(x_m)\}_{m\in\mathcal{M}}$ that solve:

$$\int_{\underline{\tilde{s}}_m(x_m)}^{h(1)} \tilde{f}_m(\tilde{s})\mathrm{d}\tilde{s} = x_m \tag{57}$$

We can then express the problem of choosing the optimal $x = \{x_m\}_{m\in\mathcal{M}}$ as:

$$\max_{x_m\in[0,\bar{x}_m],\,\forall m\in\mathcal{M}} \sum_{m\in\mathcal{M}} \int_{\underline{\tilde{s}}_m(x_m)}^{h(1)} \tilde{s}\tilde{f}_m(\tilde{s})\mathrm{d}\tilde{s} + \sum_{m\in\mathcal{M}} u_m(x_m) \quad \text{s.t.} \sum_{m\in\mathcal{M}} x_m \leq q \tag{58}$$

where a solution exists by compactness of the constraint sets and continuity of the objective. We can derive necessary and sufficient conditions on the solution(s) to this problem by considering the Lagrangian:

$$\begin{aligned}\mathcal{L}(x, \lambda, \overline{\kappa}, \underline{\kappa}) = &\sum_{m\in\mathcal{M}} \int_{\underline{\tilde{s}}_m(x_m)}^{h(1)} \tilde{s}\tilde{f}_m(\tilde{s})\mathrm{d}\tilde{s} + \sum_{m\in\mathcal{M}} u_m(x_m) \\ &+ \lambda\left(q - \sum_{m\in\mathcal{M}} x_m\right) + \sum_{m\in\mathcal{M}} \overline{\kappa}_m(\bar{x}_m - x_m) + \sum_{m\in\mathcal{M}} \underline{\kappa}_m x_m\end{aligned} \tag{59}$$

The first-order necessary conditions to this program are given by:

$$\frac{\partial\mathcal{L}}{\partial x_m} = -\underline{\tilde{s}}'_m(x_m)\underline{\tilde{s}}_m(x_m)\tilde{f}_m(\underline{\tilde{s}}_m(x_m)) + u'_m(x_m) - \lambda - \overline{\kappa}_m + \underline{\kappa}_m = 0 \tag{60}$$

$$\lambda\frac{\partial\mathcal{L}}{\partial\lambda} = \lambda\left(q - \sum_{m\in\mathcal{M}} x_m\right) = 0 \tag{61}$$

$$\overline{\kappa}_m\frac{\partial\mathcal{L}}{\partial\overline{\kappa}_m} = \overline{\kappa}_m(\bar{x}_m - x_m) = 0 \tag{62}$$

$$\underline{\kappa}_m\frac{\partial\mathcal{L}}{\partial\underline{\kappa}_m} = \underline{\kappa}_m x_m = 0 \tag{63}$$

for all $m \in \mathcal{M}$. By implicitly differentiating Equation 57, we obtain that:

$$-\underline{\tilde{s}}'_m(x_m)\tilde{f}_m(\underline{\tilde{s}}_m(x_m)) = 1 \tag{64}$$

Thus, we can simplify Equation 60 to:

$$\frac{\partial \mathcal{L}}{\partial x_m} = \underline{\tilde{s}}_m(x_m) + u'_m(x_m) - \lambda - \overline{\kappa}_m + \underline{\kappa}_m = 0 \tag{65}$$

Observe that all constraints are linear. Thus, if the objective function is strictly concave, the first-order conditions are also sufficient. Observe by Equation 64 that $\underline{\tilde{s}}_m(x_m)$ is a strictly decreasing function of x_m, and all cross-partial derivatives are zero. Therefore, the first summation is strictly concave. Moreover u'_m is a decreasing function of x_m by virtue of the assumption that u_m is concave for all $m \in \mathcal{M}$. Then the second summation is concave. Thus, the objective function is strictly concave and the optimal allocation is unique.

Thus, to verify that our claimed adaptive priority mechanism is a first-best mechanism, it suffices to show that the allocation it implements satisfies Equations 60 to 63. The adaptive priority mechanism $A_m(y_m, s) = h^{-1}\left(h(s) + u'_m(y_m)\right)$ in the transformed score space yields transformed scores $h\left(A_m(y_m, s)\right) = \tilde{s} + u'_m(y_m)$. Define x_m as the admitted measure of agents from group m under this mechanism. Agents in group $m \in \mathcal{M}$ are allocated the resource if and only if $\tilde{s} + u'_m(x_m) \geq s^C$ for some threshold s^C that solves:

$$\sum_{m \in \mathcal{M}} \int_{\max\{\min\{s^C - u'_m(x_m), h(1)\}, h(0)\}}^{h(1)} \tilde{f}_m(\tilde{s}) \mathrm{d}\tilde{s} = q \tag{66}$$

We can therefore partition $\mathcal{M}$ into three sets that are uniquely defined: (i) interior $\mathcal{M}_I = \{m \in \mathcal{M} | s^C - u'_m(x_m) \in (h(0), h(1))\}$; (ii) no allocation $\mathcal{M}_0 = \{m \in \mathcal{M} | s^C - u'_m(x_m) \geq h(1)\}$; (iii) full allocation $\mathcal{M}_I = \{m \in \mathcal{M} | s^C - u'_m(x_m) \leq h(0)\}$. For all $m \in \mathcal{M}_0$, we implement $x_m = 0$. For all $m \in \mathcal{M}_I$, we implement $x_m = \bar{x}_m$. For all $m \in \mathcal{M}_I$, we implement $x_m \in (0, \bar{x}_m)$. For any $m \in \mathcal{M}_I$, the allocation threshold is $\underline{\tilde{s}}_m(x_m) = s^C - u'_m(x_m)$. For any $m \in \mathcal{M}_0$, the allocation threshold is $h(1)$. For any $m \in \mathcal{M}_I$, the allocation threshold is $h(0)$.

We now verify that this outcome satisfies the established necessary and sufficient conditions. For all $m \in \mathcal{M}_I$, by the complementary slackness conditions we have that $\underline{\kappa}_m = \overline{\kappa}_m = 0$. Substituting the above into Equation 60 for all $m \in \mathcal{M}_I$ we obtain that:

$$s^C - \lambda = 0 \tag{67}$$

which is satisfied for $\lambda = s^C$. As $q = \sum_{m \in \mathcal{M}} x_m$, the complementary slackness condition for λ is then satisfied. For all $m \in \mathcal{M}_0$, by complementary slackness we have that $\overline{\kappa}_m = 0$ and Equation 60 is satisfied by:

$$\underline{\kappa}_m = \lambda - h(1) - u'_m(0) \tag{68}$$

For all $m \in \mathcal{M}_I$, by complementary slackness we have that $\underline{\kappa}_m = 0$ and Equation 60 is satisfied by:

$$\overline{\kappa}_m = h(0) + u'_m(\bar{x}_m) - \lambda \tag{69}$$

This completes the proof of first-best optimality of A^*. Moreover, as the optimal allocation is unique for all ω, any allocation that differs from the allocation implemented by the optimal APM at any ω would not be first-best optimal. Therefore, any first-best-optimal mechanism must implement essentially the same allocation as A^*. □

A.4 Proof of Theorem 2

Proof. First, we prove the if parts of the results. Part (i): When u_m is linear, u'_m is constant and the first-best optimal adaptive priority mechanism is a priority mechanism $P(s,m) = h^{-1}(h(s)+u'_m)$. Part (ii): When $\tilde{u}'_m(x_m) \geq k$ for $x_m \leq x_m^{\text{tar}}$ and $\tilde{u}'_m(x_m) = 0$ for $x_m > x_m^{\text{tar}}$ and $\sum_{m\in\mathcal{M}} x_m^{\text{tar}} < q$, observe that the optimal mechanism admits $x_m \geq x_m^{\text{tar}}$ for all $m \in \mathcal{M}$ in all states of the world, but conditional on $x_m \geq x_m^{\text{tar}}$ for all $m \in \mathcal{M}$ admits the highest-scoring set of agents. A quota $Q_m = x_m^{\text{tar}}$ and $Q_R = q - \sum_{m\in\mathcal{M}} x_m^{\text{tar}}$, with $D(R) = |\mathcal{M}|+1$ implements this allocation and is first-best optimal for any authority that is extremely risk-averse.

Second, we prove the only if parts of the results. Part (i): Assume the utility functions are not linear and let m denote a group where u'_m is not constant in $[0,q]$. We say that a state ω has full support if f_w has full support. A state ω has *full support in m and n* if $f_w(\cdot,m) > 0$ and $f_w(\cdot,n) > 0$ for some m and n and positive measures of only m and n. Let ω be a state that has full support in m and n. Moreover, assume both groups have a measure q of agents. We first establish that in any optimal allocation, agents from both groups are allocated the resource.

Claim 2. *If preferences are non-trivial, then the optimal allocation has $x_n, x_m > 0$.*

Proof. Toward a contradiction, suppose without loss of generality that $x_n = 0$. This implies that $x_m = q$. By the necessary first-order condition from Theorem 1 (combing Equations 60 and 63), we have that:

$$u'_m(q) + h(0) = u'_n(0) + h(1) + \underline{\kappa}_n \geq u'_n(0) + h(1) \tag{70}$$

where the inequality follows as $\underline{\kappa}_n \geq 0$. Thus, we have that:

$$u'_m(q) - u'_n(0) \geq h(1) - h(0) > u'_m(q) - u'_n(0) \tag{71}$$

where the first inequality follows by rearranging Equation 70 and the second follows by the definition of non-triviality of preferences. This is a contradiction, thus $x_n, x_m > 0$ in any optimal allocation. □

We now establish an equation relating x_n and x_m that will be useful in the steps to come.

Claim 3. *Let ω have full support in m and n, μ denote a cutoff matching with cutoffs s_m and s_n. Let x_m and x_n denote the measures of agents who are allocated the object at μ. μ is optimal if and only if $u'_m(x_m) + h(s_m) = u'_n(x_n) + h(s_n)$ and $x_n + x_m = q$.*

Proof. By the necessary and sufficient first-order conditions from Theorem 1, we again have that:

$$u'_m(x_m) + h(s_m) - \overline{\kappa}_m + \underline{\kappa}_m = u'_n(x_n) + h(s_n) - \overline{\kappa}_n + \underline{\kappa}_n \tag{72}$$

By Claim 2, we have $x_m, x_n > 0$. Thus, by the complementary slackness conditions (Equations 62 and 63), we have that $\overline{\kappa}_m = \underline{\kappa}_m = \overline{\kappa}_n = \underline{\kappa}_n = 0$. Thus, we obtain:

$$u'_m(x_m) + h(s_m) = u'_n(x_n) + h(s_n) \tag{73}$$

together with $x_n + x_m = q$, we have characterized the optimal allocation as claimed. □

Continue to let x_m and x_n denote the measures of group m and n agents at the optimal allocation under ω, and s_m and s_n denote the cutoff scores for admission. There are now two cases to consider: (i) $u'_m(x_m)$ and $u'_n(x_n)$ are locally constant. (ii) $u'_m(x_m)$ or $u'_n(x_n)$ are not locally constant. If we are in case (i), we will construct an ω' with a unique optimal allocation x'_m and x'_n where $u'_m(x'_m)$ or $u'_n(x'_n)$ is not locally constant, and then show jointly how we arrive at a contradiction in both cases (i) and (ii).

To this end, suppose that we are in case (i). Let x^*_m and x^*_n denote the measures that are closest to x_m and x_n such that $u'_m(x_m)$ and $u'_n(x_n)$ are not locally constant, *i.e.*,:

$$x^*_k = \arg\min_{x'_k} \left\{ |x_k - x'_k| \middle| u'_k(x_k) = u'_k(x'_k) \text{ and for all } \varepsilon > 0 \right. \\ \left. \text{either } u'_k(x'_k - \varepsilon) > u'_k(x_k) \text{ or } u'_k(x'_k + \varepsilon) > u'_k(x_k) \right\} \tag{74}$$

As u'_k is continuous, this minimum is attained and x^*_k is well-defined. Without loss of generality, assume $|x_m - x^*_m| \leq |x_n - x^*_n|$ and define both $\hat{x}_m = x^*_m$ and $\hat{x}_n = q - x^*_m$. We now construct a state ω' such that $\hat{x}$ is optimal:

Claim 4. *Define ω' where $F_m(1) - F_m(s_m) = \hat{x}_m$, $F_n(1) - F_n(s_n) = \hat{x}_n$ and ω' has full support in m and n. The allocation that admits the highest-scoring $\hat{x}_m$ group m agents and the highest-scoring $\hat{x}_n$ group n agents is the unique optimal allocation.*

Proof. By Claim 3, as $\hat{x}_m + \hat{x}_n = q$ by construction, $\hat{x}$ is optimal if and only if Equation 73 holds. To this end, observe that if we admit $\hat{x}$, then the cutoff scores are the same as under x as $F_m(1) - F_m(s_m) = \hat{x}_m$ and $F_n(1) - F_n(s_n) = \hat{x}_n$, by construction. Thus, we have that:

$$u'_m(\hat{x}_m) + h(s_m) = u'_m(x_m) + h(s_m) = u'_n(x_n) + h(s_n) = u'_n(\hat{x}_n) + h(s_n) \tag{75}$$

where the first equality holds by construction as $\hat{x}_m = x_m^*$ and $u'_m(x_m^*) = u'_m(x_m)$, the second equality holds by optimality of x, and the third equality holds as $|x_m - x_m^*| \leq |x_n - x_n^*|$, which implies that $u'_n(\hat{x}_n) = u'_n(x_n^*)$. Thus, Equation 73 holds, and $\hat{x}$ is optimal, as claimed. □

Observe that this construction also applies trivially in case (ii) with $x_m^* = x_m$. Thus, using this construction, we can now study cases (i) and (ii) together. In state ω', to implement this optimal allocation, we must have that $P(s, m) < P(s_n, n)$ for all but a measure zero set of s such that $s < s_m$. We will now construct another state ω'' such that any priority mechanism with this property is suboptimal.

First, suppose that $x_m^* \leq x_m$ and fix some $\varepsilon \in (0, x_m^*)$. Define $\tilde{s}_m$ as solving the following equation:

$$u'_m(\hat{x}_m - \varepsilon) + h(\tilde{s}_m) = u'_n(\hat{x}_n + \varepsilon) + h(s_n) \tag{76}$$

We then have that:

$$\tilde{s}_m = h^{-1}\left(h(s_n) + u'_n(\hat{x}_n + \varepsilon) - u'_m(\hat{x}_m - \varepsilon)\right) < h^{-1}\left(h(s_n) + u'_n(\hat{x}_n) - u'_m(\hat{x}_m)\right) = s_m \tag{77}$$

where the first equality rearranges Equation 76 and the second inequality uses the facts that $u'_n(\hat{x}_n) \leq u'_n(\hat{x}_n + \varepsilon)$ and $u'_m(\hat{x}_m) < u'_m(\hat{x}_m - \varepsilon)$. We now construct a state ω'' such that $(\hat{x}_m - \varepsilon, \hat{x}_n + \varepsilon)$ is optimal.

Claim 5. *Define ω'' where $1 - F_m(\tilde{s}_m) = \hat{x}_m - \varepsilon$, $1 - F_n(s_n) = \hat{x}_n + \varepsilon$ with full support in m and n. The allocation that admits the highest-scoring $(\hat{x}_m - \varepsilon, \hat{x}_n + \varepsilon)$ agents is the unique optimal allocation.*

Proof. Following the same steps as Claim 4, and the fact that Equation 76 holds by construction, we have that the claim holds. □

Observe that to implement this optimal allocation a priority mechanism must set $P(s, m) \geq P(s_n, n)$ for all but zero measure $s > \tilde{s}_m$. However, since $\tilde{s}_m < s_m$, this contradicts the optimality condition for state ω' that $P(s, m) < P(s_n, n)$. This is because for all but measure zero $s \in (\tilde{s}_m, s_m)$, which we have established is non-empty, we have that:

$$P(s, m) \geq P(s_n, n) > P(s, m) \tag{78}$$

which is a contradiction. To complete the proof, we need only consider the case that $x_m^* > x_m$. In this case, we can apply essentially the same steps and the result follows. Concretely, instead increasing $\hat{x}_m$ by ε and following the same steps yields the required contradiction.

We have now constructed three states $\omega, \omega', \omega''$ such that no priority mechanism can be optimal in each state when the authority is not risk-neural, completing the proof.

Part (ii): Assume that a quota policy is optimal, we now show that the authority's preferences must be extremely risk-averse. For each group $m \in \mathcal{M}$, let $c_m \in [0,1]$ and $c_m \neq c_n$ if $m \neq n$. Let ω be such that the scores of agents from each group m are uniformly distributed between $[c_m, c_m + \epsilon]$, where ϵ is chosen to be small so that there is no overlap of these supports and each group has measure q agents. Let m_ω denote the group with the highest c_m at ω. Now, compute the optimal allocation at ω and denote the measure of admitted agents from each group at the optimal allocation by $\{x^*_m(\omega)\}_{m\in\mathcal{M}}$. We first show that under any optimal quota policy, the level of the quotas must be set equal to the optimal allocation for all but the highest-scoring group:

Claim 6. *If a quota Q attains the optimal allocation, then for each $m \neq m_\omega$, $Q_m = x^*_m(\omega)$.*

Proof. If $Q_m > x^*_m(\omega)$, then we admit $x_m \geq Q_m > x^*_m(\omega)$, which is suboptimal as there is a unique optimal allocation by Theorem 1. If $Q_m < x^*_m(\omega)$ and $m \neq m_\omega$, then $x_m = Q_m$ as $c_{m_\omega} > c_m + \varepsilon$ and no agent from group m can claim a merit slot. This is suboptimal. Thus, $Q_m = x^*_m(\omega)$ for all $m \neq m_\omega$. □

Next, create ω' by changing the highest-scoring group, *i.e.*, $m_\omega \neq m_{\omega'}$. Let $x^*_{m_\omega}(\omega')$ denote the measure of admitted agents from group m_ω under ω'. Applying Claim 6, If Q attains the optimal allocation, then it must be that $Q_{m_\omega} = x^*_{m_\omega}(\omega')$. Define Q^*_m by $Q^*_m = x^*_m$ for all $m \in \mathcal{M} \setminus \{m_\omega\}$ and $Q^*_{m_\omega} = x^*_{m_\omega}(\omega')$.

Now, we have proved that if Q is an optimal policy, then $Q_m = Q^*_m$ for all $m \in \mathcal{M} \setminus \{m_\omega\}$ and $Q_{m_\omega} = Q^*_{m_\omega}$. We now establish that merit slots must be processed after any positive measure quota slots if the merit slots are of positive measure:

Claim 7. *If there is a quota policy that attains the first-best, Q, then $Q_m = Q^*_m$ and either $\sum_{m\in\mathcal{M}} Q^*_m = q$, i.e., there are no merit slots (merit slot processing does not matter), or $\sum_{m\in\mathcal{M}} Q^*_m < q$ and merit slots are processed after any positive measure quota slots.*

Proof. We have already proved $Q_m = Q^*_m$. If $\sum_{m\in\mathcal{M}} Q^*_m = q$, there are no merit slots and any processing order yields the same result. If $\sum_{m\in\mathcal{M}} Q^*_m < q$, for a contradiction, assume merit slots are processed before quota slots for group m and $Q^*_m > 0$. There are two cases, $m \neq m_\omega$ and $m = m_\omega$. We start with the first case. Note that there is a cutoff s_m for group m with $s_m < c_m + \epsilon$ and all agents from group m who score above s_m are allocated the resource. Create ω'' by taking measure $x_m/2$ of these agents who are allocated the resource and give them scores above c_{m_ω} (the highest-scoring group at ω). The scores of the remaining $x_m/2$ agents are distributed uniformly at $[c_m, c_m + \epsilon]$.

We now observe that the optimal allocations at ω and ω'' are the same. This is because increasing the scores of already admitted agents does not change the preferences of the authority of whom to admit. Moreover, the optimal allocation at ω'' cannot be attained if the quota slots for group m are processed after the merit slots. This follows as, if merit slots are processed before quota slots for group m, a strictly positive measure of them would go to group m agents at ω'' since now they have a measure of agents with the highest scores, which violates optimality.

This proves the claim for $m \neq m_\omega$. To prove the result for $m = m_\omega$, replicate the above steps with ω' where m_ω is not the highest-scoring group. □

We now use these claims to establish that if quotas are first-best optimal, then (u, h) must agree with $(\tilde{u}, \tilde{h})$ on optimal allocations.

Claim 8. *The quota first policy with $Q_m = x_m^{tar}$ maximizes the utility with respect to $\tilde{u}, \tilde{h}$.*

Proof. This is clear as for $\tilde{u}, \tilde{h}$, diversity utility dominates until x_m^{tar} and has no effect after. □

This proves the result since if there exists a first-best optimal quota policy, then it is rationalized by $(\tilde{u}, \tilde{h})$ with $x_m^{tar} = Q_m^*$. Hence, if there is a first-best quota mechanism, the authority is extremely risk-averse. □

A.5 Proof of Proposition 3

Proof. Let x_m^* denote the measure of group m agents in the optimal allocation, with $x^* = \{x_m^*\}_{m \in \mathcal{M}}$. A priority policy $P(s, m) = h^{-1}(h(s) + u_m'(x_m^*)) = A_m(x_m^*, s)$ implements the same allocation as the optimal adaptive priority mechanism and by Theorem 1, is optimal. A quota mechanism with (Q, D) where $Q_m = x_m^*$ implements x^* for all D, and is therefore optimal. □

A.6 Proof of Theorem 3

Proof. We first prove the following lemma.

Lemma 3. *Any stable matching is a cutoff matching.*

Proof. Assume that μ is a stable matching. Let $S_{m,c} = \inf_\theta \{s_c(\theta) : m(\theta) = m, \mu(\theta) = c\}$. Since μ satisfies within-group fairness, for all m and $s' > S_{m,c}$, if $m(\theta) = m$ and $s_c(\theta) = s'$, $\mu(\theta) \succeq_\theta c$. Moreover, from part (iv) of the definition of matching, this extends to the case where $s' = S_{m,c}$. Concretely, suppose that $\mu(\theta) \neq c$, $c \succ_\theta \mu(\theta)$ and $s_c(\theta) = S_{m,c}$. Consider a sequence of types $\{\theta_k\}_{k \in \mathbb{N}}$ with common group m and scores $\{s_c(\theta_k)\}_{k \in \mathbb{N}}$ such that $s_c(\theta_k) > S_{m,c}$ for all $k \in \mathbb{N}$ and $s_k(\theta) \to S_{m,c}$. Define the set $\Theta^E = \{\theta \in \Theta : c \succ \mu(\theta)\}$, which must be open by part (iv) of the definition of a matching. We have that $\theta_k \notin \Theta^E$ for all $k \in \mathbb{N}$ but $\lim_{k\to\infty} \theta_k \in \Theta^E$, which contradicts that Θ^E is open. Thus, if μ is stable, then it is also a cutoff matching. □

Therefore, to characterize stable matchings, it is enough to characterize cutoffs that induce a stable matching, which we call *stable cutoffs*.

Definition 6. *A vector S is a market-clearing cutoff if it satisfies the following:*

1. $D_c(S) \leq q_c$ *for all* c.
2. $D_c(S) = q_c$ *if* $S_{m,c} > 0$ *for some* $m \in \mathcal{M}$.

Since an authority can admit different measures of agents from different groups, there is a continuum of cutoffs that clear the market given S_{-c}, as long as $\{(0, \ldots, 0)\}$ is not the only market-clearing cutoff. Let $I(S_{-c})$ denote the set of market-clearing cutoffs. Let $I^*(S_{-c}) \subseteq I(S_{-c})$ denote the unique (by Lemma 2) cutoffs that implement the outcome under APM A_c^* when the authority faces the induced type measure over the set $\tilde{D}_c(S_{-c})$. Define the map $T_c : [0,1]^{|\mathcal{M}|\times|\mathcal{C}|} \to [0,1]^{|\mathcal{M}|}$ as $T_c(S) = I_c^*(S_{-c})$ with $T : [0,1]^{|\mathcal{M}|\times|\mathcal{C}|} \to [0,1]^{|\mathcal{M}|\times|\mathcal{C}|}$ given by $T = \{T_c\}_{c\in\mathcal{C}}$. We first show that the set of fixed points of T equals the set of stable cutoffs and that T is increasing.

Claim 9. *The set of fixed points of T equals the set of stable cutoffs.*

Proof. If S^*, with corresponding matching μ^* (by Lemma 3), is a fixed point of T, then each $c \neq c_0$ admits their most preferred measure q_c agents in $\tilde{D}_c(S^*_{-c})$ (by Theorem 1). Note that any $\hat{\Theta}$ that can block the matching must prefer c to their allocation at μ^* and therefore $\hat{\Theta} \subset \tilde{D}_c(S^*_{-c})$. Then there cannot be a $\hat{\Theta}$ that blocks μ^* at c since c already attains the first-best utility under $\tilde{D}_c(S^*_{-c})$ from the definition of $T_c(S)$ and Theorem 1. Conversely, if S^*, with corresponding matching μ^*, is a not fixed point of T, then there exists c such that $T_c(S^*) \neq S_c^*$. Let $\hat{\Theta}$ denote the set of agents who are not matched to c at μ^* but have scores greater than $T_c(S^*)$, and $\tilde{\Theta}$ denote the set of agents who are matched to c at μ^* but have scores lower than $T_c(S^*)$. From optimality of A_c^* (by Theorem 1), $\hat{\Theta}$ blocks μ^* at c with $\tilde{\Theta}$. □

Claim 10. *T is increasing.*

Proof. Fix an arbitrary $c \in \mathcal{C}$ and suppose that $S'_{-c} \geq S_{-c}$. Toward a contradiction suppose that there exists $m \in \mathcal{M}$ such that $t'_{c,m} = T_{c,m}(S') = I_c^*(S'_{-c}) < I_c^*(S_{-c}) = T_{c,m}(S) = t_{c,m}$, *i.e.,* the admissions threshold for group m at authority c goes down. Let f and f' be the induced joint densities of agents over scores at c and groups by the sets $\tilde{D}_c(S_{-c})$ and $\tilde{D}_c(S'_{-c})$, respectively. Let $\{x_{m,c}\}_{m\in\mathcal{M}}$ and $\{x'_{m,c}\}_{m\in\mathcal{M}}$ denote the measure agents who score above $t_{m,c}$ for their group (*i.e.*, admitted under A_c^*) under $\tilde{D}_c(S_{-c})$ and $\tilde{D}_c(S'_{-c})$, respectively. As $S'_{-c} \geq S_{-c}$, we have that $D^c(S_{-c}, S_c) \subseteq D^c(S'_{-c}, S_c)$ for all $S_c \in [0,1]^{|\mathcal{M}|}$. It follows that $f'(\theta_c) \geq f(\theta_c) > 0$ for all $\theta_c = (s_c, m_c) \in [0,1] \times \mathcal{M}$. As $t'_{c,m} < t_{c,m}$, f' has full support, and $f' \geq f$, we have that the measure of admitted group m agents under increases $x'_{c,m} > x_{c,m}$. But as $\sum_{k\in\mathcal{M}} x'_k = \sum_{k\in\mathcal{M}} x_k = q$, we know that there exists an $m' \in \mathcal{M}$ such that $x'_{c,m'} < x_{c,m'}$. It follows that $t'_{c,m'} > t_{c,m'}$, otherwise, if $t'_{c,m'} \leq t_{c,m'}$, then $x'_{c,m'} \geq x_{c,m'}$. But now we have shown the following:

$$\begin{aligned} h_c(t'_{c,m'}) + u'_{m',c}(x'_{c,m'}) &> h_c(t_{c,m'}) + u'_{m',c}(x_{c,m'}) \\ &\geq h_c(t_{c,m}) + u'_{m,c}(x_{c,m}) > h_c(t'_{c,m}) + u'_{m,c}(x'_{c,m}) \end{aligned} \tag{79}$$

where the first inequality follows by $t_{c,m'} < t'_{c,m'}$, $x_{c,m'} > x'_{c,m'}$, concavity of u_m and strictly increasing h_c. The second inequality follows by optimality. This is because the facts that $t_{c,m} > 0$ and $t'_{c,m'} < 1$ imply that $\overline{\kappa}_m = \underline{\kappa}_{m'} = 0$ and so Equation 65 implies that:

$$h_c(t_{c,m'}) + u'_{m',c}(x_{c,m'}) - \overline{\kappa}_{m'} = h_c(t_{c,m}) + u'_{m,c}(x_{c,m}) + \underline{\kappa}_m \tag{80}$$

with $\overline{\kappa}_{m'}, \underline{\kappa}_m \geq 0$. The final inequality follows as $t'_{c,m} < t_{c,m}$ and $x'_{c,m} > x_{c,m}$. But this contradicts the optimality condition for APM (Theorem 1), which implies that $T_c \neq I_c^*$, which is a contradiction. Hence, for all c and $m \in \mathcal{M}$, $T_{c,m}$ is an increasing function. □

As $T : [0,1]^{|\mathcal{M}|\times|\mathcal{C}|} \to [0,1]^{|\mathcal{M}|\times|\mathcal{C}|}$ is monotone and $[0,1]^{|\mathcal{M}|\times|\mathcal{C}|}$ is a lattice under the elementwise order $\geq$, Tarski's fixed point theorem implies that the set of stable matching cutoffs is a non-empty lattice.

Finally, we use the fact that the set of stable cutoffs is a complete lattice to argue that there is a unique cutoff consistent with stability.

Claim 11. *The stable matching cutoffs are unique.*

Proof. Assume that there are multiple stable cutoffs. As the set of stable cutoffs is a lattice, there exists a largest (S^+) and smallest (S^-) stable cutoffs, where $S^+ \geq S^-$, with strict inequality for some $m \in \mathcal{M}$, $c \in \mathcal{C}$ as $S^+ \neq S^-$. But then, as there is full support of agent types and authority c fills the capacity under stable cutoffs S^+, it must exceed its capacity under S^-, which is a contradiction. Hence, we have shown that there exists a unique stable matching cutoff. □

The combination of Lemma 3 and Claim 11 completes the proof. □

A.7 Proof of Proposition 4

Proof. If ϕ is equivalent to A_c^*, Claim 9 implies that ϕ is consistent with stability.

We prove consistency with stability implies that ϕ is equivalent to A^*_c by the contrapositive. To this end, suppose that ϕ is not equivalent to A^*_c. It follows that there exists a full-support density $\{\tilde{f}(s_c, m)\}_{s_c \in [0,1], m \in \mathcal{M}}$ such that ϕ yields a different allocation than A^*_c under $\tilde{f}$. The rest of the proof constructs a full-support measure F with unique stable matching μ_F such that $\tilde{f}$ is the induced density of scores and groups of the agents who demand authority c at μ_F. Given such an F, we will have that ϕ cannot be consistent with stability as it yields a different allocation than A^*_c, which itself yields $\mu_F(c)$, the set of agents c is matched to in the unique stable matching.

We first define some notation. Given a density f, for any set of types $\check{\Theta} \subseteq \Theta$, we define the marginal density of agents with score $s_c \in [0, 1]$ at authority c in group $m \in \mathcal{M}$ as:

$$f_{\text{marg}(\check{\Theta})}(s_c, m) = \int_{\check{\Theta}} \mathbb{I}[s_c(\theta) = s_c, m(\theta) = m] \, \mathrm{d}F(\theta) \tag{81}$$

To construct such an F, we proceed in three steps. First, take a full-support density f^0 that satisfies the following two conditions: i) Define $\hat{S}_c \in [0, 1]^{|\mathcal{M}|}$ as the cutoff vector that obtains by applying A^*_c to $\tilde{f}$.[37] We assume that f^0 is such that authority c's cutoff vector that is consistent with the unique stable matching, μ_{F_0}, coincides with $\hat{S}_c$; ii) for all $m \in \mathcal{M}$ and $s_c < \hat{S}_{m,c}$, $f^0_{\text{marg}(\Theta)}(s, m) < \tilde{f}(s, m)$; and iii) all authorities have strictly positive cutoffs for all groups at the unique stable matching.

Second, transform f^0 into a new density f^1 that differs from f^0 on the set of types that is matched with c under μ_{F_0}, which we call Θ_c.[38] We define the scaling factor $\iota^1(s_c, m)$ as:

$$\iota^1(s_c, m) = \frac{\tilde{f}(s_c, m)}{f^0_{\text{marg}(\Theta_c)}(s_c, m)} \tag{82}$$

Moreover, we define:

$$f^1(\theta) = \begin{cases} f^0(\theta)\iota^1(s_c(\theta), m(\theta)) & \text{if} \quad \theta \in \Theta_c, \\ f^0(\theta) & \text{otherwise.} \end{cases} \tag{83}$$

This changes the scores of the types who are allocated to c under μ_{F^0} but does not change their total measure, their composition, or their scores at any other authority. Thus, the unique stable matching under f^1, μ_{F^1}, coincides with μ_{F^0}. Moreover, by assumption i) of step 1, we have that $f^1_{\text{marg}(\Theta_c)}(s_c, m) = \tilde{f}(s_c, m)$ for all m and $s_c \geq \hat{S}_{m,c}$.

Third, transform f^1 into a new density f^2 that differs on the set of unmatched agents under f^0 (and also therefore f^1 by step 2), $\tilde{\Theta}$, and define the set of types who strictly prefer c to their assignment under μ_{F^0} (and also therefore μ_{F^1} by step 2), $\hat{\Theta}_c$.[39] We define a new

[37] Which exists as any monotone APM admits a cutoff structure (Lemma 3) and the optimal APM is monotone (Theorem 1).

[38] Formally, $\Theta_c = \{\theta : \theta \in D_c(\mu_{F^0}), s_c(\theta) \geq \hat{S}_{m(\theta),c}\}$.

[39] Formally, $\tilde{\Theta} = \{\theta : \theta \in D_c(\mu_{F^1}), s_c(\theta) < \hat{S}_{m(\theta),c}, s_{c'}(\theta) < S^{\mu_{F^1}}_{m(\theta),c'}$ for all $c' \neq c\}$, where $S^{\mu_{F^1}}_{m,c'}$ denotes

scaling factor $\iota^2(s_c, m)$ as:

$$\iota^2(s_c, m) = \frac{\tilde{f}(s_c, m) - f_{\text{marg}(\hat{\Theta}_c)}(s_c, m)}{f_{\text{marg}(\tilde{\Theta})}(s_c, m)} \tag{84}$$

which is strictly positive by assumption ii) of step 1. We then define f^2 as:

$$f^2(\theta) = \begin{cases} f^1(\theta)(1 + \iota^2(s_c(\theta), m(\theta))) & \text{if} \quad \theta \in \tilde{\Theta}, \\ f^1(\theta) & \text{otherwise.} \end{cases} \tag{85}$$

By construction, $f^2_{\text{marg}(\hat{\Theta}_c)}(s_c, m) = \tilde{f}(s_c, m)$ for all m and $s_c < \hat{S}_{m,c}$. Moreover, $\mu_{F^2} = \mu_{F^1} = \mu_{F^0}$ as all $\theta \in \tilde{\Theta}$ remain unmatched.

We have now constructed a full-support density f^2 with unique stable matching μ_{F^2} (by Theorem 3) such that the density over $D_c(\mu_{F^2})$ coincides with $\tilde{f}$. Moreover, by Claim 9, A^*_c selects $\mu_{F^2}(c)$ from $D_c(\mu_{F^2})$. As ϕ selects a different allocation from $D_c(\mu_{F^2})$ (as it has density of types $\tilde{f}$), it is inconsistent with stability. □

A.8 Proof of Lemma 1

Proof. Fix an arbitrary type $\theta \in \Theta$. Suppose that $DA^t(Q^0)(\theta)$ does not converge to some $r \in \mathcal{R}$. As $\mathcal{R}$ is a finite set, this implies that there exist three time indices $t_0, t_1, t_2 \in \mathbb{N}$ such that $t_0 < t_1 < t_2$ with the property that $DA^{t_0}(Q_0)(\theta) = DA^{t_2}(Q_0)(\theta) \neq DA^{t_1}(Q_0)(\theta)$. Moreover, we know that $DA^{t_1}(Q_0)$ must delete at least the top-ranked element of $DA^{t_0}(Q_0)(\theta)$. As DA only deletes elements, it cannot be true that $DA^{t_0}(Q_0)(\theta) = DA^{t_2}(Q_0)(\theta)$. □

A.9 Proof of Theorem 4

Let $T^t_{m,c}$ denote the step t cut-offs for group m agents at authority c at iteration t of the mapping T (see the proof of Theorem 3 for the formal definition) starting with the market-clearing cutoff being 0 at all authorities. Formally, $T^t(S) = T \circ T^{t-1}(S)$ with initial condition $T^1 = T(0)$. Similarly, let $H^t_{m,c}$ denote the step t cut-offs for group m agents at authority c at step t of the Deferred Acceptance procedure when Ch_c is the optimal APM for all c. Formally, $H^t_{m,c} = \inf_{\theta \in \{\tilde{\theta} \in Ch_c(\mathcal{T}_c(Q^t)) : m(\tilde{\theta}) = m\}} s_c(\theta)$.

It is immediate that $T^1_{m,c} = H^1_{m,c}$ from the definitions and this forms the base case for our proof by induction that $H^t \leq T^t$ for all $t \in \mathbb{N}$. To this end, we now prove the inductive step:

Claim 12. *Suppose that $H^{t'}_{m,c} \leq T^{t'}_{m,c}$ for all m and c for all $t' \leq t$. Then $H^{t+1}_{m,c} \leq T^{t+1}_{m,c}$.*

---

the group m cutoff at authority c' at the stable matching μ_{F^1}, which is strictly positive by assumption iii) of step 2. Moreover, $\hat{\Theta}_c = \{\theta : \theta \in D_c(\mu_{F^1}), s_c(\theta) < \hat{S}_{m(\theta),c}\}$.

Proof. Let $\Theta^{t+1}_{m,c}$ denote the set of agents admitted to c at iteration $t+1$ of the mapping T. Formally

$$\Theta^{t+1}_{m,c} = \{\theta \in D_c(T^t) \text{ and } s_c(\theta) \geq T^{t+1}_{m(\theta),c}\} \tag{86}$$

We now show that no authority receives an application under DA from an agent that is preferable to the agents they have admitted at step $t+1$ of the iterative algorithm T.

Claim 13. *If θ applies to c at step $t+1$ of DA and $s_c(\theta) \geq T^{t+1}_{m(\theta),c}$, then $\theta \in \Theta^{t+1}_{m,c}$.*

Proof. Suppose for a contradiction that θ applies to c at step $t+1$ of DA and $s_c(\theta) \geq T^{t+1}_{m(\theta),c}$ but $\theta \notin \Theta^{t+1}_{m,c}$. By definition, we have that $\theta \notin D_c(T^t)$. Let c' denote the authority such that $\theta \in D_{c'}(T^t)$. By the definition of D, we then have that $s_{c'}(\theta) \geq T^t_{m,c'}$. As T is monotone, $s_c(\theta) \geq T^{t+1}_{m(\theta),c} \geq T^t_{m(\theta),c}$. Thus, both $s_c(\theta) \geq T^t_{m(\theta),c}$ and $s_{c'}(\theta') \geq T^t_{m,c'}$ and both c and c' are attainable for θ at iteration t of T. As θ chooses c' over c, we know that $c' \succ_\theta c$. By the inductive hypothesis, we have that for all $t' \leq t$, $H^{t'}_{m(\theta),c'} \leq T^t_{m(\theta),c'} \leq s_{c'}(\theta)$, θ is not rejected from c' in any previous step of DA. Thus, c' is not deleted from the preference list of θ and θ applies to c' or a more preferred authority at step $t+1$. Thus, θ does not apply to c at step $t+1$ of DA, which is a contradiction. □

Now suppose for a contradiction $H^{t+1}_{m,c} > T^{t+1}_{m,c}$ for some m and c. By monotonicity of T and the inductive hypothesis, we have that $H^{t+1}_{m,c} > T^{t+1}_{m,c} \geq T^t_{m,c} \geq H^t_{m,c}$ and $H^{t+1}_{m,c} > 0$. As F has full support, the set of group m agents that rank c first and have scores in $(H^t_{m,c}, H^{t+1}_{m,c})$ has strictly positive measure. Moreover, all such agents are tentatively admitted at c at step t and rejected from c at step $t+1$. As some positive measure of agents are rejected at step $t+1$, c must fill its capacity at step $t+1$.

Let $\hat{x}^{t+1}_{m,c}$ denote the measure of group m agents admitted to c at step $t+1$ of DA and $\tilde{x}^{t+1}_{m,c}$ denote the measure of agents in $\tilde{D}_c(T^t_{-c})$ with $s_c(\theta) \geq T^{t+1}_{m,c}$. Combining the contradictive hypothesis that $H^{t+1}_{m,c} > T^{t+1}_{m,c}$ with Claim 13, we obtain that $\hat{x}^{t+1}_{m,c} < \tilde{x}^{t+1}_{m,c}$, where the strict inequality follows from the full-support assumption. As c fills its capacity at step $t+1$ of DA, there must be m' such that $\hat{x}^{t+1}_{m',c} > \tilde{x}^{t+1}_{m',c}$.

Moreover, Claim 13 and $\hat{x}^{t+1}_{m',c} > \tilde{x}^{t+1}_{m',c}$ imply that there exists θ' admitted to c at step $t+1$ of DA such that $m(\theta') = m'$ and $s_c(\theta') < T^{t+1}_{m',c}$. Further, as $H^{t+1}_{m,c} > H^t_{m,c}$ and $H^{t+1}_{m,c} > T^{t+1}_{m,c}$, there exists θ that applies to and is rejected by c at step $t+1$ of DA such that $m(\theta) = m$ and $s_c(\theta) \geq T^{t+1}_{m,c}$. Then we must have

$$A^*_{m',c}(\tilde{x}^{t+1}_{m',c}, T^{t+1}_{m',c}) > A^*_{m',c}(\hat{x}^{t+1}_{m',c}, s_c(\theta')) > A^*_{m,c}(\hat{x}^{t+1}_{m,c}, s_c(\theta)) > A^*_{m,c}(\tilde{x}^{t+1}_{m,c}, T^{t+1}_{m,c}) \tag{87}$$

where the first and third inequalities holds by monotonicity of A^* and second holds as θ' is accepted while θ is rejected at step $t+1$ of DA where c was using the optimal APM. Thus, by full support, there exists a strictly positive measure of types with $m(\theta) = m'$ who have scores $s(\theta) \in (A^*_{m,c}(\tilde{x}^{t+1}_{m,c}, T^{t+1}_{m,c}), A^*_{m',c}(\tilde{x}^{t+1}_{m',c}, T^{t+1}_{m',c}))$ and who are not assigned to c at step $t+1$ of the iteration of T. This is a contradiction as these agents must be admitted to c under the APM. □

Having shown that $H^t \leq T^t$ for all $t \in \mathbb{N}$, we now show that H^t is increasing:

Claim 14. *$H^t_{m,c}$ is increasing for all m and c.*

Proof. If c does not fill its capacity at time t, then $H^t_{m,c} = H^{t-1}_{m,c} = \ldots = H^1_{m,c} = 0$ and so $H^t_{m,c} \geq H^{t-1}_{m,c}$. Suppose now that c fills its capacity at time t.

Let $x^t_{m,c}$ denote the measure of agents from group m that are accepted by c at step t of DA. First, if $x^t_{m,c} = x^{t+1}_{m,c}$ for all m, then as all accepted agents reapply in $t+1$ and under the optimal APM, Ch_c chooses higher scoring agents before lower scoring ones, $H^{t+1}_{m,c} \geq H^t_{m,c}$.

Second, suppose that $x^t_{m',c} \neq x^{t+1}_{m',c}$ for some m'. As c fills its capacity, there exists m such that $x^{t+1}_m < x^t_m$. Moreover, as all agents admitted in step t reapply in step $t+1$ of DA, we have that $H^{t+1}_{m,c} > H^t_{m,c}$. Suppose for a contradiction there is an m'' such that $H^{t+1}_{m'',c} < H^t_{m'',c}$. As all agents admitted in step t reapply in step $t+1$, we have $x^{t+1}_{m''} \geq x^t_{m''}$. However, this contradicts that Ch_c is the optimal APM as

$$A^*_{m,c}(x^{t+1}_m, H^{t+1}_{m,c}) > A^*_{m,c}(x^t_m, H^t_{m,c}) = A^*_{m',c}(x^t_{m''}, H^t_{m'',c}) > A^*_{m'',c}(x^{t+1}_{m''}, H^{t+1}_{m'',c}) \tag{88}$$

where the first and third inequalities hold as A^* is a monotone APM and the second equality holds the adaptive priority of the cutoff agent from each group must be equal by the definition of an adaptive priority mechanism. □

Let $T^* = \lim_{t\to\infty} T^t$ denote the cut-offs in the unique stable matching, μ. Claims 12 and 14 imply that $H^t \to H^*$ with $H^* \leq T^*$. Moreover, as $H^t \leq T^*$ for all t, no agent is rejected from their match under μ at any step of DA, which implies that all agents are matched to a weakly more preferred authority under the DA outcome μ^{DA} compared to μ.

Claim 15. *$\mu^{DA} = \mu$.*

Proof. It is enough to show that no θ strictly prefers $\mu^{DA}(c)$ to their match under μ. Suppose that there is a θ such that $c = \mu^{DA}(\theta)$, $c' = \mu(\theta)$, $m(\theta) = m$, and $c \succ_\theta c'$. This implies that $s_c(\theta) < T^*_{m,c}$ as otherwise θ would be matched to c in μ. Fix a $\varepsilon \in (0, \min_{m,c} T^*_{m,c})$, which must exist as $T^* > 0$. Now consider the set of types θ' such that $s_c(\theta') \in (s_c(\theta), T^*_{m,c})$, $s_{\hat{c}}(\theta') < \varepsilon$ for all $\hat{c} \neq c$, $m(\theta') = m$ and c is $\succ_{\theta'}$-maximal. This set of types has strictly positive measure, have $\mu(\theta') = c_0$, and apply to c in the first round of DA. As $c = \mu^{DA}(\theta)$ and $s_c(\theta') > s_c(\theta)$, we have that $c = \mu^{DA}(\theta')$. But as all authorities fill their capacities at T^*, this implies that there exists some positive measure set of types such that $\mu(\tilde{\theta}) = c'' \neq c_0$

and $\mu^{DA}(\tilde{\theta}) = c_0$. But this contradiction as $\mu(\tilde{\theta}) \succ_{\tilde{\theta}} \mu^{DA}(\tilde{\theta})$ and all agents must weakly prefer their assignment under μ^{DA} to their assignment under μ. $\square$

B Additional Results for the Example (Section 2)

B.1 Formal Equivalence Between Prices *vs.* Quantities and Priorities *vs.* Quotas

The structure of the comparative advantage of priorities over quotas from Section 2 hints at a more formal relationship between our analysis of affirmative action policies and Weitzman's analysis of price and quantity regulation. In Weitzman's model, there is a single firm producing a quantity of a single good $x \in \mathbb{R}$ with production costs $C(x, \zeta)$ and benefits $B(x, \zeta')$:

$$\begin{aligned} C(x,\zeta) &= a_0(\zeta) + (C' + a_1(\zeta))(x - \hat{x}) + \frac{C''}{2}(x-\hat{x})^2 \\ B(x,\zeta') &= b_0(\zeta') + (B' + b_1(\zeta'))(x - \hat{x}) + \frac{B''}{2}(x-\hat{x})^2 \end{aligned} \tag{89}$$

where $B', C', C'' > 0$, $B'' < 0$, and ζ and ζ' are random variables. The regulator can either set a price that the firm must charge (after which the firm chooses its optimal production quantity) or mandate the production of a given quantity. The comparative advantage of prices over quantities Δ^{Weitzman} is then defined as the difference between expected benefits net of costs under the optimal price regime minus the corresponding net benefits under the optimal quantity regime. This comparative advantage is given by:

$$\Delta^{\text{Weitzman}} = \frac{C''^{-1}}{2}\left(1 + C''^{-1}B''\right)\text{Var}[a_1(\zeta)] \tag{90}$$

The intuition for this formula is that when benefits are more curved than costs $|B''| > C''$, reducing variability in production is more valuable than the gain of having producers minimize costs. Thus, quantities are preferred. On the other hand, when costs are more curved than benefits, prices are preferred as there is greater production when producers have the lowest marginal costs of production.

These trade-offs are, in a certain sense, formally analogous to those that we have highlighted between priorities and quotas. In particular, under the mapping $C''^{-1} \mapsto \kappa$, $B'' \mapsto -\gamma\beta$, $\text{Var}[\omega] \mapsto \text{Var}[a_1(\zeta)]$, we have that $\Delta^{\text{Weitzman}} = \Delta$. The intuition for this is that C''^{-1} in the Weitzman framework determines how sensitive production is to changes in marginal cost, while κ in our framework determines how sensitive the admitted measure of minority students is to the relative scores. Moreover, B'' corresponds to curvature in the benefits of production while $\gamma\beta$ corresponds to curvature in the benefits of admitting more minority students. Finally, $\text{Var}[a_1(\zeta)]$ corresponds to the authority's uncertainty in the level of marginal costs of production while $\text{Var}[\omega]$ corresponds to the authority's uncertainty regarding the marginal cost of admitting more minority students in terms of lost total score. Thus, the positive selection effect whereby priorities admit more minority students in the states of the world where they score highest is directly analogous to the effect that price regulation gives rise to the greatest production in states where the firm's marginal cost is lowest. Moreover, the guarantee effect whereby quotas prevent variation in the measure of admitted minority students across states of the world is analogous to the ability of quantity regulation to stabilize the level of production. Importantly, our results therefore allow one to apply established price-theoretic intuition for the benefits of price *vs* quantity choice to matching markets without an explicit price mechanism.

B.2 Beyond Affirmative Action: Medical Resource Allocation

The lessons of this paper apply not only to affirmative action in academic admissions, but also more broadly to settings in which centralized authorities must allocate resources to various groups. One prominent such context is the allocation of medical resources during the Covid-19 pandemic. An important issue faced by hospitals is how to prioritize health workers (doctors, nurses and other staff) in the receipt of scarce medical resources: hospitals wish to both treat patients according to clinical need and ensure the health of the frontline workers needed to fight the pandemic. To map this setting to our example, suppose that the score s is an index of clinical need for a scarce medical resource available in amount q, the measure of frontline health workers is κ, and ω indexes the level of clinical need in the patients currently (or soon to be) treated by the hospital, which is unknown. The risk aversion of the authority $\gamma\beta$ corresponds to both a fear of not treating sufficiently many frontline workers and excluding too many clinically needy members of the general population.

In practice, both priority systems and quota policies have been used, as detailed extensively by Pathak, Sönmez, Unver, and Yenmez (2021).[40] The primary concern that has been voiced is that if a priority system is used, some groups (or characteristics) may be completely shut out of allocation of the scarce resource and that this is unethical, so quotas should be preferred. Our framework can be used to understand this argument: if there is an unusually high draw of ω, a priority system would lead to the allocation of very few resources to frontline workers, and vice-versa. Our Proposition 1 implies that if the authority is very averse to such outcomes ($\gamma\beta$ is high), quotas will be preferred and for exactly the reasons suggested. However, we also highlight a fundamental benefit of priority systems in inducing positive selection in allocation: when ω is high, it is beneficial that fewer resources go to the less sick medical workers and more to the relatively sicker general population. More generally,

[40]Some other papers that study the allocation of scarce medical resources are Akbarpour, Budish, Dworczak, and Kominers (2021), Grigoryan (2021) and Dur, Thayer, and Phan (2022).

we argue that an adaptive priority mechanism that awards frontline workers a score subsidy that depends on the number of more clinically needy frontline workers could further improve outcomes.

An important additional consideration in this context arises if the hospital or authority must select a regime (priorities or quotas) before it understands the clinical need of its frontline workers κ, after which it can decide exactly how to prioritize these workers or set quotas, but before ultimate demand for medical resources ω is known. It follows from Proposition 1 that the comparative advantage of priorities over quotas is:

$$\mathbb{E}[\Delta] = \frac{1}{2}\left(\mathbb{E}[\kappa] - (\text{Var}[\kappa] + \mathbb{E}^2[\kappa])\gamma\beta\right)\text{Var}[\omega] \tag{91}$$

Thus, an increase in uncertainty $\text{Var}[\kappa]$ regarding the need of frontline workers leads to a greater preference for quotas. This highlights a further advantage of quotas in settings where a clinical framework must be adopted in the face of uncertainty regarding the clinical needs of frontline workers, as was the case at the onset of the Covid-19 pandemic.

B.3 Optimal Precedence Orders

Thus far we have modelled quotas by first allocating minority students to quota slots and then allocating all remaining students according to the underlying score. However, we could have instead allocated $q - Q$ places to all agents according to the underlying score and then allocated the remaining Q places to minority students. The order in which quotas are processed is called the *precedence order* in the matching literature and their importance for driving outcomes has been the subject of a growing literature (see *e.g.,* Dur, Kominers, Pathak, and Sönmez, 2018; Dur, Pathak, and Sönmez, 2020; Pathak, Rees-Jones, and Sönmez, 2022). Our framework can be used to understand which precedence order is optimal, a question that has not yet been addressed.

In this example, the same factors that determine whether one should prefer priorities or quotas determine whether one should prefer processing quotas second or first. By virtue of uniformity of scores, it can be shown in the relevant parameter range that a priority subsidy of α is equivalent to a quota policy of $\kappa\alpha$ when quotas are processed second. Thus, the comparative advantage of priorities over quotas is exactly equal to the comparative advantage of processing quotas second over first. The intuition is analogous: processing quotas second allows for positive selection while processing quotas first fixes the number of admitted minority students. Thus, on the one hand, when the authority is more risk-averse, they should process quota slots first to reduce the variability in the admitted measure of minority students. On the other hand, when they are less risk-averse, they should process quotas second to take advantage of the positive selection effect such policies induce.

Corollary 1. *The optimal quota-second policy achieves the same value as the optimal priority policy; quota-second policies are preferred to quota-first policies if and only if* $\frac{1}{\kappa} \geq \gamma\beta$.

Proof. We show that a quota-second policy Q is equivalent to a priority subsidy of $\alpha(Q) = \frac{Q}{\kappa}$. A quota-second policy admits the highest-scoring $x = \kappa(1-\omega)+Q$ minority students, floored by zero and capped by $\min\{\kappa, q\}$. A priority policy $\alpha(Q) = \frac{Q}{\kappa}$ admits the highest-scoring $x = \kappa(1 + \alpha(Q) - \omega) = \kappa(1 - \omega) + Q$ minority students, floored by zero and capped by $\min\{\kappa, q\}$. Thus, state-by-state, quota-second policy Q and priority subsidy $\alpha(Q) = \frac{Q}{\kappa}$ yield the same allocation. The claims then follow from Proposition 1. □

We emphasize that this equivalence is a result of the uniform distribution of scores and merely illustrates the similarity between priority policies and processing quotas second. This result does not hold in the more general model we study in the remainder of the paper. Indeed, in Theorem 2, we show that for any quota policy to be optimal in the presence of uncertainty, it must process quotas first.

C Additional Quantitative Results

In this Appendix, we describe both the methodology and results of the two robustness exercises that are not discussed in full detail in the main text. First, we estimate the gains from APM when we assume that CPS sets one tier size for all tiers rather than separately optimizing the sizes of the four tiers. Second, we estimate the gains from APM under alternative utility functions that differentially penalize underrepresentation and overrepresentation.

C.1 Estimation with Homogeneous Reserves

As we have motivated, in this section we estimate an alternative model, where CPS chooses a single reserve size, r, instead of separate reserve sizes for all tiers. Formally, we replace the vector of reserve sizes of the four socioeconomic tiers, $r = (r_1, r_2, r_3, r_4)$ by $r = (r, r, r, r)$. In this setting, we define the marginal benefit of increasing reserve size as

$$G(r, \Lambda; \beta, \gamma) = \frac{\partial}{\partial r}\Xi(r, \Lambda; \beta, \gamma) \tag{92}$$

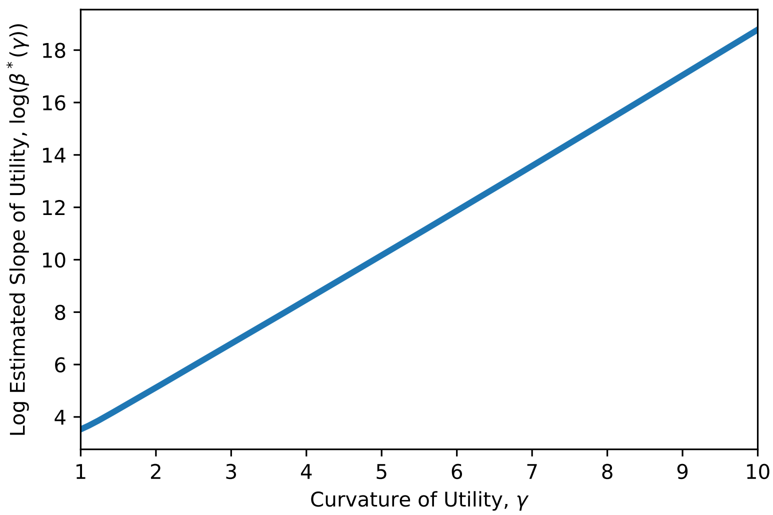
As in the general model, any (interior) reserve policy r^* must satisfy $G(\hat{r}^*, \hat{\Lambda}; \beta, \gamma) = 0$. This first-order condition yields one moment, and so we can estimate one parameter. To this end, we fix γ, and for each $\gamma \in [1, 10]$, and we estimate $\beta^*(\gamma)$ as the exact solution to the following empirical moment condition:

$$G(\hat{r}^*, \hat{\Lambda}; \beta^*(\gamma), \gamma) = 0 \tag{93}$$

Figure 7 plots the logarithm of the estimated $\beta^*(\gamma)$. The estimated $\beta^*(\gamma)$ is increasing in γ. As the loss term $|x_t - 0.25|$ is in $(0, 1)$, $\beta^*(\gamma)$ is increasing and convex in γ, where $\beta^*(1) = 34$ and $\beta^*(10) = 1.436 \times 10^8$. In Figure 8, we plot the gains as a function of γ, which shows that even though the estimated value for β moves quite a lot, the empirical gains range from 2 to 4 points. This also shows that the estimated gain from APM of 2.1 under our benchmark specification is close to the lower bound of the estimated gains under the alternative specification with homogeneous reserves.

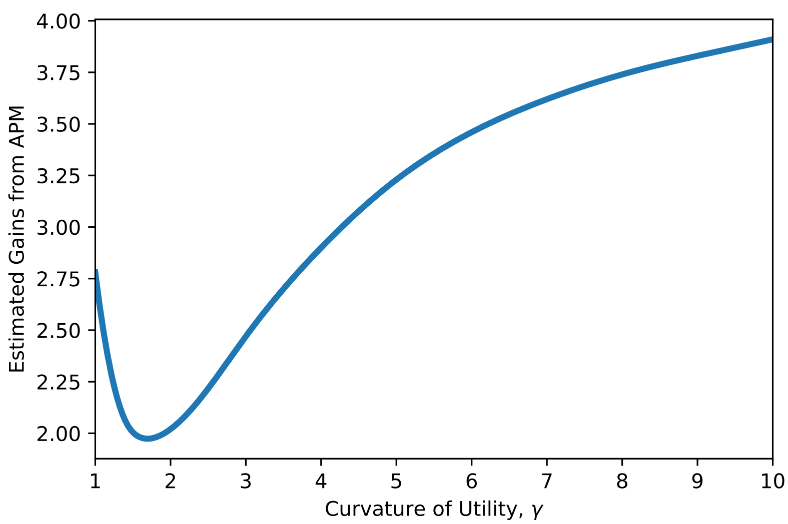
Finally, we benchmark these gains as a fraction of loss from underrepresentation under the CPS policy, where the loss of underrepresentation is calculated under the estimated parameter values. In Figure 9, we plot the gains under APM as a percentage of diversity loss under the CPS policy. These range from 26% to 300%. Our baseline percentage gain estimate of 37.5% is again close to the lower bound that we estimate under the alternative specification with homogeneous reserves.

Figure 7: Estimated Slope of Utility Under Homogeneous Reserves



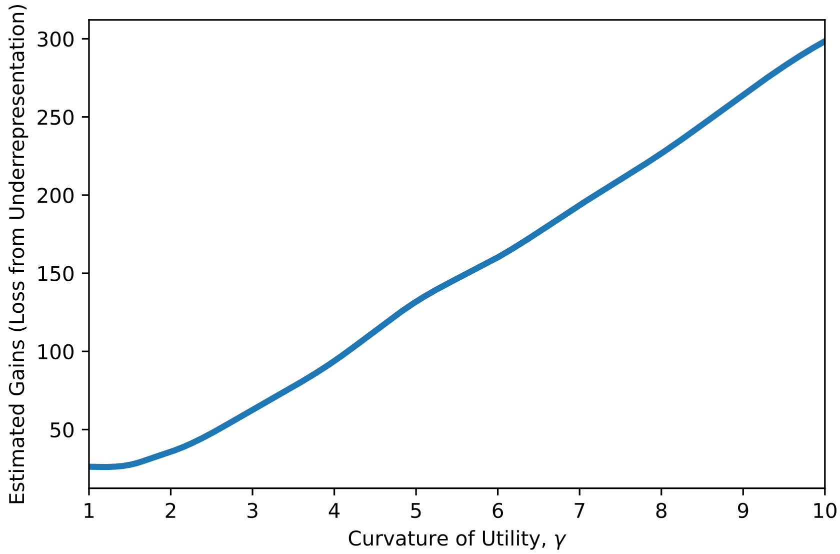
Note: This graph plots the estimated logarithm of the slope of utility $\log \beta^*(\gamma)$ in the homogeneous reserve case as we vary the curvature of utility $\gamma \in [1, 10]$.

Figure 8: Payoff Gains from APM Under Homogeneous Reserves



Note: This graph plots the estimated difference in payoffs in the homogeneous reserves case between the optimal APM and the CPS policy as we vary the curvature of utility $\gamma \in [1, 10]$.

Figure 9: The Gains from APM as a Fraction of the Loss From Underrepresentation Under Homogeneous Reserves



Note: This graph plots the estimated difference between the payoffs under the optimal APM under homogeneous reserves as a fraction of loss from underrepresentation as we vary the curvature of utility $\gamma \in [1, 10]$.

C.2 Gains from APM Under Different Utility Functions

In this section, as we have motivated, we estimate alternative objective functions to investigate the robustness of our findings.

First, we analyze a setting that includes a loss term only for underrepresented tiers (and does not penalize overrepresentation of any tier). To this end, we replace the term $|0.25 - x_t|$ with $\min\{0, (0.25 - x_t)\}$ and perform the same estimation with the following parametric utility function:

$$\xi(\bar{s}, x; \beta, \gamma) = \bar{s} + \beta \sum_{t=1}^{4} (\min\{0, (0.25 - x_t)\})^{\gamma} \tag{94}$$

The estimated parameter values are $\beta^* = -52058$ and $\gamma^* = 3.87467$. We compute the difference between the empirical payoffs under APM and the CPS reserve policy to be 0.262, which is significantly lower than our estimate of 2.1. However, the reason for this is that the diversity domain is estimated to be less important under this specification, and the diversity loss under the CPS policy is 2.71. Thus, improvements from APM correspond to 9.6% of the loss from underrepresentation, which is attenuated relative to our baseline specification, but remains non-negligible.

Second, we allow CPS to care differentially about underrepresentation and overrepresentation by considering a utility function with separate coefficients for underrepresented and overrepresented tiers. To this end, we define the following loss function:

$$f(x_t, \beta_l, \beta_h, \gamma) = \begin{cases} \beta_l(0.25 - x_t)^\gamma & \text{if } x_t \leq 0.25 \\ \beta_h(x_t - 0.25)^\gamma & \text{if } x_t > 0.25 \end{cases} \tag{95}$$

where β_l indexes the loss from underrepresentation of a tier, while β_h indexes the loss from overrepresentation. We then perform the same estimation with the following parametric utility function:

$$\xi(\bar{s}, x; \beta, \gamma) = \bar{s} + \sum_{t=1}^{4} f(x_t, \beta_l, \beta_h, \gamma) \tag{96}$$

This yields the following estimated values: $\beta_l^* = -1362270$, $\beta_h^* = -12278$, $\gamma^* = 5.28021$. We compute the difference between the empirical payoffs under APM and the CPS reserve policy to be 0.195 and the loss from underrepresentation under the CPS policy to be 2.24. Thus, we conclude that improvement from APM corresponds to 8.7% of loss from underrepresentation under the CPS policy, which is similar to what we obtain under the specification in which there is no loss from overrepresentation.

D Extension to More General Authority Preferences

In this Appendix, we relax Assumption 1 to allow for (i) non-separable diversity preferences, (ii) non-separable score and diversity preferences, (iii) non-differentiable preferences, and (iv) non-concave diversity preferences. We show how these changes in assumptions lead to certain modified APM mechanisms becoming first-best optimal.

D.1 Non-Separable Diversity Preferences

First, we relax Assumption 1 and instead suppose that the authority's preferences satisfy the following assumption:

Assumption 2. *The authority's utility function can be represented as:*

$$\xi\left(\bar{s}_h, x\right) \equiv g\left(\bar{s}_h + u(x)\right) \tag{97}$$

for some continuous, strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ and a concave, partially differentiable u in each argument.

In this environment, we define a *non-separable APM* $\tilde{A} = \{\tilde{A}_m\}_{m\in\mathcal{M}}$ where $\tilde{A}_m : \mathbb{R}^{|\mathcal{M}|} \times [0,1] \to \mathbb{R}$. This implements allocation μ in state ω as per Definition 2 (under the modification of point 1 in Definition 2 to allow A_m to depend on x rather than just x_m).

We generalize Theorem 1 to show that the following non-separable APM uniquely implements the first-best optimal allocation:

Proposition 5. *The non-separable APM $\tilde{A}^*_m(y,s) \equiv h^{-1}(h(s) + u^{(m)}(y))$ and uniquely implements the first-best optimal allocation.*[41]

Proof. Follow every step in the proof of Theorem 1 with $\sum_{m\in\mathcal{M}} u_m(x_m)$ replaced by $u(x)$ and $u'_m(x_m)$ replaced by $u^{(m)}(x)$. □

Thus, allowing for non-separable diversity preferences does not substantially change the analysis of adaptive priority mechanisms. One must simply adapt the APM to be non-separable to allow cross-group diversity concerns to shape the marginal benefits of admitting agents from various groups. The main difference is that this a non-separable APM does not necessarily allow the simple implementation of Algorithm 1. This is because, in the presence of cross-group adaptive priorities, it is no longer enough to rank agents within their own group. A small adaptation to this algorithm that dynamically admits agents, starting from the highest-scoring agents in each group, would naturally implement the unique first-best optimal allocation.

[41] Where we define $u^{(m)}(y) = \frac{\partial}{\partial y_m} u(y)$.

D.2 Non-Separable Score and Diversity Preferences

Second, we relax Assumption 1 and instead suppose that the authority's preferences are represented by:

Assumption 3. *The authority's Bernoulli utility function can be represented as:*

$$\xi\left(\bar{s}_h, x\right) \tag{98}$$

where ξ is monotone, differentiable, and concave.

We define a state-dependent APM $\hat{A} = \{\hat{A}_m\}_{m\in\mathcal{M}}$ where $\hat{A}_m : \mathbb{R}^{|\mathcal{M}|} \times [0,1] \times \Omega \to \mathbb{R}$. This implements allocation μ in state ω as per Definition 2 (where point 1 in Definition 2 is modified to allow A_m to depend on both x and ω).

In this more general setting, we now find a state-dependent APM that implements the optimal allocation.

Proposition 6. *The following state-dependent APM implements a first-best optimal allocation:*

$$A_m(y, s, \omega) \equiv h^{-1}\left(h(s) + \frac{\xi_{x_m}\left(\bar{s}_h(y,\omega), y\right)}{\xi_{\bar{s}_h}\left(\bar{s}_h(y,\omega), y\right)}\right) \tag{99}$$

where $\bar{s}_h(y,\omega)$ is the score index in state ω when the highest-scoring $y = \{y_m\}_{m\in\mathcal{M}}$ agents of each attribute are allocated.

Proof. Follow every step in Theorem 1 with $\sum_{m\in\mathcal{M}} \int_{\underline{\tilde{s}}_m(x_m)}^{h(1)} \tilde{s}\tilde{f}_m(\tilde{s})\mathrm{d}\tilde{s} + \sum_{m\in\mathcal{M}} u_m(x_m)$ replaced with $\xi\left(\bar{s}_h(y,\omega), x\right)$ where $\bar{s}_h(y,\omega) = \sum_{m\in\mathcal{M}} \int_{\underline{\tilde{s}}_m(x_m)}^{h(1)} \tilde{s}\tilde{f}_{m,\omega}(\tilde{s})\mathrm{d}\tilde{s}$. □

There are two substantial differences in this optimal policy from our baseline APM. First, the policy depends on the joint distribution of agents in the population. Thus, specifying it *ex ante* is likely to be extremely challenging in any practical setting. This is necessary because the marginal rate of substitution between diversity and scores depends on the level of scores, which depends on the distribution of agents. Second, without assumptions on the shape of the distribution of agents, there is no guarantee that this policy is monotone and thus no guarantee that it implements a unique policy.

Thus, while Proposition 5 showed that cross-group separability is largely inessential for our main conclusions, separability between score and diversity preferences is key to the power of APM.

D.3 Non-Differentiable Preferences

In this section, we retain the majority of Assumption 1, where we instead suppose that the authority's diversity preferences $\{u_m\}_{m\in\mathcal{M}}$ are potentially non-differentiable at finitely many points.

As u_m is concave, the left and right derivatives of u_m, u_m^- and u_m^+, exist. The definition of our first-best APM is not applicable to this case since u_m' might not exist. Therefore, we define the following generalized optimal APM $A_m^*(y_m, s) \equiv h^{-1}(h(s)+u_m^-(y_m))$, which simply replaces u_m' with u_m^- in the definition. By concavity of u_m, u_m^- is monotone decreasing. Thus, this generalized optimal APM (as it is a montone APM) implements a unique allocation by Proposition 2. Moreover, the unique allocation that it implements is an optimal allocation:

Proposition 7. *Let μ^* denote the allocation implemented by the generalized optimal APM. μ^* is an optimal allocation.*

Proof. We first prove a claim. An allocation in this setting is a cutoff allocation if there exists cutoffs $\{s_m\}_{m\in\mathcal{M}}$ such that an agent θ is assigned the resource if and only if $s(\theta) \geq s_m$ and $m(\theta) = m$.

Claim 16. *There exists a unique optimal allocation μ' in the sense that all other allocations that attain the optimal payoff differ from μ' on at most a measure zero set of types. Moreover, there exists an optimal allocation that is a cutoff allocation.*

Proof. In the setting of Theorem 1, observe that $\underline{\tilde{s}}_m(x_m)$ is strictly decreasing in x_m. This, together with the concavity of u implies that the objective is strictly concave and constraints are linear. Therefore an optimal allocation exists and is unique up to measure zero transformations. Given this allocation μ' (with measures x_m), an optimal cutoff allocation is obtained by the cutoff scores s_m^* that satisfy

$$s_m' = \sup\left\{s_m \in [0,1] : \int_{s_m}^1 \tilde{f}_m(\tilde{s})\mathrm{d}\tilde{s} = x_m\right\} \tag{100}$$

□

Using this claim, toward a contradiction, assume there exists another allocation μ', which gives the authority a strictly higher utility. Moreover, take μ' to be an optimal cutoff allocation (which must exist by the claim). As μ' differs from μ^* and both are cutoff allocations, we have that there exist two groups $m, n \in \mathcal{M}$ such that: (i) $s_m' > s_m^*$ and $x_m' < x_m^*$ and (ii) $s_n' < s_n^*$ and $x_n' > x_m^*$. We have that:

$$A_m^*(x, s) \geq A_m^*(x_m^*, s) > A_m^*(x_m^*, s_m^*) \geq A_n^*(x_n^*, \hat{s}) \geq A_n^*(\hat{x}, \hat{s}) \tag{101}$$

for all $s \in (s_m^*, s_m')$, $\hat{s} \in (s_n', s_n^*)$, $x \leq x_m^*$, $\hat{x} \geq x_n^*$. The first inequality follows by concavity of u_m, the second follows by the fact that h is strictly increasing, the third follows by the definition of APM and the fact that $\mu^*(s_m^*, m) = 1$ and $\mu^*(\hat{s}, n) = 0$, and the fourth follows from concavity of u_n. Thus, we have that, for all $s \in (s_m^*, s_m')$, $\hat{s} \in (s_n', s_n^*)$, $x \leq x_m^*$, $\hat{x} \geq x_n^*$:

$$u_m^-(x) + h(s) > u_n^-(\hat{x}) + h(\hat{s}) \tag{102}$$

Thus, the total marginal utility obtained by replacing any positive measure type m agents with scores $s \in (s_m^*, s_m')$ with an identical measure of type n agents with scores $\hat{s} \in (s_n', s_n^*)$ is positive. But this contradicts the optimality of μ'. Thus, if $\tilde{\mu}$ is optimal, then $\tilde{\mu} = \mu^*$ (up to a measure zero set). □

D.4 Non-Concave Preferences

In this section, we relax the assumption that the u_m are concave.

Proposition 8. *If μ is an optimal allocation, then μ is implemented by A^*.*

Proof. Without concavity, the optimal allocation characterized in the proof of Theorem 1 is no longer unique. However, the Lagrangian conditions we have derived are still necessary for any optimal allocation $x = \{x_m\}_{m \in \mathcal{M}}$. Thus, any optimal allocation is implemented by A^*. □

This result shows that any optimal allocation is implemented by the optimal APM. However, when $\{u_m\}_{m \in \mathcal{M}}$ are not concave, A^* is not necessarily monotone. Therefore, A^* does not necessarily implement a unique allocation. Indeed, it is possible that A^* implements suboptimal allocations, as it will implement any locally optimal allocation. Therefore, a mechanism defined by an arbitrary selection from the allocations implemented by A^* would not be first-best optimal. However, A^* may still help decision-making in this setting as it implements any optimal allocation.

E APM Are Dominant Under Decentralized Admissions

In this Appendix, we study which mechanisms are optimal under decentralized admissions. We consider a setting in which the agents apply sequentially to the authorities, who then decide which agents to admit. We index the stage of the game by $t \in \mathcal{T} = \{1, \ldots, |\mathcal{C}| - 1\}$. Each stage corresponds to a (non-dummy) authority $I(t)$, where $I : \mathcal{T} \to \mathcal{T}$. At each stage t, any unmatched agents choose whether to apply to authority $I(t)$. Given the set of applicants, authority $I(t)$ chooses to admit a subset of these agents, who are then matched to the authority. Given this, histories are indexed by the path of the measure of agents who have not yet matched, $h^{t-1} = (F, F_1, \ldots, F_{t-1}) \in \mathcal{H}^{t-1}$. Given each history h^{t-1} and set of applicants $\Theta_c^A \subseteq \Theta$, a strategy for an authority returns a set of agents $\Theta_c^G \subseteq \Theta$ whom they will admit such that $\Theta_c^G \subseteq \Theta_c^A$ and $F_t(\Theta_c^G) \leq q_c$ for each time at which they could move $t \in \mathcal{T}$, $a_{c,t} : \mathcal{H}^{t-1} \times \mathcal{P}(\Theta) \to \mathcal{P}(\Theta)$, where $\mathcal{P}(\Theta)$ is the power set over Θ.[42] A strategy for an agent returns a choice of whether to apply to authorities at each history and time $t \in \mathcal{T}$ for all agent types $\theta \in \Theta$, $\sigma_{\theta,t} : \mathcal{H}^{t-1} \to [0, 1]$.

Within this context, our notion of equilibrium is that of subgame perfect equilibrium:

Definition 7 (Equilibrium)**.** *A strategy profile $\Sigma = \{\{a_{c,t}\}_{c\in\bar{\mathcal{C}}}, \{\sigma_{\theta,t}\}_{\theta\in\Theta}\}_{t\in\mathcal{T}}$ is a subgame perfect equilibrium if $a_{c,t}$ maximizes authority utility given Σ for all $c \in \bar{\mathcal{C}}$ and $t \in \mathcal{T}$ and $\sigma_{\theta,t}$ is maximal according to agent preferences for all $\theta \in \Theta$ and $t \in \mathcal{T}$.*

We moreover say that a strategy $a_{\tilde{c},t}$ for an authority $\tilde{c}$ at time t is *dominant* if it maximizes authority utility regardless of the strategies of all other authorities and agents, $\{\{a_{c,t}\}_{c\in\bar{\mathcal{C}}\setminus\{\tilde{c}\}}, \{\sigma_{\theta,t}\}_{\theta\in\Theta}\}_{t\in\mathcal{T}}$, and the order in which authorities admit agents, I. Moreover, an equilibrium Σ is in *dominant strategies* if $a_{c,t}$ is dominant for all $c \in \bar{\mathcal{C}}$ and $t \in \mathcal{T}$. We denote the unique probabilistic allocation of agents to authorities induced by Σ as $\mu_\Sigma : \Theta \to \Delta(\mathcal{C})$. A probabilistic allocation μ_Σ is deterministic if $\mu_\Sigma(\theta)$ is a Dirac measure on some authority $c \in \mathcal{C}$ for all $\theta \in \Theta$. A deterministic allocation μ_Σ corresponds to a matching μ if $\mu_\Sigma(\theta)$ is a Dirac measure on $\mu(\theta)$ for all $\theta \in \Theta$.

We now establish that the single-authority optimal APM characterizes dominance.

Theorem 5. *A mechanism implements a dominant strategy for an authority if and only if it implements essentially the same allocations as A_c^*.*

[42] Formally, so that $F_t(\Theta_c^G)$ is well defined, we require that authorities' strategies be measurable in the Borel sigma algebra over Θ.

Proof. We prove that APM A_c^* implements a dominant strategy by backward induction. Consider the terminal time $t = |\mathcal{C}| - 1$. Some measure of agents λ applies to the authority. Regardless of the measure λ, by Theorem 1 we have that the APM A_c^* is first-best optimal (to see this more concretely, simply index λ by an arbitrary $\omega \in \Omega$ and apply Theorem 1). Thus, A_c^* is dominant. Moreover, from Theorem 1, any strategy that differs from A_c^* on a strictly positive measure set cannot be optimal. Thus any dominant strategy implements essentially the same allocation as A_c^*. Consider now any time $t < |\mathcal{C}| - 1$, precisely the same argument applies and A_c^* is (essentially uniquely) dominant. □

The intuition behind this result is that each authority takes as given the set of agents that will accept it. Thus, given this measure of agents, they can do no better than to employ the same APM that a single authority would, which is A_c^* by Theorem 1.

Theorem 5 provides a powerful rationale for focusing on APMs in decentralized markets at both positive and normative levels. Normatively, this result allows an analyst to advise an authority regarding how it should conduct its admissions. This is important because any policy that does not coincide with the APM we derive — such as the popular priority and quota mechanisms outside of the cases delimited by Theorem 2 — will disadvantage an authority. Positively, this result allows a sharp prediction that the equilibrium matching between agents and authorities will be the unique stable matching (as per Theorem 3):

Proposition 9. *For all equilibria Σ^* where authorities use A^*, the allocation μ_{Σ^*} is deterministic and corresponds to the unique stable matching of this economy.*

Proof. We first prove the following claim.

Claim 17. *μ_{Σ^*} is (almost surely) a deterministic allocation that corresponds to a cutoff matching μ^*.*

Proof. Since there is a continuum of agents, under any Σ^*, with probability 1, any authority c faces a given set of agents who apply Θ_c^{A,Σ^*} with induced measure $\lambda_c^{\Sigma^*}$. As c uses APM A_c^*, with probability 1, any agent θ is admitted to an authority if and only if $s_c(\theta) \geq S_{m,c}^{\Sigma^*}$, where $S_{m,c}^{\Sigma^*}$ denotes the cutoffs when APM A_c^* is applied to agent measure $\lambda_c^{\Sigma^*}$. Since the agents have strict preferences, in any equilibrium, each agent applies to the $\succeq_\theta$ −maximal authority in $\{c : s_c(\theta) \geq S_{m,c}^{\Sigma^*}\}$, and is admitted, which establishes that μ_{Σ^*} is (almost surely) deterministic allocation that corresponds to a cutoff matching with cutoffs $S_{m,c}^{\Sigma^*}$. □

We now establish that μ_{Σ^*} is the unique stable matching of the economy.

Claim 18. *μ^* is the unique stable matching of this economy.*

Proof. For a contradiction, assume μ_{Σ^*} is not stable. Let S denote the unique cutoffs associated with μ_{Σ^*}. Since μ_{Σ^*} is not stable, by Claim 9, S is not a fixed point of T. Let $t_c = T_c(S)$. Since S is not a fixed point of T, there exists $m \in \mathcal{M}$ and $c \in \mathcal{C}$ such that $t_{m,c} \neq S_{m,c}$. Moreover, let $\{x^t_{m,c}\}_{m\in\mathcal{M}}$ and $\{x^s_{m,c}\}_{m\in\mathcal{M}}$ denote the measure of agents in $\tilde{D}_c(S_{-c})$ who are above the admission thresholds for authority c under t_c and S_c. As in Claim 10, note that if there exists m, c such that $t_{m,c} > S_{m,c}$, then from full support, we have that $x^s_{m,c} > x^t_{m,c}$. Since the authority fills its capacity in both cases, there must exists m' such that $x^t_{m',c} > x^s_{m',c}$ which is only possible if $S_{m,c} > t_{m,c}$. By an identical argument, if there is m, c such that $t_{m,c} < S_{m,c}$, then there exists m' $S_{m',c} < t_{m',c}$. Therefore, whenever $t_{m,c} \neq S_{m,c}$, there exists c and m, m' such that $t_{m,c} > S_{m,c}$ and $S_{m',c} > t_{m',c}$. But now we have shown the following:

$$h_c(S_{c,m'}) + u_{m'}(x^s_{c,m'}) > h_c(t_{c,m'}) + u_{m'}(x^t_{c,m'}) \geq h_c(t_{c,m}) + u_m(x^t_{c,m}) > h_c(S_{c,m}) + u_m(x^s_{c,m})$$

where the first inequality follows by $t_{c,m'} < S_{c,m'}$, $x^s_{c,m'} < x^t_{c,m'}$, and concavity of u_m. The second inequality follows by optimality. This is because the facts that $t_{c,m} > 0$ and $t'_{c,m'} < 1$ imply that the Lagrange multipliers in the proof of Theorem 1 $\overline{\kappa}_m = \underline{\kappa}_{m'} = 0$. The final inequality follows since $t_{m,c} > S_{m,c}$ and $x^t_{m,c} < x^s_{m,c}$. However, this is a contradiction since $h_c(S_{c,m'}) + u_{m'}(x^s_{c,m'}) > h_c(S_{c,m}) + u_m(x^s_{c,m})$ implies that there exists $\varepsilon > 0$, an agent θ with score $s_c(\theta) = S_{c,m'} - \epsilon$ and type $m(\theta) = m$ has higher score under A^* than the agent θ' with score $s_c(\theta') = S_{c,m}$ and type $m(\theta') = m$. Since θ' is admitted to c, θ would be if it applied to c. Moreover, from full support, there is such θ whose top choice is c and the strategy of this agent is not a best response, which is a contradiction. □

The combination of Claims 17 and 18 completes the proof. □

The intuition is that if an equilibrium matching under A^* was not the unique stable matching, then it must be that some agents are applying suboptimally and failing to select the most preferred authority that they can attend, which contradicts that the outcome is consistent with equilibrium. Proposition 9 also shows that the allocation implemented in the dominant strategy equilibrium is the same for all possible orderings of authorities, I.

F Efficient Mechanisms with Multiple Authorities

We have characterized stability and the decentralized outcome (and shown that they are the same), but two natural questions remain. First, is the stable outcome efficient for the authorities?[43] Second, if not, what kind of centralized solution can remedy any inefficiency? We show that the stable outcome is generally inefficient and that a modified, centralized APM mechanism restores efficiency.

F.1 Inefficiency of the Decentralized Outcome

The notion of efficiency that we will consider is utilitarian efficiency over authorities. A mechanism in the multi-authority setting is a function $\phi : \Omega \to \mathcal{U}$, where $\mathcal{U}$ is the set of matchings (which, by definition, encodes the feasibility requirement imposed in the single authority setting). We define the total authority value Ξ_T of a mechanism ϕ under distribution $\Lambda \in \Delta(\Omega)$ as the total expected utility of the allocations induced by that mechanism:

$$\Xi_T(\phi, \Lambda) = \sum_{c \in \bar{\mathcal{C}}} \Xi_c(\phi, \Lambda) \tag{103}$$

A mechanism is efficient if it maximizes total authority value for all possible distributions:

Definition 8 (Efficiency)**.** *A mechanism ϕ^* is efficient if:*

$$\Xi_T(\phi^*, \Lambda) = \sup_{\phi} \Xi_T(\phi, \Lambda) \tag{104}$$

for all $\Lambda \in \Delta(\Omega)$.

For this section, so that scores are directly comparable across authorities and allocations are interior, we impose the following assumption:

Assumption 4. *Scores and preferences are such that $s_c(\theta) = s_{c'}(\theta)$, $h_c = h$ and $g_c = Id$, where Id is the identity function, for all $c, c' \in \bar{\mathcal{C}}$ and $\theta \in \Theta$. Moreover, $\lim_{x \to^+ 0} u'_{m,c}(x) = \infty$, $u_{m,c}$ is strictly concave for all $m \in \mathcal{M}$ and $c \in \bar{\mathcal{C}}$, and for all $\theta \in \Theta$, c_0 is less preferred than c for all $c \in \bar{\mathcal{C}}$.*

Assumption 4 makes scores a common numeraire across authorities and is akin to the standard quasi-linearity assumption in mechanism design. For example, it may be suitable in settings where the score is derived from a common index of academic attainment, such as in Chicago Public Schools. This assumption does not impose that all authorities have common marginal rates of substitution between scores and diversity, as they are allowed unrestricted

[43] We have shown that the decentralized outcome corresponds to the unique stable matching. Naturally, stable allocations need not be efficient for the agents. Here, we will see if they are efficient for the authorities.

heterogeneity in diversity preferences. We add the Inada condition for analytical tractability. We argue that it is also reasonable to assume that failing to admit any individuals from a given group is intolerable for authorities. We add that the outside option is ranked lower than all authorities so that any agent is willing to be assigned to any of the authorities. When authorities control highly desirable resources, such as elite school or university seats or essential medical supplies, we argue that this is a reasonable assumption.

With the efficiency benchmark defined, we now demonstrate that the decentralized equilibrium outcome can fail to be efficient. We prove this result via an explicit example with two authorities, c and c' of capacity $\frac{1}{4}$, and two groups of agents, m and m' of measure $\frac{1}{2}$. All agents in group m prefer c' to c and all agents in group m' prefer c to c'. Authority c values admitting group m agents more on the margin than group m' agents, and authority c' values admitting group m' agents more on the margin than group m agents. Using the optimal APMs, both authorities admit more agents of the group whose admissions they value relatively less than the efficient benchmark. The intuition for this is that both authorities "steal" the high-scoring agents of the group whom they relatively less value from the other authority, an externality that they do not internalize.

Proposition 10 (Equilibrium Inefficiency)**.** *All authorities using the privately optimal APMs* $\{A_c^*\}_{c\in\mathcal{C}}$ *is not necessarily efficient.*

Proof. We prove the result by explicitly constructing an economy in which the optimal APMs lead to inefficiency. There are two authorities, c and c', both with capacity $1/2$ and two groups of agents, m and m'. Both agent groups have a measure of 1 and their scores are uniformly distributed in $[1/2, 1]$. Authorities' utility functions are given by

$$\xi_c\left(\bar{s}_h, x\right) \equiv \bar{s}_h + \frac{1}{4}\sqrt{x_m} + \frac{1}{8}\sqrt{x_{m'}} \tag{105}$$

$$\xi_{c'}\left(\bar{s}_h, x\right) \equiv \bar{s}_h + \frac{1}{4}\sqrt{x_{m'}} + \frac{1}{8}\sqrt{x_m} \tag{106}$$

with $h(x) \equiv x$ while all agents of type m prefer authority c' to c while all agents of type m' prefer authority c to c'.[44]

We will now derive the stable outcome of this economy, which is (up to measure zero transformation) the unique outcome implemented when the authorities use the optimal APM. Let x_m^c denote the measure of type m agents at authority c. First, note that higher-scoring agents from the same group go to the more preferred authority. To see why this is true,

[44]This assumption on the preferences and the distribution of scores violate our full support assumption, but adding an arbitrarily small full support density to all types makes arbitrarily small changes in the utility under the stable matching and optimal allocation but complicates the calculation, so we omit it for expositional clarity.

note that if $m(\theta) = m(\theta') = m$, $s(\theta) > s(\theta')$ and $\mu(\theta) = c$ while $\mu(\theta') = c'$, c and θ would violate within group fairness since θ has higher priority at c than θ' regardless of the allocation. As a result, in any stable allocation μ, the highest-scoring $x^c_{m'}$ type m' agents are assigned to c and the next highest-scoring $x^{c'}_{m'}$ agents are assigned to authority c', while rest of the type m' agents are not assigned to any authority. The allocation for type m agents is analogous. Moreover, since $q = 1/2$ for both authorities, $x^{c'}_{m'}$=1/2-$x^c_{m'}$ and x^c_m=1/2-$x^{c'}_m$ and the allocation is completely determined by the measures $x^c_{m'}$ and $x^{c'}_m$.

Next, note that at μ, the adaptive priority of the lowest-scoring type m and m' agents must be equal at both authorities. To see why this is true, take authority c without loss of generality. Let $s^c_{m'} = 1 - x^c_{m'}$ and $s^c_m = 1 - x^c_{m'} - x^c_m$ denote the scores of the lowest-scoring type m and m' agents and A_m denote the optimal APM. For a contradiction, assume $A_m(x^c_m, s^c_m) > A_{m'}(x^c_{m'}, s^c_{m'})$. Since agents of type m' with scores lower than s^c_m are unassigned at μ, for small enough ϵ, a type m agent with score $s^c_m - \epsilon$ and authority c blocks the matching. Similarly, assume $A_m(x^c_m, s^c_m) < A_{m'}(x^c_{m'}, s^c_{m'})$. Since agents of type m' with scores lower than $s^{c'}_m$ are assigned to authority c or unmatched at μ, a type m' agent with score $s^c_{m'} - \epsilon$ and authority c blocks the matching μ. Thus, the following equations must be satisfied:

$$A_m(x^c_m, s^c_m) = A_{m'}(x^c_{m'}, s^c_{m'}) \text{ and } A_m(x^{c'}_m, s^{c'}_m) = A_{m'}(x^{c'}_{m'}, s^{c'}_{m'}) \tag{107}$$

As the optimal APM in this setting is given by:

$$A^*_{\hat{m},\hat{c}}(y_{\hat{m}}, s) \equiv s + u'_{\hat{m},\hat{c}}(y_{\hat{m}}) \tag{108}$$

for all $\hat{m} \in \{m, m'\}$ and $\hat{c} \in \{c, c'\}$, we have that:

$$1 - x^c_{m'} + \frac{1}{8}\frac{1}{\sqrt{x^c_{m'}}} = 1 - x^c_{m'} - x^c_m + \frac{1}{4}\frac{1}{\sqrt{1/2 - x^c_{m'}}} \tag{109}$$

and:

$$1 - x^{c'}_m + \frac{1}{8}\frac{1}{\sqrt{x^{c'}_m}} = 1 - x^{c'}_m - x^{c'}_{m'} + \frac{1}{4}\frac{1}{\sqrt{1/2 - x^{c'}_m}} \tag{110}$$

These equations are identical up to relabelling and so $x^c_{m'} = x^{c'}_{m'} = x^*$ for some x^*. Thus, we need to find the solution to the following single equation to characterize the allocation:

$$1 - x^* + \frac{1}{8}\frac{1}{\sqrt{x^*}} = \frac{1}{2} + \frac{1}{4}\frac{1}{\sqrt{1/2 - x^*}} \tag{111}$$

Observe that this equation can be rewritten as the fixed point equation:

$$x^* = \frac{1}{2} + \frac{1}{8}\frac{1}{\sqrt{x^*}} - \frac{1}{4}\frac{1}{\sqrt{1/2 - x^*}} \tag{112}$$

We observe that the RHS satisfies the following properties: (i) $\lim_{x^*\to 0} \text{RHS}(x^*) = \infty$, (ii)

$\lim_{x^* \to \frac{1}{2}} \text{RHS}(x^*) = -\infty$, and (iii) $\text{RHS}'(x^*) < 0$ for all $x^* \in (0, \frac{1}{2})$. Thus, there exists a unique solution. Moreover, we can guess-and-verify that this solution is $x^* = \frac{1}{4}$.

In summary, if both authorities use the optimal APM, then the outcome is

$$\mu(\theta) = \begin{cases} c & \text{if } m(\theta) = m, s(\theta) \in [1/2, 3/4) \text{ or } m(\theta) = m', s(\theta) \in [3/4, 1] \\ c' & \text{if } m(\theta) = m', s(\theta) \in [1/2, 3/4) \text{ or } m(\theta) = m, s(\theta) \in [3/4, 1] \\ \theta & \text{otherwise} \end{cases} \tag{113}$$

In this outcome, both authorities have an average score of 3/4 and admit measure 1/4 agents from both groups, giving them a utility of 15/16. Thus, total utilitarian welfare is 15/8 under the decentralized outcome.

We now show that this does not attain the efficient benchmark. A necessary condition for the (utilitarian) efficient outcome is that for c:

$$\frac{1}{4} \frac{1}{\sqrt{x_m^c}} = \frac{1}{8} \frac{1}{\sqrt{1/2 - x_m^c}} \tag{114}$$

and for c':

$$\frac{1}{4} \frac{1}{\sqrt{x_{m'}^{c'}}} = \frac{1}{8} \frac{1}{\sqrt{1/2 - x_{m'}^{c'}}} \tag{115}$$

This implies that $x_m^c = x_{m'}^{c'} = 4/10$ and $x_{m'}^c = x_m^{c'} = 1/10$. In this case, the same set of agents is admitted overall, so the score contribution to utility remains 3/4 on average across the authorities. Total utilitarian welfare is now:

$$3/2 + 1/2 \times \sqrt{4/10} + 1/4 \times \sqrt{1/10} \approx 1.895 > 1.875 = 15/16 \tag{116}$$

Completing the proof. □

F.2 An Efficient Centralized Mechanism

The inefficiency of each authority using a decentralized APM stems from the implicit incompleteness of markets: if we added the ability for authorities to pay each other for agents, then they would have willingness-to-pay to do so at the equilibrium allocation. A centralized mechanism can remedy this issue by ensuring the cross-sectional allocation of agents to authorities is optimal.

We propose the following augmentation of an APM to solve this problem, an *adaptive priority mechanism with quotas* (APM-Q). The idea behind this hybrid mechanism is to use aggregate, market-level priorities with authority-specific quotas. To this end, an APM-Q comprises an aggregate non-separable APM $\tilde{A} = \{\tilde{A}_m\}_{m \in \mathcal{M}}$ with $\tilde{A}_m : \mathbb{R}^{|\mathcal{M}|} \times [0, 1] \to \mathbb{R}$ and a profile of quota functions $Q = \{Q_{m,c}\}_{m,c \in \mathcal{M}}$ with with $Q_{m,c} : \mathbb{R}^{|\mathcal{M}|} \to \mathbb{R}_+$. Intuitively, the aggregate APM pins down the aggregate measures of allocations of each group to *any* authority $\{x_m\}_{m\in\mathcal{M}}$, where $x_m = \sum_{c\in\bar{\mathcal{C}}} x_{m,c}$. The non-separability of this APM simply means that the measures of all groups matter for the adaptive priority of any agent. Given the aggregate measure of allocation for group m, the quota function for authority c assigns $Q_{m,c}(\{x_m\}_{m\in\mathcal{M}})$ agents of type m to authority c.

Definition 9 (Adaptive Priority Mechanism with Quotas)**.** *An adaptive priority mechanism with quotas $(\tilde{A}, Q)$ comprises a non-separable APM $\tilde{A}$ and a quota function Q. An APM-Q implements allocation μ if the following are satisfied:*

1. *Aggregate allocations are in order or priorities: $\mu(\theta) \in \bar{\mathcal{C}}$ if and only if for all θ' with $\mu(\theta') = c_0$, we have that:*

$$\tilde{A}_{m(\theta)}(\{x_m(\mu)\}_{m\in\mathcal{M}}, s(\theta)) > \tilde{A}_{m(\theta')}(\{x_m(\mu)\}_{m\in\mathcal{M}}, s(\theta')) \tag{117}$$

2. *Authority-level allocations are given by the corresponding quota functions:*

$$x_{m,c}(\mu) = Q_{m,c}(\{x_m(\mu)\}_{m\in\mathcal{M}}) \tag{118}$$

3. *The resources are fully allocated:*

$$\sum_{m\in\mathcal{M}} x_{m,c}(\mu) = q_c \tag{119}$$

By appropriate choice of the APM and quota functions, we can derive an APM-Q that is efficient. To this end, define the optimally-allocated aggregate utility from diversity:

$$\begin{aligned} \tilde{u}(\{x_m\}_{m\in\mathcal{M}}) &= \max_{\{x_{m,c}\}_{c\in\mathcal{C}}} \sum_{c\in\mathcal{C}} \sum_{m\in\mathcal{M}} u_{m,c}(x_{m,c}) \\ \text{s.t.} \sum_{c\in\mathcal{C}} x_{m,c} &\le x_m, \ \sum_{m\in\mathcal{M}} x_{m,c} \le q_c, \ \forall m\in\mathcal{M}, c\in\mathcal{C} \end{aligned} \tag{120}$$

Moreover, define the marginal value of aggregate group m admissions $\tilde{u}^{(m)}(y) = \frac{\partial}{\partial y_m}\tilde{u}(y)$ and the marginal value of authority capacity $\tilde{u}_{q_c}(y) = \frac{\partial}{\partial q_c}\tilde{u}(y)$. Using these marginal values, we can design an efficient APM-Q that combines market-level APMs with authority-level quotas:

Theorem 6 (Efficient APM-Q)**.** *Every allocation induced by the following APM-Q $(\tilde{A}, Q)$ is efficient:*

1. *The non-separable APM is given by $\tilde{A}_m(y, s) = h^{-1}\left(h(s) + \tilde{u}^{(m)}(y)\right)$*
2. *The quota functions are given by $Q_{m,c}(y) = \left(u'_{m,c}\right)^{-1}\left(\tilde{u}^{(m)}(y) + \tilde{u}_{q_c}(y)\right)$*

Proof. First, we define a fictitious *composite authority* with utility function defined over vectors of total scores $\bar{s}_h = \{\bar{s}^c_h\}_{c\in\mathcal{C}}$, and aggregate allocation to each group $x = \{x_m\}_{m\in\mathcal{M}}$.

To do this, we define:

$$\begin{aligned}\tilde{u}(\{x_m\}_{m\in\mathcal{M}}) &= \max_{\{x_{m,c}\}_{c\in\mathcal{C}}} \sum_{c\in\mathcal{C}}\sum_{m\in\mathcal{M}} u_{m,c}(x_{m,c}) \\ \text{s.t.} &\sum_{c\in\mathcal{C}} x_{m,c} \le x_m,\ \sum_{m\in\mathcal{M}} x_{m,c} \le q_c,\ \forall m\in\mathcal{M}, c\in\mathcal{C}\end{aligned} \tag{121}$$

and $\tilde{\bar{s}}_h = \sum_{c\in\mathcal{C}} \bar{s}_h^c$. We write the utility function of this composite authority as

$$\tilde{\xi}\left(\tilde{\bar{s}}_h, x\right) = \tilde{\bar{s}}_h + \tilde{u}(x) \tag{122}$$

We first establish that $\tilde{u}$ satisfies the properties necessary to invoke Proposition 5, which establishes the optimality of the claimed APM for the fictitious authority.

Claim 19. *The function $\tilde{u}$ is concave and partially differentiable in each argument.*

Proof. First, we establish concavity. That is, for all $\lambda \in [0,1]$ and $x, x' \in \mathbb{R}_+^{|\mathcal{M}|}$, we have that $\tilde{u}(\lambda x' + (1-\lambda)x) \ge \lambda\tilde{u}(x') + (1-\lambda)\tilde{u}(x)$. Let $\{x_{m,c}^*\}_{m\in\mathcal{M},c\in\mathcal{C}}$ and $\{x_{m,c}^{*'}\}_{m\in\mathcal{M},c\in\mathcal{C}}$ correspond to optimal values under x and x'. Under $\tilde{x} = \lambda x' + (1-\lambda)x$, we have that $\lambda x_{m,c}^{*'} + (1-\lambda)x_{m,c}^*$ is feasible for all $m\in\mathcal{M}$ and $c\in\mathcal{C}$. Thus, we have that:

$$\begin{aligned}\tilde{u}(\tilde{x}) &\ge \sum_{m\in\mathcal{M}}\sum_{c\in\mathcal{C}} u_{m,c}(\lambda x_{m,c}^{*'} + (1-\lambda)x_{m,c}^*) \\ &\ge \sum_{m\in\mathcal{M}}\sum_{c\in\mathcal{C}} \lambda u_{m,c}(x_{m,c}^{*'}) + (1-\lambda)u_{m,c}(x_{m,c}^*) \\ &= \lambda\tilde{u}(x') + (1-\lambda)\tilde{u}(x)\end{aligned} \tag{123}$$

where the second inequality is by concavity of $u_{m,c}$ for all $m\in\mathcal{M}, c\in\mathcal{C}$.

Second, we establish partial differentiability in each argument. That is, for all $x \in \mathbb{R}_{++}^{|\mathcal{M}|}$, $\frac{\partial}{\partial x_m}\tilde{u}(x) = \tilde{u}^{(m)}(x)$ exists. This follows by Corollary 5 in Milgrom and Segal (2002). Concretely, the domain of optimization can be taken to be a compact and convex subset of a normed vector space – a sufficiently large cube in $\mathbb{R}_+^{|\mathcal{M}|\times|\mathcal{C}|}$ equipped with the Euclidean norm, for example. The objective function does not depend on x, and constraints are linear in x (and therefore both continuous and continuously differentiable). Moreover, as $x \gg 0$, there exists a $\{x_{m,c}\}$ that satisfies all constraints with strict inequality.

□

It follows that the objective function of the composite authority satisfies Assumption 2, and so Proposition 5 implies that the non-separable APM $\tilde{A}_m(y,s) = h^{-1}\left(h(s) + \tilde{u}^{(m)}(y)\right)$ uniquely implements the first-best optimal allocation for the composite authority.

It remains to establish that the quota functions implement the optimal allocation $\{x_{m,c}\}$ conditional on $\{x_m\}$. Let λ_m be the Lagrange multiplier on the x_m constraint, γ_c be the Lagrange multiplier on the q_c constraint and $\underline{\kappa}_{m,c}$ be the Lagrange multiplier on the positivity constraint. Under our maintained Inada condition, we have that $\underline{\kappa}_{m,c} = 0$. Moreover, by Corollary 5 in Milgrom and Segal (2002), we have that $\tilde{u}^{(m)}(x) = \lambda_m$, $\tilde{u}_{q_c}(x) = \gamma_c$, and $u'_{m,c}(x^*_{m,c}) = \lambda_m + \gamma_c - \underline{\kappa}_{m,c}$. Hence, we obtain that:

$$x^*_{m,c} = \left(u'_{m,c}\right)^{-1}\left(\tilde{u}^{(m)}(x) + \tilde{u}_{q_c}(x)\right) \tag{124}$$

Thus, the following profile of quota functions implements the optimal cross-sectional allocation:

$$Q_{m,c}(x) = \left(u'_{m,c}\right)^{-1}\left(\tilde{u}^{(m)}(x) + \tilde{u}_{q_c}(x)\right) \tag{125}$$

Completing the proof. □

The proof of this result constructs a fictitious aggregate authority in our single object setting. The claimed APM is optimal for this aggregate authority by a non-separable adaptation of Theorem 1. The substantial step in this proof establishes that $\tilde{u}$ is increasing, concave, and differentiable by employing the restrictions provided by Assumption 4. Then, given the allocation induced by this APM, we construct the quota function to optimally allocate the level of aggregate admissions induced by the APM.

Intuitively, this mechanism remedies inefficiency by "completing markets." There is a common "market price" for each group given by $\mathcal{P}_m = \tilde{u}^{(m)}(x)$ and an authority-level "shadow price of admissions" $\mathcal{P}_c = \tilde{u}_{q_c}(x)$. Authorities are allocated agents so that the marginal benefit of additional agents equals the sum of the market price and shadow price of admissions $u'_{m,c}(x_{m,c}) = \mathcal{P}_m + \mathcal{P}_c$. Hence, through the completion of markets, a centralized planner can allocate agents efficiently and internalize the externalities that prevented efficiency under the decentralized outcome. Notice that this market involves relatively few prices as it involves only $|\mathcal{M}|+|\mathcal{C}|$ shadow prices rather than the full set of $|\mathcal{M}|\times|\mathcal{C}|$ marginal values.

G Extension of the Main Results to Discrete Economies

In this Appendix, we extend the results in the main text to discrete economies and thereby establish that the core of our analysis generalizes from the continuum framework. Concretely, we show that appropriate analogs of Theorems 1, 2, and 4 carry over to discrete economies. Together, these establish the optimality of APM and characterize the (sub)-optimality of priorities and quotas. However, as discrete economies do not necessarily admit a unique stable matching (as is well known), the first part of Theorem 3 does not hold (uniqueness). This notwithstanding, we demonstrate that agent-proposing DA, when combined with the optimal APM, implements the agent-optimal stable allocation.

G.1 Primitives

An authority has q resources to allocate. At each state ω, the economy the authority faces corresponds to agents $\Theta^\omega = \{\theta_1, \ldots, \theta_{N(\omega)}\}$ where $q \leq |N(\omega)|$. As in the continuum case, $\theta \in [0,1] \times \mathcal{M}$ denotes the type of an agent who has score s and belongs to group m. We denote the score and group of any type θ by $s(\theta)$ and $m(\theta)$, respectively. For simplicity, we assume that no two agents have the same score at any ω, formally, if $\{\theta, \theta'\} \subseteq \Theta^\omega$, then $s(\theta) \neq s(\theta')$.

An allocation $\mu : \Theta \to \{0,1\}$ specifies for any type $\theta \in \Theta$ whether they are assigned to the resource. The set of possible allocations is $\mathcal{U}$ and Ω is the set of all possible economies. An allocation is feasible if it allocates no more than measure q of the resource. A mechanism is a function $\phi : \Omega \to \mathcal{U}$ that returns a feasible allocation for any possible Θ^ω.

The authority believes ω has distribution $\Lambda \in \Delta(\Omega)$. $x(\mu, \omega) = \{x_m(\mu, \omega)\}_{m \in \mathcal{M}}$ denotes the number of agents of each group allocated the resource at matching μ, while $\bar{s}_h(\mu, \omega) = \sum_{\theta \in \Theta^\omega} \mu(\theta) h(s(\theta))$ denotes the utility the authority derives from scores at μ. The preferences of the authority are given by $\xi : \mathbb{R}^{|\mathcal{M}|+1} \to \mathbb{R}$:

$$\xi\left(\bar{s}_h, x\right) \equiv \bar{s}_h + \sum_{m \in \mathcal{M}} u_m(x_m) \tag{126}$$

where h is continuous and strictly increasing and u_m is concave for all $m \in \mathcal{M}$.

G.2 Optimal Mechanisms in Discrete Economies

We adapt our definition of the Adaptive Priority Mechanisms to the discrete setting. An *adaptive priority policy* $A = \{A_m\}_{m \in \mathcal{M}}$, where $A_m : \mathbb{R} \times [0,1] \to \mathbb{R}$. The adaptive priority policy assigns priority $A_m(y_m, s)$ to an agent with score s in group m when y_m of agents of the same group is allocated the object. Given an adaptive priority policy, an APM implements allocations in the following way:

Definition 10 (Adaptive Priority Mechanism)**.** *An adaptive priority mechanism, induced by an adaptive priority A, implements an allocation μ in state ω if the following are satisfied:*

1. *Allocations are in order of priorities: $\mu(\theta) = 1$ if and only if*
   (i) for all θ' with $m(\theta') \neq m(\theta)$ and $\mu(\theta') = 0$,
   $$A_{m(\theta)}(x_{m(\theta)}(\mu, \omega), s(\theta)) \geq A_{m(\theta')}(x_{m(\theta')}(\mu, \omega) + 1, s(\theta')) \tag{127}$$
   (ii) for all θ' with $m(\theta') = m(\theta)$ and $\mu(\theta') = 0$, $s(\theta) > s(\theta')$
2. *The resource is fully allocated:*
   $$\sum_{m \in \mathcal{M}} x_m(\mu, \omega) = q \tag{128}$$

Definition 10 makes two modifications relative to the continuum model. First, the measures of agents from each group are replaced by the number of agents from each group. Second, when $m(\theta) \neq m(\theta')$, the adaptive priority of θ' is now evaluated in the case where an extra agent from $m(\theta')$ is assigned the resource.[45] Unlike the continuum case, it is possible for a monotone APM to implement two different allocations, since it can assign the same priority to two different agents, which could happen only for a zero-measure set of agents in the continuum model.

Define $A^*_m(y_m, s) \equiv h(s) + u_m(y_m) - u_m(y_m - 1)$, which will turn out to be the optimal APM. We first show that A^* is monotone, and all allocations that A^* implements give the authority the same utility.

Lemma 4. *A^* is monotone. Moreover, if A^* implements μ and $\mu' \neq \mu$ in state ω, then $\xi(\mu, \omega) = \xi(\mu', \omega)$.*

Proof. Monotonicity is immediate from the definition of A^* and concavity of u_m. Assume that A^* implements two different allocations, μ and μ' at ω. Let x_l and x'_l denote the number of group $l \in \mathcal{M}$ agents assigned the resource at μ and μ'. Since A^* is monotone and $\mu \neq \mu'$, there are m and n such that $x_m > x'_m$ and $x'_n > x_n$. Let $\tilde{\theta}_l$ and $\tilde{\theta}'_l$ denote the lowest-scoring type l agent assigned the resource at μ and μ', respectively. Similarly, let $\hat{\theta}_l$ and $\hat{\theta}'_l$ denote the highest-scoring type l agents who is not assigned the resource at μ and μ', respectively. Let $\tilde{\mu}$ denote the matching given by: $\tilde{\mu}(\theta) = \mu(\theta)$ if $\theta \notin \{\tilde{\theta}_m, \hat{\theta}'_n\}$, $\tilde{\mu}(\tilde{\theta}_m) = 0$ while $\mu(\hat{\theta}_n) = 1$. $\tilde{\mu}$ starts with μ, takes the resource away from the lowest-scoring group m agent who has it, $\tilde{\theta}_m$, and allocates it to the highest-scoring group n agent who does not have it, $\hat{\theta}_n$. Note

[45] This was not the case in the continuum model since all types of agents have measure 0 and therefore replacing θ with θ' has no effect the evaluation of diversity.

that since A^* is monotone, from $x_m > x'_m$ and $x'_n > x_n$, under μ', $\hat{\theta}_n$ is already allocated the resource while $\tilde{\theta}_m$ is not.

Claim 20. *$\tilde{\mu}$ is implemented under A^* in state ω and $\xi(\mu,\omega) = \xi(\tilde{\mu},\omega)$.*

Proof. Since A^* implements μ and $\mu(\hat{\theta}_n) = 0$, we have that $A^*_m(s(\tilde{\theta}_m), x_m) \geq A^*_n(s(\hat{\theta}_n), x_n + 1)$. Conversely, since A^* also implements μ' and $\mu'(\hat{\theta}'_m) = 0$, we have that $A^*_n(s(\tilde{\theta}'_n), x'_n) \geq A^*_m(s(\hat{\theta}'_m), x'_m + 1)$. Moreover, since $x_m > x'_m$ and $x'_n > x_n$, we have that $s(\hat{\theta}'_m) \geq s(\tilde{\theta}_m)$ and $s(\hat{\theta}_n) \geq s(\tilde{\theta}'_n)$. From this, it follows that:

$$\begin{aligned} A^*_n(s(\hat{\theta}_n), x_n + 1) &\geq A^*_n(s(\tilde{\theta}'_n), x_n + 1) \geq A^*_n(s(\tilde{\theta}'_n), x'_n) \\ &\geq A^*_m(s(\hat{\theta}'_m), x'_m + 1) \geq A^*_m(s(\hat{\theta}'_m), x_m) \geq A^*_m(s(\tilde{\theta}_m), x_m) \end{aligned} \tag{129}$$

where the first inequality holds as $s(\hat{\theta}_n) \geq s(\tilde{\theta}'_n)$, the second inequality holds as $x'_n > x_n$ (which implies $x'_n \geq x_n + 1$) and A^*_n is decreasing in its second argument, the third inequality holds as A^* also implements μ' (as stated above), the fourth inequality holds as $x'_m < x_m$ (which implies $x'_m + 1 \leq x_m$) and A^*_n is decreasing in its second argument, and the fifth inequality holds as $s(\hat{\theta}'_m) \geq s(\tilde{\theta}_m)$. Thus, $A^*_m(s(\tilde{\theta}_m), x_m) \leq A^*_n(s(\hat{\theta}_n), x_n + 1)$. This shows that $A^*_m(s(\tilde{\theta}_m), x_m) = A^*_n(s(\hat{\theta}_n), x_n + 1)$, which implies that $\tilde{\mu}$ is implemented under A^* and $\xi(\mu,\omega) = \xi(\tilde{\mu},\omega)$. □

Note that Claim 20 shows that starting from a matching μ which is implemented by A^*, taking away the object from a particular agent who does not have it in μ' and allocating it to a particular agent who has it in μ', we arrive at another matching $\tilde{\mu}$ that is implemented under A^* and gives the authority the same payoff. Therefore, starting from any μ that is implemented by A^* and repeating this construction (by replacing μ at step i with $\tilde{\mu}$ at step $i-1$) where at each step we take the resource from an agent who is not allocated the resource at μ' and assign it to an agent who is, in finitely many steps we arrive at μ'. Since the payoff stays the same at each step, μ' gives the authority the same payoff as μ. □

Theorem 7. *If μ is implemented by A^*, then μ is an optimal matching.*

Proof. First, note that an optimal matching exists since the economy (and therefore the set of matchings) is finite. We first show the following lemma.

Lemma 5. *If μ is not implemented by A^*, then there exists μ' that gives the authority a strictly higher payoff.*

Proof. If μ is not implemented by A^*, then there exists θ and θ' such that $\mu(\theta) = 0$, $\mu(\theta') = 1$ and either $m(\theta) = m(\theta')$ and $s(\theta) > s(\theta')$ or $m(\theta) \neq m(\theta')$ and

$$\begin{aligned} &h(s(\theta)) + u_{m(\theta)}(x_{m(\theta)}(\mu) + 1) - u_{m(\theta)}(x_{m(\theta)}(\mu)) > \\ &h(s(\theta')) + u_{m(\theta')}(x_{m(\theta')}(\mu)) - u_{m(\theta')}(x_{m(\theta')}(\mu) - 1) \end{aligned} \tag{130}$$

However, in both cases, a μ' that allocates the resource to θ instead of θ' (while not changing any other agent's matching) strictly improves the utility of the authority. □

Lemma 5 proves that the optimal matching cannot be a matching that is not implemented by A^*. Since the optimal matching exists, then it is implemented by A^*. From Lemma 4, all matchings implemented by A^* give the authority the same payoff, proving the result. □

Note that Lemma 4 and Theorem 7 imply that any mechanism that is defined by an arbitrary singleton selection from the set of matchings that A^* implements would achieve an optimal matching under any ω and therefore would be first-best optimal.

G.3 Priorities *vs.* Quotas in Discrete Economies

Now, we define Priority and Quota Mechanisms in the discrete model and extend our (sub)optimality results to discrete economies.

A *priority policy* $P : \Theta \to [0, 1]$ awards an agent of type $\theta \in \Theta$ a priority $P(\theta)$.

Definition 11 (Priority Mechanisms). *A priority mechanism, induced by a priority policy P, allocates the resource in order of priorities until measure q has been allocated, with ties broken uniformly and at random.*

A *quota policy* is given by (Q, D), where $Q = \{Q_m\}_{m\in\mathcal{M}}$ and $D : \mathcal{M} \cup \{R\} \to \{1, 2, \ldots, |\mathcal{M}|+1\}$ is a bijection. The vector Q reserves Q_m objects for agents in group m, with residual capacity $Q_R = q - \sum_{m\in\mathcal{M}} Q_m$ open to agents of all types. The bijection D (often called the precedence order) determines the order in which the groups are processed.

Definition 12 (Quota Mechanisms). *A quota mechanism, induced by a quota policy (Q, D), proceeds by allocating $Q_{D^{-1}(k)}$ objects to agents from group $D^{-1}(k)$ (if there are sufficient agents from this group) to the resource in ascending order of k, and in descending order of score within each k. If there are insufficiently many agents of any group to fill the quota, the residual capacity is allocated to a final round in which all agents are eligible.*

We also extend the definitions of risk-neutrality and high risk aversion to the discrete setting. Authority preferences are *non-trivial* if for all $m, n \in \mathcal{M}$:

$$h(1) + (u_n(1) - u_n(0)) > h(0) + (u_m(q) - u_m(q - 1)) \tag{131}$$

The authority is *risk-neutral* if for all $m \in \mathcal{M}$, $u_m(x) = c_m x$ for some $c_m \geq 0$ and all $x \in \{0, 1, \ldots, q\}$. Define $\tilde{u}$ and $\tilde{h}$ as follows: there exists x_m^{tar} such that $\tilde{u}_m(x_m+1)-\tilde{u}_m(x_m) = 0$ for all $x_m \geq x_m^{\text{tar}}$ and $\tilde{u}_m(x_m+1)-\tilde{u}_m(x_m) \geq h(1)-h(0)$ for $x_m < x_m^{\text{tar}}$ and where $\sum_{m\in\mathcal{M}} x_m^{\text{tar}} \leq q$. Let $\tilde{\xi}$ denote the preferences of the authority under $\tilde{u}$ and $\tilde{h}$. The authority with preferences ξ is *extremely risk-averse* if the set of optimal allocations under ξ and $\tilde{\xi}$ coincide for all ω.

Theorem 8. *The following statements are true:*

1. *If there is no uncertainty, then there exist first-best priority and quota mechanisms.*
2. *Suppose that the authority has non-trivial preferences. There exists a first-best priority mechanism if and only if the authority is risk-neutral. This mechanism is given by* $P(s, m) = s + u_m(1) - u_m(0)$.
3. *Suppose that the authority has non-trivial preferences. There exists a first-best quota mechanism if and only if the authority is extremely risk-averse. This mechanism is given by* $Q_m = x_m^{tar}$ *and* $D(R) = |\mathcal{M}| + 1$.

Proof. Part (1):

Claim 21. *Let* μ *denote an optimal allocation at* ω*. Then* μ *is a cutoff matching.*

Proof. If μ is not a cutoff matching, then there exists (s, m) and (s', m) where $\mu(s, m) = 1$, $\mu(s', m) = 0$ and $s' > s$. Define μ' by setting: $\mu'(s, m) = 0$, $\mu(s', m) = 1$ and $\mu(\tilde{s}, \tilde{m}) = \mu(\tilde{s}, \tilde{m})$ for all $(\tilde{s}, \tilde{m})$ such that $(\tilde{s}, \tilde{m}) \notin \{(s, m), (s', m)\}$. Observe that, $\xi(\mu', \omega) - \xi(\mu, \omega) = s' - s > 0$. Therefore, μ is not an optimal allocation, which is a contradiction. □

Let μ denote an optimal allocation under ω, $\{\hat{s}_m(\mu, \omega)\}_{m \in \mathcal{M}}$ denote the cutoff scores at μ and s^* denote an arbitrary number. Any priority policy that assigns $P(\hat{s}_m(\omega), m) = s^*$ for all $m \in \mathcal{M}$ and is strictly increasing in the first argument allocates the resource to any agent who has a higher score than the cutoff for their group and implements the optimal allocation.

Let x_m denote the number of group m agents who are allocated the resource at an optimal allocation under ω. Then a quota policy that sets $Q_m = x_m$ allocates the resource to any agent who has a higher score than the cutoff for their group and implements the optimal allocation.

Part (2): The if part of the result follows from observing the priority policy $P(s, m) = s + u_m(1) - u_m(0)$ is equivalent to the optimal APM A^* under risk neutrality since $u_m(1) - u_m(0) = u_m(y_m + 1) - u_m(y_m)$ for all m, y_m. Thus, by Theorem 7, $P(s, m) = s + u_m(1) - u_m(0)$ is first-best optimal.

To prove the only if part, assume risk neutrality does not hold and let m denote a group such that u_m does not satisfy risk neutrality. For a contradiction, assume that P is an optimal priority policy. First, we observe that $P(s, m)$ must be strictly increasing in s for all m. To see why, assume $P(s, m) = P(s', m)$ where $s > s'$ and just consider an ω where there are $q - 1$ group m agents with scores strictly higher than s, and no other agents. Clearly, the optimal allocation would be to allocate the resource to all agents but (s', m), while P allocates the resource to (s', m) with at least probability 1/2.

Second, let m denote a group such that u_m does not satisfy risk neutrality. Take another arbitrary group n. We have the following:

Claim 22. *Either (i) there exists $t < q$, s_m, s_n such that*

$$u_m(t+1) - u_m(t) + h(s_m) = u_n(q-t) - u_n(q-t-1) + h(s_n) \tag{132}$$

or (ii) there exists $t < q$ such that

$$u_m(t+1) - u_m(t) + h(1) < u_n(q-t) - u_n(q-t-1) + h(0) \tag{133}$$

$$u_m(t) - u_m(t-1) + h(0) > u_n(q-t+1) - u_n(q-t) + h(1) \tag{134}$$

Proof. From non-triviality, we know that $u_m(1) - u_m(0) + h(1) > u_n(q) - u_n(q-1) + h(0)$ and $u_n(1) - u_n(0) + h(1) > u_m(q) - u_m(q-1) + h(0)$. The result then follows from the fact that h is continuous and strictly increasing and u_m and u_n are concave. □

We first prove the result under case (ii). Fix two agents with scores $s_m \in (0,1)$, who belong to group m and $s_n \in (0,1)$, who belong to group n. Assume that there are $t-1$ group m agents and $q-t$ group n agents with higher scores than $\max\{s_n, s_m\}$, so a total of t group m agents and $q-t+1$ group n agents. Note that in this case, only one agent will not be allocated the resource in the optimal allocation, and that would be either (s_m, m) or (s_n, n). From equation 134, (s_m, m) is more preferred than (s_n, n) and therefore it must be that $P(s_n, n) < P(s_m, m)$, as otherwise P would not be optimal. Next, assume that there are t group m agents and $q-t-1$ group n agents with higher scores than $\max\{s_n, s_m\}$. From equation 133, (s_n, n) is more preferred than (s_m, m) and therefore it must be that $P(s_m, m) < P(s_n, n)$, which is a contradiction.

We now prove the result under case (i).

Claim 23. *In case (i), any optimal priority policy P must satisfy $P(s_m+\epsilon, m) > P(s_n, n)$ for all $\epsilon > 0$ and $P(s_m - \epsilon, m) < P(s_n, n)$ for all $\epsilon > 0$*

Proof. From Equation 132, we see that when there are t group m agents and $q-t-1$ group n agents with higher scores, $(s_m+\epsilon, m)$ is strictly preferred to (s_n, n), which is strictly preferred to $(s_m - \epsilon, m)$. □

Since u_m is not linear, there exists an l such that $u_m(l+1) - u_m(l) < u_m(l) - u_m(l-1)$. There are two possibilities: $l \le t$ or $l > t$. First, suppose that $l \le t$. We have that:

$$u_m(l) - u_m(l-1) + h(s_m) > u_m(l+1) - u_m(l) + h(s_m) \ge u_n(q-l) - u_n(q-l+1) + h(s_n) \tag{135}$$

where the first inequality follows from $u_m(l+1) - u_m(l) < u_m(l) - u_m(l-1)$ and the second inequality follows as $u_m(t+1) - u_m(t) + h(s_m) = u_n(q-t) - u_n(q-t-1) + h(s_n)$, u_m and

u_n are concave, and $l \leq t$. Thus, for sufficiently small $\epsilon > 0$, we have that:

$$u_m(l) - u_m(l-1) + h(s_m - \epsilon) > u_n(q-l) - u_n(q-l+1) + h(s_n) \tag{136}$$

Given this inequality, we see that when there are $l-1$ group m agents and $q-l$ group n agents with higher scores, $(s_m - \epsilon, m)$ is strictly preferred to (s_n, n). Thus, to implement the optimal allocation, it must be that $P(s_m - \epsilon, m) \geq P(s_n, n)$, which is a contradiction to Claim 23.

Second, suppose that $l > t$. We know that:

$$u_m(t+1) - u_m(t) + h(s_m) = u_n(q-t) - u_n(q-t-1) + h(s_n) \tag{137}$$

As $l > t$, from concavity of u_m and u_n,

$$u_m(l) - u_m(l-1) + h(s_m) \leq u_n(q-l+1) - u_n(q-l) + h(s_n) \tag{138}$$

From concavity of u_n and u_m:

$$u_m(l+1) - u_m(l) + h(s_m) < u_n(q-l) - u_n(q-l-1) + h(s_n) \tag{139}$$

Thus, for sufficiently small $\epsilon > 0$, we have that:

$$u_m(l+1) - u_m(l) + h(s_m + \epsilon) < u_n(q-l) - u_n(q-l-1) + h(s_n) \tag{140}$$

Given this inequality, we see that when there are l group m agents and $q-l-1$ group n agents with higher scores, (s_n, n) is strictly preferred to $(s_m + \epsilon, m)$. Thus, to implement the optimal allocation, it must be that $P(s_m + \epsilon, m) \leq P(s_n, n)$, which is a contradiction to Claim 23.

Part (3): To prove the if part, fix an ω and let μ^* denote the optimal allocation under ω. Let x^*_m denote the number of group m agents allocated the resource at μ^* and $x_m(\omega)$ denote the total number of group m agents under ω.

Claim 24. *If the authority is extremely risk-averse, then* $x^*_m \geq \min\{x_m(\omega), x^{tar}_m\}$

Proof. Assume for a contradiction this is not the case. Then $x^*_m < x_m(\omega)$ and $x^*_m < x^{tar}_m$. Since $\sum_{m \in \mathcal{M}} x^{tar}_m \leq q$ and $x^*_m < x^{tar}_m$, there exists $n \in \mathcal{M}$ such that $x^*_n > x^{tar}_n$. Let s_n denote the score of the lowest-scoring group n agent who is allocated the resource, and let s_m denote the score of any group m agent who is not allocated the resource, which exists as $x^*_m < x_m(\omega)$. Since the authority is extremely risk-averse, we have the following:

$$h(s_m) + u_m(x^*_m + 1) - u_m(x^*_m) > h(s_n) - u_n(x^*_n) + u_m(x^*_n - 1) \tag{141}$$

However, this contradicts the optimality of μ^* and proves the claim. □

Claim 25. *If the authority is extremely risk-averse,* $x^*_m > x^{tar}_m$ *and* $x^*_n > x^{tar}_n$*,* $\mu^*(s, m) = 0$ *and* $\mu^*(s', n) = 1$*, then* $s' > s$*.*

Proof. Assume for a contradiction that $s > s'$.[46] The difference in the utility of the authority when allocating the resource to (s, m) rather than (s', n) is given by

$$h(s) + u_m(x_m^* + 1) - u_m(x_m^*) - (h(s') - u_n(x_n^*) + u_m(x_n^* - 1)) = h(s) - h(s') > 0 \quad (142)$$

which is a contradiction to optimality of μ^*. □

The previous two claims show that under any ω, the optimal allocation admits (i) the highest-scoring x_m^{tar} agents from each group (provided that they exist) and (ii) highest-scoring agents who are not in (i), until the capacity is exhausted. Clearly, the quota policy $Q_m = x_m^{\text{tar}}$ and $D(R) = |\mathcal{M}| + 1$ implements this outcome at every ω.

To prove the only if part, assume that $\{Q_m\}_{m \in \mathcal{M}}$ is part of an optimal quota policy.

Claim 26. *For and each $m, n \in \mathcal{M}$ and any t, l such that $t \leq Q_m$, $Q_m > 0$ and $l \geq Q_n$, we have that:*

$$u_m(t) - u_m(t-1) + h(0) \geq u_n(l+1) - u_n(l) + h(1) \quad (143)$$

Proof. Assume that at ω, there are t group m agents, one of which one has score 0 and $l+1$ group n agents with scores higher than $1 - \epsilon_1$ and q agents from other groups who have scores higher than $1 - \epsilon_2$, where $\epsilon_1 > \epsilon_2 > 0$. As $t \leq Q_m$ and $Q_n < l+1$, t group m agents and $Q_n < l+1$ group n agents are admitted under Q. Since Q is optimal for all ϵ_1, we must have that:

$$u_m(t) - u_m(t-1) + h(0) \geq u_n(l+1) - u_n(l) + h(1 - \epsilon_1) \quad (144)$$

The statement then follows from continuity of h by taking the limit $\epsilon_1 \to 0$. □

Claim 27. *Merit slots are processed last at the optimal quota policy.*

Proof. For a contradiction, assume there is a merit slot that is processed before a quota slot. Let l denote the last merit slot that precedes a quota slot. Let m denote a group that has a quota slot after l. We consider a state in which: (i) there are q group n agents with scores $\hat{s} - \epsilon_i$, where $\epsilon_i > 0$ for all $i \in \{1, \ldots, q\}$ (let $\underline{\hat{s}}$ denote the score of the highest-scoring agent from this group), (ii) there are Q_m group m agents with scores $\hat{s} + \epsilon_j$ for $j \in \{1, \ldots, Q_m\}$ (let $\overline{\hat{s}}$ denote the score of lowest-scoring agent from this group) and one with score $\hat{s}/2$, and (iii) q agents from other groups with scores in $(\underline{\hat{s}}, \overline{\hat{s}})$. A group m agent with score $\hat{s} + \epsilon_k$ for some k is matched to l, thus $(\hat{s}/2, m)$ is matched to a later quota slot, while some agents with type $(\hat{s} - \epsilon_j, n)$ are rejected for some j. Let $\hat{s} - \epsilon_{j'}$ be the score of the highest-scoring such agent. From the optimality of the quota policy we have that

$$u_m(Q_m + 1) - u_m(Q_m) + h(\hat{s}/2) \geq u_n(Q_n + 1) - u_n(Q_n) + h(\hat{s} - \epsilon_{j'}) \quad (145)$$

[46] Remember that $s' = s$ was ruled out by assumption.

Let s^* be the score of the lowest-scoring group n agent (*i.e.*, $s^* = \min_{i\in\{1,\dots,q\}} \hat{s} - \epsilon_i$). Next, consider the modified version of the above state, all group n agents are the same, but all of the other Q_m group m agents as well as q agents from other groups now have scores in $(s^* - \hat{\epsilon}, s^*)$ and the group m agent who had a score of $\hat{s}/2$ now has a score of $\hat{s}/2 + \hat{\epsilon}$ for $\hat{\epsilon} > 0$. Note that now the group n agent with score $\hat{s} - \epsilon_{j'}$ is allocated the slot l or an earlier slot, while the agent $(\hat{s}/2 + \hat{\epsilon}, m)$ is not allocated to any slot. Thus

$$u_m(Q_m + 1) - u_m(Q_m) + h(\hat{s}/2 + \hat{\epsilon}) \leq u_n(Q_n + 1) - u_n(Q_n) + h(\hat{s} - \epsilon_{j'}) \tag{146}$$

which, since h is strictly increasing, implies that $u_m(Q_m + 1) - u_m(Q_m) + h(\hat{s}/2) < u_n(Q_n + 1) - u_n(Q_n) + h(\hat{s} - \epsilon_{j'})$. This contradicts Equation 145, proving the claim. □

Given the previous two claims, the following claim proves the result.

Claim 28. *If merit slots are processed last, then for all $l \geq Q_m$ and $j \geq Q_n$*

$$u_m(l + 1) - u_m(l) = u_n(j + 1) - u_n(j) \tag{147}$$

Proof. Assume for a contradiction this does not hold. Without loss of generality, assume $u_m(l + 1) - u_m(l) > u_n(j + 1) - u_n(j)$ and define δ as

$$\delta = (u_m(l + 1) - u_m(l)) - (u_n(j + 1) - u_n(j)) \tag{148}$$

Consider a state with $q - 1$ agents with scores higher than s^*, of which exactly Q_m are group m agents and Q_n are group n agents. Moreover, there is one more group m agent with score $s' < s^*$ (denote this agent by θ_m) and one more group n agent with score $s'' \in (s', s^*)$ where $h(s'') - h(s') < \delta$ (denote this agent by θ_n). Note that all agents apart from θ_m and θ_n are allocated the resource before the final merit slot. Moreover, since θ_n has a higher score, she obtains the final merit slot. However, this is a contradiction to the optimality of Q as $h(s'') - h(s') < \delta$ and allocating that resource to θ_m gives the authority higher utility. This proves the claim. □

Taken together, claims 26 and 28 prove that a fictitious authority that is extremely risk-averse with $x_m^{\text{tar}} = Q_m$ agrees with the authority on the optimal allocation, for all ω. To see this, observe that claim 26 implies that diversity preferences dominate any concern for scores when a group is allocated less than Q_m. Moreover, conditional on being allocated at least Q_m, it is as if there is no residual diversity preference, by claim 28. This proves the only if part of (3), which finishes the proof of the result. □

G.4 Characterization of Stable Allocations in Discrete Economies

In this section, we extend our discrete model to the multiple authority case. Of course, in discrete models, there can be multiple stable matchings. This notwithstanding, we show that agent-proposing DA, when combined with the optimal APM, implements the agent-optimal stable allocation.

Let Θ_0 denote the set of agents. $\mathcal{C} = \{c_0, c_1, \ldots, c_{|\mathcal{C}|-1}\}$ denotes the set of authorities. q_c denotes the capacity of authority c and $q_{c_0} \geq |\Theta_0|$. $\theta = (s, m, \succ) \in [0,1]^{|\mathcal{C}|} \times \mathcal{M} \times \mathcal{R} = \Theta$, where $\mathcal{R}$ is set of all complete, transitive, and strict preference relations over $\mathcal{C}$ such that c_0 is less preferred than all $c \in \mathcal{C}$. For each type θ, $s_c(\theta)$ denotes the score of θ at authority c and $m(\theta)$ denotes the group of θ.

A matching in this environment is a function $\mu : \mathcal{C} \cup \Theta \to 2^{\Theta} \cup \mathcal{C}$ where $\mu(\theta) \in \mathcal{C}$ is the authority any type θ is assigned and $\mu(c) \subseteq \Theta$ is the set of agents assigned to authority c, which satisfies $|\mu(c)| \leq q_c$ for all c. $x_c(\mu) = \{x_{m,c}(\mu)\}_{m \in \mathcal{M}}$ denotes the number of agents of each group assigned to authority c at μ while $\bar{s}_{h_c}(\mu) = \sum_{\theta \in \mu(c)} h(s(\theta))$ denotes the score utility the authority derives from μ. The preferences of the authority are given by:

$$\xi_c(\bar{s}_{h_c}, x_c) = \bar{s}_{h_c} + \sum_{m \in \mathcal{M}} u_{m,c}(x_{m,c}) \tag{149}$$

where h_c is continuous and strictly increasing and $u_{m,c} : \mathbb{R} \to \mathbb{R}$ is concave for all $m \in \mathcal{M}$ and $c \in \bar{\mathcal{C}}$.

In Theorem 3, we showed that in the continuum model, there is stable matching and this matching is a cutoff matching. It is well known that in discrete models there may be multiple stable matchings, so the first part of the result does not hold. However, we can extend the second part of Theorem 3. Recall that in the discrete setting, a matching μ is stable if there are no blocking pairs, that is, there does not exist an agent θ and an authority $c \in \mathcal{C}$ (which includes the dummmy authority) such that $c \succ_\theta \mu(\theta)$ and either (i) c does not fill its capacity or (ii) there exists $\theta' \in c$ such that $\xi_c(\bar{s}_{h_c}(\mu'), x_c(\mu')) > \xi_c(\bar{s}_{h_c}(\mu), x_c(\mu))$, where $\mu'(c) = \mu(c) \setminus \theta' \cup \theta$.

Proposition 11. *If μ is a stable matching, then it is a cutoff matching.*

Proof. If μ is not a cutoff matching, then there exist $\theta = (s, m)$, $\theta' = (s', m)$ and $c \in \overline{\mathcal{C}}$ such that $\mu(\theta') = c$, $c >_\theta \mu(\theta)$ and $s > s'$. Define μ' as follows. $\mu'(c) = \mu(c) \setminus \theta' \cup \theta$ and $\mu'(c') = \mu(c')$ for all $c' \neq c$. As $s > s'$ and h_c is strictly increasing, $\xi_c(\bar{s}_{h_c}(\mu'), x_c(\mu')) > \xi_c(\bar{s}_{h_c}(\mu), x_c(\mu))$, which contradicts the stability of μ. □

In discrete markets, there may be multiple stable matchings. We now show that when all authorities use the optimal APM, Agent-Proposing Deferred Acceptance implements the agent-optimal stable matching, in other words, Theorem 4 holds if we replace the unique stable matching with agent optimal stable matching. For this result, we also assume that the authorities preferences are strict in the sense that given the finite economy Θ and two different matchings μ and μ' such that $\mu(c) \neq \mu(c')$, $\xi_c(\bar{s}_{h_c}(\mu), x_c(\mu)) \neq \xi_c(\bar{s}_{h_c}(\mu'), x_c(\mu'))$.

Theorem 9. *Agent-Proposing Deferred Acceptance implements the agent-optimal stable matching when all authorities use the optimal APM.*

Proof. We first define substitutable preferences in this setting, following Definition 6.2 in Roth and Sotomayor (1990). A choice rule satisfies substitutes if an agent θ is chosen from a set of agents Θ and $\theta' \neq \theta$, then θ must be chosen from $\Theta' \equiv \Theta \setminus \theta'$.

We first show that under our assumptions on the preferences, the optimal APM for an authority satisfies this property. Suppose that θ is chosen from some Θ under the optimal APM for some authority. For a contradiction, suppose that θ is not chosen from Θ'. As the optimal APM admits higher-scoring agents before lower-scoring ones, this means that there are strictly fewer $m(\theta)$ agents chosen from Θ'. Let n_m and n'_m denote the number of m group m agents chosen from Θ and Θ'. Then we have $n_{m(\theta)} > n'_{m(\theta)}$. As θ is not chosen from Θ', there must be a group m' such that more m' agents are chosen from Θ' compared to Θ. Let $\hat{\theta}$ denote the highest scoring m' agent that is not chosen from Θ and $\tilde{\theta}$ denote the lowest scoring m' agent chosen from Θ'. Note that $s(\hat{\theta}) \geq s(\tilde{\theta})$, which yields the following inequalities:

$$\begin{aligned} &A_{m(\theta)}(n_{m(\theta)}, s(\theta)) > A_{m(\hat{\theta})}(n_{m(\hat{\theta})}, s(\hat{\theta})) \geq A_{m(\tilde{\theta})}(n'_{m(\tilde{\theta})}, s(\tilde{\theta})) \\ &> A_{m(\theta)}(n'_{m(\theta)}, s(\theta)) \geq A_{m(\theta)}(n_{m(\theta)}, s(\theta)) \end{aligned} \tag{150}$$

where the first inequality holds as θ is chosen from Θ while $\hat{\theta}$ was not chosen (and the preferences were assumed to be strict), the second inequality holds as $n'_{m(\tilde{\theta})} > n'_{m(\hat{\theta})}$ and $s(\hat{\theta}) \geq s(\tilde{\theta})$, the third inequality holds as $\tilde{\theta}$ is chosen from Θ' while θ was not chosen (and the preferences were assumed to be strict), and the fourth inequality holds as $n_{m(\theta)} > n'_{m(\theta)}$, which is a contradiction—as the first and final terms are identical.

Given this, the theorem follows from Theorem 6.8 in Roth and Sotomayor (1990): when authorities have substitutable preferences (and preferences are strict), the agent-proposing deferred acceptance algorithm produces the agent-optimal stable matching. □

G.5 Dominance of APM in Sequential Discrete Economies

We finally extend the dominance of APM to discrete economies. As in Appendix E, agents apply to the authorities sequentially, who decide which agents to admit. We index the stage of the game by $t \in \mathcal{T} = \{1, \ldots, |\mathcal{C}| - 1\}$. Each stage corresponds to an authority $I(t)$, where $I : \mathcal{T} \to \mathcal{T}$. At each stage t, any unmatched agents choose whether to apply to authority $I(t)$. Given the set of applicants, authority $I(t)$ chooses to admit a subset of these agents. Given this, histories are indexed by the path of the remaining of agents who have not yet matched, $h^{t-1} = (\Theta_0, \Theta_1, \ldots, \Theta_{t-1}) \in \mathcal{H}^{t-1}$. Given each history h^{t-1} and set of applicants $\Theta_c^A \subseteq \Theta$, a strategy for an authority returns a set of agents $\Theta_c^G \subseteq \Theta$ whom they will admit such that $\Theta_c^G \subseteq \Theta_c^A$ and $|\Theta_c^G| \leq q_c$ for each time at which they could move $t \in \mathcal{T}$, $a_{c,t} : \mathcal{H}^{t-1} \times \mathcal{P}(\Theta) \to \mathcal{P}(\Theta)$, where $\mathcal{P}(\Theta)$ is the power set over Θ. A strategy for an agent returns a choice of whether to apply to authorities at each history and time for all agent types $\theta \in \Theta$, $\sigma_{\theta,t} : \mathcal{H}^{t-1} \to [0, 1]$. We moreover say that a strategy $a_{\tilde{c},t}$ for an authority $\tilde{c}$ at time t is *dominant* if it maximizes authority utility regardless of $\{\{a_{c,t}\}_{c \in \mathcal{C}/\{\tilde{c}\}}, \{\sigma_{\theta,t}\}_{\theta \in \Theta}\}_{t \in \mathcal{T}}$ and I.

Theorem 10. *The APM A_c^* is a dominant strategy for all authorities.*

Proof. We prove that APM A_c^* implements a dominant strategy for all authorities in all stages by backward induction. Consider the terminal time $t = |\mathcal{C}| - 1$. Some set of agents $\hat{\Theta} \subseteq \Theta$ applies to the authority. Regardless of $\hat{\Theta}$, by Theorem 7 we have that the set of agents chosen under any selection from APM A_c^* is first-best optimal. Thus, A_c^* is dominant. Consider now any time $t < |\mathcal{C}| - 1$, precisely the same argument applies and A_c^* is dominant. □

References

Akbarpour, M., E. B. Budish, P. Dworczak, and S. D. Kominers (2021): "An economic framework for vaccine prioritization," *Available at SSRN 3846931*.

Dur, U., S. D. Kominers, P. A. Pathak, and T. Sönmez (2018): "Reserve design: Unintended consequences and the demise of Boston's walk zones," *Journal of Political Economy*, 126(6), 2457–2479.

Dur, U., P. A. Pathak, and T. Sönmez (2020): "Explicit vs. statistical targeting in affirmative action: theory and evidence from Chicago's exam schools," *Journal of Economic Theory*, p. 104996.

Dur, U., M. Thayer, and W. Phan (2022): "Allocating Medical Resources During a Pandemic," *Working Paper*.

Grigoryan, A. (2021): "Effective, fair and equitable pandemic rationing," *Available at SSRN 3646539*.

Milgrom, P., and I. Segal (2002): "Envelope theorems for arbitrary choice sets," *Econometrica*, 70(2), 583–601.

Pathak, P. A., A. Rees-Jones, and T. Sönmez (2022): "Immigration lottery design: engineered and coincidental consequences of H-1B reforms," *Review of Economics and Statistics*, pp. 1–43.

Pathak, P. A., T. Sönmez, M. U. Unver, and M. B. Yenmez (2021): "Leaving No Ethical Value Behind: Triage Protocol Design for Pandemic Rationing," Discussion paper.