Appendix A. The prediction operator and associated error covariance matrix

The core field and secular variation evolve according to

$$g_{\ell,k+1}^{m} = \alpha_\ell \left(g_{\ell,k}^{m} + \Delta t \dot{g}_{\ell,k}^{m}\right) + \omega_c \tag{A.1}$$

$$\dot{g}_{\ell,k+1}^{m} = \dot{\alpha}_\ell \, \dot{g}_{\ell,k}^{m} + \dot{\omega}_c \tag{A.2}$$

where ω_c and $\dot{\omega}_c$ are zero mean errors. To the first order their variances, v_{ω_c} and $\dot{v}_{\omega_c}$ respectively, are proportional to that of the coefficients of the secular acceleration, $\ddot{g}_\ell^m$: $v_{\omega_c} = \frac{\Delta t^4}{4} \ddot{v}_{g_\ell^m}$ and $\dot{v}_{\omega_c} = \Delta t^2 \, \ddot{v}_{g_\ell^m}$. Unfortunately, the variance of the secular acceleration coefficients, $\ddot{v}_{g_\ell^m}$, is not part of the priors we set. To circumvent this difficulty, we define a time scale $\tilde{\tau}_{sa} = \sqrt{\frac{\dot{v}^0_{g_\ell^m}}{\ddot{v}_{g_\ell^m}}}$, where $\dot{v}^0_{g_\ell^m}$ is the prior variance of the secular variation coefficients. This definition differs from the usual τ_{sa} time scale definition (see e.g. – Christensen et al. (2012)). Nonetheless we assume that $\tilde{\tau}_{sa}$ takes similar values, which are around 10-15 years up to spherical harmonics degree 13 and get smaller for higher spherical harmonics degrees (Christensen et al. 2012). Therefore we set $\tilde{\tau}_{sa} = 11$years, independently of the spherical harmonics degree, even if we know that this value is likely too large for the largest spherical harmonics degrees of our secular variation model. Similarly we defined $\tilde{\tau}_{sv} = \sqrt{\frac{v^0_{g_\ell^m}}{\dot{v}^0_{g_\ell^m}}}$ where $v^0_{g_\ell^m}$ is the prior variance of the core field coefficients. With these definitions, the variances v_{ω_c} and $\dot{v}_{\omega_c}$ reduces to:

$$v_{\omega_c} = \frac{1}{4}\left(\frac{\Delta t}{\tilde{\tau}_{sv}}\right)^2 \left(\frac{\Delta t}{\tilde{\tau}_{sa}}\right)^2 v^0_{g_\ell^m}$$

and

$$\dot{v}_{\omega_c} = \left(\frac{\Delta t}{\tilde{\tau}_{sa}}\right)^2 \dot{v}^0_{g_\ell^m},$$

respectively.

We turn now to the estimation of the α_ℓ and $\dot{\alpha}_\ell$ values in equations (A.1), (A.2). Starting with the core field evolution, we compute α_ℓ from eq. (A.1) and from the stationary hypothesis. The latter implies that the variance $v_{g_\ell^m}$ is independent of time. Therefore equation (A.1) gives:

$$v^0_{g_\ell^m} = \alpha_l^2 \left(v^0_{g_\ell^m} + \Delta t^2 v^0_{\dot{g}_\ell^m}\right) + v_{\omega_c}, \tag{A.3}$$

and therefore:

$$\alpha_\ell = \sqrt{\frac{1 - \frac{1}{4}\left(\frac{\Delta t}{\tilde{\tau}_{sv}}\right)^2 \left(\frac{\Delta t}{\tilde{\tau}_{sa}}\right)^2}{1 + \left(\frac{\Delta t}{\tilde{\tau}_{sv}}\right)^2}}. \tag{A.4}$$

Now, for the evolution of the SV, we evaluate the value of $\dot{\alpha}_\ell$ from eq. (A.2), and with the stationary hypothesis it gives:

$$\dot{v}^0_{g^m_\ell} = \dot{\alpha}_l^2\, \dot{v}^0_{g^m_\ell} + \dot{v}_{\omega_c},$$

leading to

$$\dot{\alpha}_\ell = \sqrt{1 - \left(\frac{\Delta t}{\tilde{\tau}_{sa}}\right)^2}. \tag{A.5}$$

It is now possible to describe precisely the prediction operator $\mathbf{P}$. This operator can be presented as a block diagonal matrix, where there is a single block for the core and its SV, and a block for each of the other individual sources. These individual sources evolve as an AR1 process (see section Prediction step). Therefore the blocks are diagonal and their diagonal coefficients are the α from equation (11).

For the core field and secular variation block, we define 4 diagonal sub-blocks: $\mathbf{P}^{cc}$, $\mathbf{P}^{ss}$, $\mathbf{P}^{cs}$ and $\mathbf{P}^{sc}$, such that the block can be presented as:

$$\begin{pmatrix} \mathbf{P}^{cc} & \mathbf{P}^{cs} \\ \mathbf{P}^{sc} & \mathbf{P}^{ss} \end{pmatrix}.$$

$\mathbf{P}^{cc}$ (reps. $\mathbf{P}^{ss}$) relates the core field coefficients (resp. secular variation coefficients) at time step k to those at time step $k+1$. Similarly, $\mathbf{P}^{sc}$ (reps. $\mathbf{P}^{cs}$) relates the core field coefficients (resp. secular variation coefficients) at time step k to those of the secular variation (resp. core field) at time step $k+1$. The diagonal elements of $\mathbf{P}^{cc}$ are the α_ℓ from eq. (A.1) and those of $\mathbf{P}^{ss}$ are the $\dot{\alpha}_\ell$ from eq. (A.2). $\mathbf{P}^{sc}$ is an empty sub-block, and $\mathbf{P}^{cs}$ diagonal elements are $(\Delta t\, \alpha_\ell)$ as it can be derived from equation (A.1).

We explicit the error covariance matrix $\mathbf{C_w}$, which has also a block diagonal structure. It is composed of one block matrix for each source. For all sources, we introduce the diagonal matrix $\mathbf{D}^{n_s}$ where n_s designates the source. The diagonal terms of $\mathbf{D}^{n_s}$, $d_i^{n_s}$ are defined as follows:

$$\begin{aligned} d_i^{cc} &= \frac{1}{2}\frac{\Delta t^2}{\tilde{\tau}_{sa}} && \text{for the core} \\ d_i^{ss} &= \frac{\Delta t}{\tilde{\tau}_{sa}} && \text{for the SV} \\ d_i^{n_s} &= \sqrt{1-\alpha_{n_s}^2} && \text{for all the other sources} \end{aligned} \tag{A.6}$$

where Δt is the time step, $\tilde{\tau}_{sa}$ is the time scale defined above, and α_{n_s} is the factor governing the evolution of AR1 processes, defined in eq. (12). The superscripts cc and ss designates the core field and SV respectively. The superscript n_s stands for any of the other sources, that we recall evolve like AR1 processes (eq. (11)).

We also denote $\widetilde{\mathbf{C}}_0^{n_s}$ the block of $\widetilde{\mathbf{C}}_0$ corresponding to the source n_s. We recall that $\widetilde{\mathbf{C}}_0$ is introduced in section Analysis step. It contains the prior information on the spatial behaviour of every source modelled, through block matrices that can be diagonals (for all sources in the case of the HS model) or full (for the core field and SV in the case of the CE model).

For source n_s, the corresponding block of $\mathbf{C}_{\mathbf{w}}^{n_s}$ is therefore:

$$\begin{aligned} \mathbf{C}_{\mathbf{w}}^{cc} &= \mathbf{D}^{cc}\,\widetilde{\mathbf{C}}_0^{ss}(\mathbf{D}^{cc})^t && \text{for the core} \\ \mathbf{C}_{\mathbf{w}}^{ss} &= \mathbf{D}^{ss}\,\widetilde{\mathbf{C}}_0^{ss}(\mathbf{D}^{ss})^t && \text{for the SV} \\ \mathbf{C}_{\mathbf{w}}^{n_s} &= \mathbf{D}^{n_s}\,\widetilde{\mathbf{C}}_0^{n_s}(\mathbf{D}^{n_s})^t && \text{for all the other sources} \end{aligned} \tag{A.7}$$

where the super-script t denotes the transpose.

References

U. Christensen, I. Wardinski, and V. Lesur. Time scales of geomagnetic secular acceleration in satellite field models and geodynamo models. Geophys J. Int., 190:243–254, 2012. DOI: 10.1111/j.1365-246X.2012.05508.x.