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Some phenomenological theoretical aspects of intrinsic superfluid critical velocities

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We use a phenomenological theory of superfluid critical velocities in small orifices to explain three phenomena: (1) The low-temperature intrinsic critical velocity v_c in dc experiments should depend logarithmically on the driving pressure head; the logarithmic increment in v_c is the width of the velocity distribution function for phase slips. (2) The critical velocity measured in dc experiments is related to that seen in ac experiments through the quantum number of the phase slips involved. (3) It is possible that quantum tunneling processes may be significant at temperatures as high as 0.3 K.

For over half a century physicists have recognized that dissipation-free superflow can exist only below some critical velocity, v_c. Measurements of this critical velocity have been made in many laboratories[1] exploring the effects of various parameters on the apparent breakdown of superfluidity. Out of this wealth of observational data, several theories have emerged to explain the observed critical velocity.

A prediction of what that velocity should be was made by Landau, who demonstrated that, above about 60 m/s, superflow would decay by means of creation of elementary excitations. Although the Landau velocity is a presumed upper limit to critical velocities, it is now understood that motion of quantized vortex lines can lead to dissipation mechanisms at velocities considerably less than the Landau limit. For example, the complex motion of a tangle of quantum turbulence can lead to dissipation processes in long channels.[2] Also, near the transition temperature T_λ, thermal activation of small quantized vortex rings can lead to finite dissipation.[3]

In 1965 Anderson described[4] the fundamental processes that might lead to flow dissipation when a pure superfluid (e.g., ^{4}He below 1 K) passes through a small orifice of area s in a very thin wall. He suggested that, at the critical velocity v_c, an instability could occur in the flow which would lead to the motion of a quantized vortex line across the orifice, crossing all the enclosed streamlines. In such a process, the order-parameter phase difference across the hole would change by 2π and a fixed amount of energy would be removed from the flow field. The energy decrement[4,5] due to a 2π phase slip is $\Delta E = \kappa \rho_s v_c s$, where κ is the quantum of circulation and ρ_s is the fluid's mass density.

A confirmation of the Anderson phase-slip picture came recently in the experiments of Avenel and Varoquaux (AV).[6] They studied the time evolution of a superfluid Helmholtz oscillator which contained a submicrometer orifice within the superfluid flow path. When the velocity in the orifice reaches a critical value, the oscillation amplitude abruptly drops by an amount corresponding to the energy removed by an Anderson 2π phase slip. This observation has recently been confirmed elsewhere.[7]

For several decades before the AV experiment, the predominant method for studying the critical velocity problem involved trying to measure the largest flow velocity at which fluid could pass through an orifice without any accompanying pressure drop.[1] Typically an initial pressure difference, ΔP, is applied across the orifice and, as ΔP relaxes to zero, the correlation between average mass current and pressure head is recorded. In the Anderson picture, the ideal current versus ΔP characteristic would exhibit a vertical line at $\Delta P = 0$ extending to a critical current. The curve would then follow an almost horizontal line, thus demonstrating that the flow is saturated at the critical value. For finite ΔP, the number of 2π phase slips per second would be given by[4] the Josephson-Anderson frequency, $f_{\text{JA}} = \Delta P / \rho_s \kappa$.

Actual data differs in several ways from the above ideal picture. The current versus ΔP curve exhibits a rapid but smooth transition to the dissipative regime. In addition, the mass current does not seem to totally saturate even when the pressure head is substantially higher than zero.[8,9] Commonly the observed current is noticeably different for the two directions of flow through the hole.[9,10] Finally, there appears to be two different critical current regimes: one being large and temperature dependent and a second smaller current which is temperature independent. How can such seemingly complex behavior be reconciled with the phase-slip picture that seems so clearly demonstrated in the AV experiment?

The central clues to answer this question come from several experimental observations. The first fact is that the phase-slip critical velocity, measured below 1 K in the small orifice experiments, is a linearly decreasing function of increasing temperature:

$$v_c = v_{c0}(1 - T/T_0) . \tag{1}$$

Here v_{c0} is approximately 10 m/s and $T_c = 2.5$ K. The second observation is that the velocity at which a phase slip occurs has a small random variation about the mean value.[1,7]

A final point comes from a recent experiment using an AV-type oscillator with one single orifice. It is observed that, although the critical velocity is the same for flow in both directions through the hole, slips may be all of size 2π for one direction yet be $2\pi n$ (where n is integer) for the other direction.[7]

The first point we wish to address is the explanation of why dc intrinsic critical velocities (i.e., the temperature dependent v_c) depend on the magnitude of the driving pressure head. We will derive the form of the v_c versus ΔP characteristic curve and show that its departure from the simplified picture mentioned above is due to the statistical nature of the of the phase-slip events.

It has been pointed out that,[11] in ac experiments, the observed linear temperature dependence of v_c, given by Eq. (1), can be deduced by assuming that the phase slip is initiated by a thermally activated process. The linear temperature dependence of v_c will follow if the activation energy E_a is a linearly decreasing function of superfluid velocity:

$$E_a = E_0(1 - v/v_{c0}) , \tag{2}$$

where the energy E_0 and the velocity v_{c0} are phenomenological constants connected to the fluctuation process.

The statistical nature of the thermal activation process leads to a finite width in the observed distribution function characterizing the critical velocity. The authors of Ref. 12 show that, if thermal activation is the sole source of phase-slip nucleation, then the observed values of the phase-slip critical velocity, v_c, should have a width, Δv_c, which is proportional to temperature. The actual data displayed in Ref. 12 suggest that thermal process are (especially at low temperatures) not the only mechanisms involved in the phase-slip problem. Nevertheless, a fit to the data, in a temperature range where the data appear linear, yields the value[12] of $E_0/k_B = 106$ K. Although it is not known whether this is a universal value characterizing the energy barrier, we will use it to make numerical estimates in what follows.

To derive the v_c versus ΔP curve for dc flow, we start from the same statistical assumptions and calculate the observed critical velocity seen in a pressure-driven flow experiment on a hole characterized by 2π phase slips. The important point in the derivation is that the total probability for a phase slip to occur near a velocity v, $g(v)dv$, depends on two factors. The first is the conditional probability rate, $\lambda(t)$, that the system experiences the slip at time t, provided that it did not occur previously. This function depends on the velocity $v(t)$. Using the activation energy given in Eq. (2),

$$\lambda(t) = \Gamma \exp\{-(\beta E_0)[1 - v(t)/v_{c0}]\} , \tag{3}$$

where Γ is an attempt frequency and $\beta = (k_B T)^{-1}$.

The second factor is the time the system spends near a given velocity, a time that is determined by the acceleration of the fluid. We consider the situation where the fluid in the orifice with effective length, L, experiences a constant pressure-driven acceleration $a = \Delta P/\rho_s L$. The probability density, $g(v)$, for having a phase slip at velocity v if the fluid is accelerated from an initial velocity v_i is given by[13,14]

$$g(v) = \left[\frac{\lambda(v)}{a}\right] \exp\left[-\int_{v_i}^{v} \frac{\lambda(v')}{a} dv'\right] . \tag{4}$$

Performing the integral gives, for $v > v_i$,

$$g(v) = A e^{\beta E_0 v/v_0} \exp[-(A v_{c0}/\beta E_0) \times (e^{\beta E_0 v/v_0} - e^{\beta E_0 v_i/v_0})] \tag{5}$$

with $A = (\Gamma/a)e^{-\beta E_0}$).

For the kind of orifices that are used in real experiments, a phase slip always makes the fluid velocity drop well below the values for which the probability density is significantly larger than zero. Thus, v_i in Eq. (5) can be replaced by zero without affecting the result of the following calculations.

By multiplying Eq. (5) by v and integrating over all possible velocities, we calculate the mean velocity at which a phase slip occurs, $\langle v_c \rangle$. After some effort one finds

$$\langle v_c \rangle = v_{c0}[1 - (k_B T/E_0)\ln(a_1 \rho_s L/\Delta P)] , \tag{6}$$

where the number

$$a_1 = 1.78(k_B T/E_0)\Gamma v_{c0} \tag{7}$$

and the numerical factor comes from evaluating a definite integral.

To the extent that the temperature dependence of the logarithmic term is very weak, Eq. (6) recovers the observed linear temperature dependence with $T_0 = (E_0/k_B)/\ln(a_1 \rho_s L/\Delta P)$.

One feature, very relevant for dc flow experiments, is the explicit dependence of $\langle v_c \rangle$ on driving pressure head. The logarithmic increment in critical velocity is given by

$$\langle v_c \rangle^{-1} d\langle v_c \rangle / d(\ln \Delta P) = (k_B T/E_0)/(1 - T/T_0) . \tag{8}$$

Using, in Eq. (8), the values of E_0 and T_0 given by experiment, we predict that the magnitude of the logarithmic increment, at 1.39 K, is 0.029. In a dc orifice-flow experiment, Hess reported[8] that, in the temperature range where Eq. (1) is obeyed, the logarithmic increment of the average velocity is 0.026±0.003 at $T = 1.39$ K. Though this velocity is not exactly a measurement of $\langle v_c \rangle$ (see below), the agreement with the prediction of Eq. (8) is suggestive although possibly coincidental.

A second prediction from Eq. (6) is that, in a dc experiment, the observed temperature intercept T_0 will depend logarithmically on ΔP. In the AV type of oscillator experiment one can directly measure the distribution function of phase slips and its associated width Δv_c. The analysis in Ref. 12 predicts this width to be

$$\Delta v_c / \langle v_c \rangle = (2/\ln 2)(k_B T/E_0)/(1 - T/T_0) , \tag{9}$$

which, except for the numerical factor, is precisely the same as the expression for the logarithmic increment Eq. (8). This explicitly shows that the increase in average critical velocity due to pressure head is due to the statistical width of the phase-slip distribution function.

At a sufficiently low temperature, quantum tunneling processes can dominate thermal fluctuations to produce a temperature-independent distribution width.[19] By the simplest argument one might estimate the crossover temperature to the quantum limit. By equating the characteristic Boltzmann factor exponent $E_0/k_B T$ to the exponent for tunneling through a square barrier of width w and energy height E_0, we find the characteristic quantum temperature T_Q:

$$T_Q=(hE_0^{1/2})/(2\pi w m^{1/2}) . \tag{10}$$

It is not obvious what the value of the mass m or length w should be in this equation. In Ref. 1 it is suggested that the characteristic length scale in the vortex nucleation process is on the order of 10^{-8} m. If we use this length and let the mass be the bare ^{4}He mass, one finds a value of $T_Q=0.35$ K. At such a temperature one would expect that the fractional width in the distribution function would be determined approximately half by thermal fluctuations and half by quantum fluctuations.

In actual measurements of the distribution width, Ref. 12 reports a linear temperature dependence superimposed on a temperature-independent part. The two parts are comparable at about 0.5 K. The similarity between this temperature and the estimate of T_Q given above may be coincidental. Clearly additional experiments and more detailed theory are needed to determine the role of quantum tunneling in the vortex nucleation problem.

Another point we will discuss is the relation between the average critical velocity measured in a dc experiment and the critical velocity characteristic of 2π phase slips. In the simplest dissipation event, a small vortex segment is created through some thermal fluctuation.[1] After creation the segment evolves in the flow field, crossing all the flow lines and annihilating at a boundary. It is also possible for the thermally created vortex to evolve in a more complicated manner, perhaps undergoing a few twists and reconnections before finally annihilating. This will lead to a $2\pi n$ phase slip and the energy dissipated will be n times greater than the 2π event.

The magnitude of the temperature-dependent phase-slip critical velocity $\langle v_c \rangle$ is symmetric with respect to flow direction. This is because, for pure potential flow in the absence of vortices, the flow pattern depends only on the boundaries and must be independent of direction. However, once a vortex is created, its motion can be very dependent on direction of flow and geometry.

An oscillator experiment of the AV type measures directly the velocity which initiates the phase slip. This critical velocity, $\langle v_c \rangle$, is independent of the n value of the slip. In contrast, a pressure-driven dc flow experiment measures the average mass current through the orifice. The average velocity $\langle v \rangle$, which characterizes the mass current, is not simply the critical velocity $\langle v_c \rangle$.

It is straightforward to calculate the relation between $\langle v \rangle$ and $\langle v_c \rangle$. We assume that the average current through an aperture is characterized by phase slips of size $2\pi n$. The applied pressure head ΔP causes the instantaneous velocity in the hole to increase linearly with time with acceleration $a=\Delta P/\rho_s L$, where L is the effective length of the orifice.[15] When the velocity reaches the critical velocity, the phase slip of $2\pi n$ causes the velocity to drop from the $\langle v_c \rangle$ to $\langle v_c \rangle - n\kappa/L$. (Here κ enters from the expression for the energy dissipated in a 2π phase slip.) The instantaneous velocity thus consists of periods of constant acceleration to $\langle v_c \rangle$ followed by abrupt drops. The process repeats at the frequency f_{JA}/n.

The time average of such a pattern is easily shown to be

$$\langle v \rangle = \langle v_c \rangle - n\kappa/2L . \tag{11}$$

Thus, the current through the orifice, which is proportional to $\langle v \rangle$ is not an indicator of $\langle v_c \rangle$ unless the multiple number n is known. If the slip number, n is a random variable, depending on T and ΔP, the dependence of $\langle v \rangle$ on ΔP would not necessarily reflect the simple temperature dependence given in Eq. (8). The disparity between some dc flow experiments may be contained in Eq. (11). For a typical effective length, $L=10^{-6}$ m, and a typical $\langle v_c \rangle = 5$ m/s, the second term in Eq. (11) is about an $n\%$ change in $\langle v \rangle$ relative to $\langle v_c \rangle$.

In recent experiments[7] using a submicron orifice, it was observed that, using the oscillator technique of AV, flow through the hole in one direction always produced single 2π phase slips, whereas flow in the opposite direction exhibited slips of $n>1$ at the same $\langle v_c \rangle$. In a dc flow measurement on the same hole, the flow in the direction characterized by single phase slips exhibited an average velocity very close to the $\langle v_c \rangle$ in the oscillator experiment. However, dc flow in the opposite direction displays an average velocity substantially less than that of the former case. In the latter case, expansion of the sensitivity of the flow detector actually permits observation of the individual large deceleration events. It is found that, when the current is small, the fluid is undergoing deceleration events characterized by large-n values.

The discussion above considers dissipation arising from discrete phase-slip events which are thermally initiated and finite in time. However, it is possible for dissipation to occur in quite a different manner. If the hole is sufficiently large, it is possible for a single, thermally activated vortex to grow indefinitely into a tangle of twisted vorticity.[16] In such an instance, the vortex tangle removes large amounts of energy continuously from the pressure drive. The average velocity in the orifice will then be small and temperature independent. This is the type of dissipation typically seen in holes larger than 10^{-6} m. The limiting velocity of such flow has no connection with the thermal activation $\langle v_c \rangle$ and can only be characterized by complex numerical simulations.

Some experiments have reported seeing spontaneous and uncontrollable switching between a large temperature-dependent flow to a small temperature-independent flow.[9,10] This would fit with the discussion above.[17]

It is interesting to note that, even in an oscillator experiment with a hole larger than 10^{-6} m, if the actual onset event leading to turbulence is detected, the critical velocity for this event is the large temperature-dependent value, even though subsequent motion of the oscillator is heavily damped due to the sustained vortex tangle near the orifice.[18]

We summarize the above comments by pointing out that we have explained several important points related to the understanding of the superfluid critical velocity. We have shown that the average velocity where phase slips occur is a function of the pressure head across the orifice. The effect arises due to the finite width of the phase-slip distribution function. We showed that the current through an orifice is characterized by both the critical velocity *and* the size of the phase slip. Differing dc measurements can be reconciled by assuming one critical velocity but differing multiple numbers for the slips. We point out that the existence of two different critical current regimes is consistent with the picture that the temperature-independent regime is characterized by sustained vortex evolution.[16]

The important remaining questions dealing with the onset of dissipation at low temperatures concern the (possibly tractable) problem to determine the size of v_{c0} and E_0 as well as the probably more complex question of trying to predict the time evolution of nascent vorticity within an orifice. The role of possible quantum fluctuations is also of interest.

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