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Version Accepted Manuscript

Citation (published version) Lang, Hans-Dieter, Alon Ludwig, and Costas D. Sarris. "Convex optimization of wireless power transfer systems with multiple transmitters." IEEE Transactions on Antennas and Propagation 62, no. 9 (2014): 4623-4636.

DOI https://doi.org/10.1109/TAP.2014.2330584

Convex Optimization of Wireless Power Transfer Systems with Multiple Transmitters

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***Abstract*—Wireless power transfer systems with multiple transmitters promise advantages of higher transfer efficiencies and focusing effects over single transmitter systems. From the standard formulation, straightforward maximization of the power transfer efficiency is not trivial. By reformulating the problem, a convex optimization problem emerges, which can be solved efficiently. Further, using Lagrangian duality theory, analytical results are found for the achievable maximum power transfer efficiency and all parameters involved. With these closed-form results, planar and coaxial wireless power transfer setups are investigated.**

***Index Terms*—Convex Optimization, Maximum Transfer Efficiency, Multiple Transmitters, Wireless Power Transfer.**

I. Introduction

WIRELESS power transfer (WPT) continues to gain much attention ever since Tesla's ideas [1] were rediscovered, realized and extended a few years ago [2]–[4]. Besides many applications, also the fundamentals, particularly the limits of the transfer efficiency and enhancements thereof have been studied [5]–[7].

More recently, systems with multiple transmitters were investigated, both theoretically and experimentally [8]–[10]. Multiple transmitters provide more degrees of freedom of the primary field or current distribution and, therefore, promise the possibility of enhanced transfer efficiency as compared to single transmitter systems. In this paper, systems with multiple transmitters and a single receiver are investigated. In this context, they will be called MISO (for multiple-input single-output) systems, similar as, for example, in communication systems. In contrast, systems with only one active transmitter and a single receiver are called SISO (single-input, single-output) WPT systems.

Optimization of WPT systems with multiple transmitters (MISO WPT systems), particularly the required excitations (voltages) and the capacitive loading to maximize the achievable power transfer efficiency is not trivial. Numerous tests have shown that even more elaborate and generally powerful global optimization algorithms, such as pattern search and genetic algorithms as well as particle swarm optimizers have problems finding the best parameters — or they lead to infeasible computational cost and long optimization periods. In addition, using these methods, physical insight into the problem and its characteristics is not necessarily available in a straightforward manner, but usually can only be obtained by backtracking the end results.

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Manuscript received January xx, 2014; revised Month xx, 2014.

In this paper, a physically intuitive design procedure using convex optimization is presented. Besides being able to obtain results such as the maximum power transfer efficiency and the required excitations and capacitive loading efficiently, each step of the formulation is physically meaningful. Furthermore, using Lagrangian duality theory, analytical solutions for all parameters involved are found. Applied to the WPT systems with one or more transmitters, various interesting conclusions can be drawn, for example in relation to subwavelength focusing [11]–[13].

It is well understood that impedance matching of the receiver load to the total WPT system as source is important to achieve maximum transfer efficiency [14]–[16]. However, although it might be feasible to tune capacitors during operation (e.g. using varactors), adjustment of the load resistance via (tunable) matching network is difficult and results in additional loss. Therefore, the general question to be answered within this paper is: What is the maximum achievable power transfer efficiency for a particular WPT geometry/setup to a load R_L and how can it be achieved. Additionally, by analyzing the closed-form expression of the maximum transfer efficiency with respect to R_L, the optimum $R_L^\star$ as well as the maximum achievable power transfer efficiency are found.

The outline of this paper is as follows. First, in Section II, the problem setup is introduced and the general mathematical relations are given. Second, in Section III, the convex optimization procedure is derived. In Section IV, using Lagrangian duality theory, analytical solutions to the optimization problems are formulated for all parameters involved. Section V discusses and compares some basic results and performances of MISO and SISO WPT systems. Finally, in Section VI, a summary and conclusions are given.

II. Problem Statement

A remark on the notation: Italicized letters represent variables, bold small letters refer to vectors, bold capital letters are matrices (and similarly for functions resulting in vectors and matrices); $\mathbf{v}^T$ stands for the transpose of the vector $\mathbf{v}$ while $\mathbf{v}^H$ stands for its Hermitian (conjugate transpose). Real parts of complex numbers are denoted by $(\cdot)'$, imaginary parts by $(\cdot)''$, and the conjugate complex is denoted by $(\cdot)^*$, not to be confused with the optimum solution $(\cdot)^\star$ to a particular optimization problem.

A. System geometry and electrical setup

The WPT systems considered here consist of transmitter loops which are located at positions (x_n, y_n, z_n), for $n = 1, ..., N_{\text{tx}}$ (where N_{tx} stands for the total number of transmitters, the total number of loops is $N = N_{\text{tx}} + 1$) and one receiver loop, located at (x_N, y_N, z_N), as shown in Fig. 1.

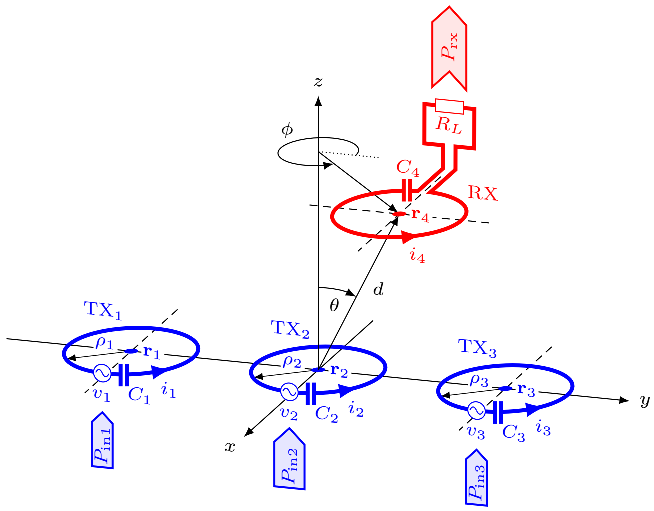


Fig. 1. Illustration of the problem setup: multiple transmitters (here three loops TX$_1$ to TX$_3$, $N_{\text{tx}} = 3$), at the positions (x_n, y_n, z_n), $n = 1, ..., N_{\text{tx}}$ and one receiver at $(x_N, y_N, z_N) = (d \sin\theta \cos\phi, d \sin\theta \sin\phi, d \cos\theta)$. Each loop is loaded by a capacitor C_n, the receiver end also contains the resistive load R_L.

Each loop is loaded with a capacitor C_n (corresponding to the respective capacitive reactance of interest at the operating frequency). Additionally, the receiver loop is loaded with the resistive load R_L. Each transmitter loop is excited by a voltage source $v_n \in \mathbb{C}$.

Note that, although here only cases with aligned (parallel) loops are considered, this is not a requirement. In fact, the derivation is general and can be used for impedance matrices of arbitrary power transfer systems with one or more transmitters and a single receiver.

The idea is to assess the maximum achievable power transfer efficiency of a given WPT system in a given direction. Thus, the receiver loop is positioned at various points in space, scanning over a region around the transmitter loops. At each position, the capacitors and the excitations are to be optimized so as to maximize the total power transfer efficiency η.

B. Initial problem formulation

Using a voltage vector $\mathbf{v} = \left[v_1, v_2, ..., v_N\right]$ and a current vector $\mathbf{i} = \left[i_1, i_2, ..., i_N\right]$ (all $v_n, i_n \in \mathbb{C}$), where the last (Nth) entries correspond to the receiver loop, the full system can be formulated as

$$\mathbf{v} = (\mathbf{Z} - j\mathbf{X}_c + \mathbf{R}_L)\,\mathbf{i} = \mathbf{Z}_{\text{load}}\mathbf{i}\,, \tag{1}$$

where $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$, with $\mathbf{Z} \in \mathbb{C}^{N\times N}$ and $\mathbf{R}, \mathbf{X} \in \mathbb{R}^{N\times N}$, is the unloaded impedance matrix of the WPT setup and $\mathbf{Z}_{\text{load}} = \mathbf{Z}'_{\text{load}} + j\mathbf{Z}''_{\text{load}} = \mathbf{R}_{\text{load}} + j\mathbf{X}_{\text{load}}$ is the fully loaded impedance matrix, to be optimized at a later point. The capacitive loading comes from $\mathbf{X}_c = \mathbf{Diag}(\mathbf{x}_c)$, a diagonal matrix containing the (real) reactances of the capacitors,

$$\mathbf{x}_c = \left[x_1, ..., x_{N_{\text{tx}}}, x\right] = \left[\frac{1}{\omega C_1}, ..., \frac{1}{\omega C_{N_{\text{tx}}}}, \frac{1}{\omega C_N}\right]\,, \tag{2}$$

where all x_n, x and C_n are real variables and ω is the angular frequency of operation. Note that the last reactance value is denoted by x instead of x_N to simplify the notation throughout the derivation later on. Finally, the load matrix

$$\mathbf{R}_L = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & R_L \end{bmatrix} \tag{3}$$

is zero everywhere but at the very last element, corresponding to the receiver; $R_L > 0$ denotes the (real) load resistance at the receiver end.

In more detail, the total system is

$$\underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_{N-1} \\ 0 \end{bmatrix}}_{\mathbf{v}} = \underbrace{\begin{bmatrix} z_{11} - \frac{j}{\omega C_1} & z_{12} & \cdots & z_{1N} \\ z_{21} & z_{22} - \frac{j}{\omega C_2} & \cdots & z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ z_{N1} & z_{N2} & \cdots & z_{NN} - \frac{j}{\omega C_N} + R_L \end{bmatrix}}_{\mathbf{Z}_{\text{load}}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix}}_{\mathbf{i}} \tag{4}$$

where z_{nm} denotes the elements from the unloaded impedance matrix $\mathbf{Z}$. Note that $v_N = 0$, since the load R_L is part of the loaded impedance matrix $\mathbf{Z}_{\text{load}}$.

The goal is to find the optimal capacitive loading $\mathbf{x}_c$ and currents $\mathbf{i}$ so as to maximize the total transfer efficiency, given by

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}}\,, \tag{5}$$

where

$$P_{\text{in}} = \sum_{n=1}^{N_{\text{tx}}} P_{\text{in},n} = \frac{1}{2}\sum_{n} \text{Re}\left[v_n i_n^*\right] = \frac{1}{2}\,\text{Re}\left[\mathbf{i}^H \mathbf{v}\right] \tag{6}$$

and

$$P_{\text{out}} = \frac{1}{2}\,|i_N|^2 R_L = \frac{1}{2}\,\mathbf{i}^H \mathbf{R}_L \mathbf{i}\,, \tag{7}$$

under the constraint that the actually transferred power is non-zero, or more precisely larger than some threshold value, $P_{\text{out}} \geq P_{\text{out,min}} \geq 0$.

Various tests have confirmed that it is difficult to optimize η from Eqns. (1), (6) and (7) directly, for example using global optimization methods such as genetic and pattern search algorithms as well as particle swarm optimizers. Often, the results vary greatly between receiver positions arbitrarily close to each other (e.g. between points at distances much smaller than a quarter-wavelength), or they are unsymmetrical, even though the problem geometry is symmetric, as shown in Fig. 2 (see setups in Fig. 3). These numerical artifacts originate from both Re[·] being a non-analytic operation and the non-convex (as well as nonlinear and highly resonant) form of Eqn. (5) using Eqns. (1), (6) and (7) in general.

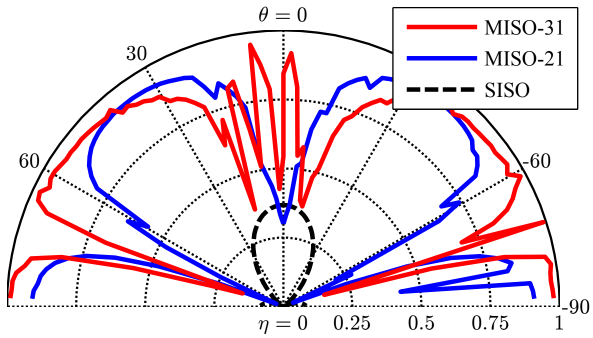


Fig. 2. Typical result of optimizing the power transfer efficiency directly from its standard formulation (real and imaginary parts of voltages, capacitor values) using genetic algorithm (population size: 50, maximum number of generations: 1000). The resulting efficiency pattern is spiky and unsymmetric (even though the geometry is symmetric).

III. Formulation of a Convex Optimization Problem for Maximizing MISO WPT Efficiency

Before the convex optimization problem can be derived, the voltages and currents are put into a vector form of separated real and imaginary parts to obtain a fully real-valued formulation.

The total input power over all transmitters is calculated by

$$P_{\text{in}} = \frac{1}{2}\,\text{Re}\left[\mathbf{i}^H\mathbf{v}\right] = \frac{1}{2}\sum_n (i'_n v'_n + i''_n v''_n) = \frac{1}{2}\,\mathbf{c}^T\mathbf{w}\,, \quad (8)$$

where $\mathbf{c} = \left[\mathbf{i}'^T, \mathbf{i}''^T\right]^T$ and $\mathbf{w} = \left[\mathbf{v}'^T, \mathbf{v}''^T\right]^T$ are real vectors ($\mathbf{c}, \mathbf{w} \in \mathbb{R}^{2N}$) containing the separated real and imaginary parts of the currents and voltages, respectively.

In similar fashion, for the output power, it is obtained that

$$P_{\text{out}} = \frac{1}{2}\,\mathbf{i}^H\mathbf{R}_L\mathbf{i} = \frac{1}{2}\,\mathbf{c}^T\tilde{\mathbf{R}}_L\mathbf{c}\,, \quad (9)$$

where

$$\tilde{\mathbf{R}}_L = \begin{bmatrix}\mathbf{R}_L & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_L\end{bmatrix}\,. \quad (10)$$

Using these expressions, the efficiency can be given in standard form, involving only real-valued unknowns

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{\mathbf{i}^H\mathbf{R}_L\mathbf{i}}{\text{Re}\left[\mathbf{i}^H\mathbf{v}\right]} = \frac{\mathbf{c}^T\tilde{\mathbf{R}}_L\mathbf{c}}{\mathbf{c}^T\mathbf{w}}\,, \quad (11)$$

where the (separated) currents and voltages are related by

$$\mathbf{c} = \begin{bmatrix}\mathbf{R}_{\text{load}} & -\mathbf{X}_{\text{load}}\\ \mathbf{X}_{\text{load}} & \mathbf{R}_{\text{load}}\end{bmatrix}^{-1}\mathbf{w}$$

$$= \begin{bmatrix}\mathbf{Z}' + \mathbf{R}_L & -\mathbf{Z}'' + \mathbf{X}_c\\ \mathbf{Z}'' - \mathbf{X}_c & \mathbf{Z}' + \mathbf{R}_L\end{bmatrix}^{-1}\mathbf{w} \quad (12)$$

similar to $\mathbf{i} = \mathbf{Z}_{\text{load}}^{-1}\mathbf{v}$. Note that at this point only the capacitor reactances on the diagonal of $\mathbf{X}_c$ and the voltages $\mathbf{w}$ are unknowns; the currents $\mathbf{c}$ follow from Eqn. (12).

A. Convex optimization formulation

In the following, the problem is reformulated using the currents $\mathbf{c}$ instead of the voltages $\mathbf{w}$. Combined with the assumption of unit received power, a convex optimization problem is obtained.

1) Implicit voltages: The voltages $\mathbf{w}$ in the denominator of Eqn. (11) are replaced by their relation to the currents Eqn. (12). From the loaded impedance matrix $\mathbf{v} = \mathbf{Z}_{\text{load}}\mathbf{i}$, it follows for Eqns. (6) and (8), since $\mathbf{Z}_{\text{load}} = \mathbf{Z}_{\text{load}}^T$,

$$P_{\text{in}} = \frac{1}{2}\,\text{Re}\left[\mathbf{i}^H\mathbf{v}\right] = \frac{1}{2}\,\text{Re}\left[\mathbf{i}^H\mathbf{Z}_{\text{load}}\mathbf{i}\right] = \frac{1}{2}\,\mathbf{i}^H(\text{Re}\,\mathbf{Z}_{\text{load}})\mathbf{i}$$

$$= \frac{1}{2}\left[\mathbf{i}'^T, \mathbf{i}''^T\right]\begin{bmatrix}\mathbf{R}_{\text{load}} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_{\text{load}}\end{bmatrix}\begin{bmatrix}\mathbf{i}'\\ \mathbf{i}''\end{bmatrix} = \frac{1}{2}\,\mathbf{c}^T\tilde{\mathbf{R}}_{\text{load}}\mathbf{c}\,, \quad (13)$$

where $\tilde{\mathbf{R}}_{\text{load}}$ is used to denote the matrix of the real parts of $\mathbf{Z}_{\text{load}}$ (i.e. whereas $\mathbf{R}_{\text{load}} = \text{Re}\,\mathbf{Z}_{\text{load}}$, so relates $\tilde{\mathbf{R}}_{\text{load}}$ similarly to $\tilde{\mathbf{Z}}_{\text{load}}$).

The power transfer efficiency is then given by

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{\mathbf{c}^T\tilde{\mathbf{R}}_L\mathbf{c}}{\mathbf{c}^T\tilde{\mathbf{R}}_{\text{load}}\mathbf{c}}\,, \quad (14)$$

which only contains $\mathbf{c} \in \mathbb{R}^{2N}$ as unknowns, since the capacitor values are not part of $\tilde{\mathbf{R}}_{\text{load}}$. However, as will be shown later, one of them is part of the equality constraints of the optimization.

It should be noted that this form of η is not generally concave (or $1/\eta$ not convex) and it does not account for the zero voltage at the receiver, $v_N = 0$, yet.

2) Convex reformulation of the objective function: By setting the received power to a constant $P_{\text{out}} = \mathbf{c}^T\tilde{\mathbf{R}}_L\mathbf{c} = 1$, meaning that 1 W power is received at the receiver load, the objective function for minimization according to

$$\eta_{\max} = \max\eta = \left[\min\frac{1}{\eta}\right]^{-1} = \left[\min_{\mathbf{c}} f(\mathbf{c})\right]^{-1} \quad (15)$$

(with $\eta \in (0, 1]$, since $P_{\text{out}} \neq 0$) becomes

$$f(\mathbf{c}) = \mathbf{c}^T\tilde{\mathbf{R}}_{\text{load}}\mathbf{c} \quad (16)$$

and is convex in $\mathbf{c}$, provided that $\tilde{\mathbf{R}}_{\text{load}} \geq 0$ (positive semi-definite), which is always true (apart from some numerical issues, addressed in the Appendix A) for impedance matrices of passive networks [22]. In fact, since WPT systems are always lossy (there is at least some small loss due to radiation), the real part of the impedance matrices has to be positive definite, $\text{Re}\,\mathbf{Z}_{\text{load}} = \mathbf{R}_{\text{load}} > 0$, which also leads to $\tilde{\mathbf{R}}_{\text{load}} > 0$. Thus, this objective function is well-suited for convex optimization (minimization).

3) Constraints: As mentioned, the constraints have to account for the voltage at the receiver port being zero, since the load is part of the impedance matrix, as shown in Eqn. (4). Thus,

$$v_N = \sum_k z_{Nk}^{\text{load}} i_k = \mathbf{z}_{\text{load}\,N}\,\mathbf{i} = 0\,, \quad (17)$$

with $\mathbf{z}_{\text{load}\,N}$ being the Nth (last) row of the loaded impedance matrix $\mathbf{Z}_{\text{load}}$, corresponding to the receiver voltage v_N. Separating real from imaginary parts, this translates to

$$v'_N = w_N = \left[\mathbf{z}'_{\text{load}\,N}, -\mathbf{z}''_{\text{load}\,N}\right]\mathbf{c} = 0 \quad (18a)$$

$$v''_N = w_{2N} = \left[\mathbf{z}''_{\text{load}\,N}, \mathbf{z}'_{\text{load}\,N}\right]\mathbf{c} = 0\,. \quad (18b)$$

Additionally, one is free to chose the real and imaginary parts of the receiver current $i_N = c_N + jc_{2N}$, as long as the received power remains $P_{\text{out}} = \mathbf{c}^T \tilde{\mathbf{R}}_L \mathbf{c} = \left(c_N^2 + c_{2N}^2\right) R_L = 1$. The natural choice is $i_N \in \mathbb{R}$ (zero phase at the receiver), and thus

$$i'_N = c_N = \frac{1}{\sqrt{R_L}} \tag{19a}$$
$$i''_N = c_{2N} = 0 \,, \tag{19b}$$

which will have to be accounted for in the constraints during the optimization process.

These four constraints, Eqns. (18a) and (18b) as well as (19a) and (19b), are then cast into the matrix form

$$\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\\ \mathbf{a}_4\end{bmatrix} \mathbf{c} = \begin{bmatrix}b_1\\ b_2\\ b_3\\ b_4\end{bmatrix} \quad \Leftrightarrow \quad \mathbf{A}_{\text{eq}}\mathbf{c} = \mathbf{b}_{\text{eq}} \,, \tag{20}$$

which is well-known to be affine in $\mathbf{c}$ [17], where $\mathbf{A}_{\text{eq}} \in \mathbb{R}^{4\times 2N}$ and $\mathbf{b}_{\text{eq}} \in \mathbb{R}^4$. Note that all constraints for this problem are of the equality type; there are no inequality constraints.

In more detail, the first two constraints in Eqn. (20) are

$$\mathbf{a}_1\mathbf{c} = b_1 = 0 \tag{21a}$$
$$\mathbf{a}_2\mathbf{c} = b_2 = 0 \,, \tag{21b}$$

where $\mathbf{a}_1 = \left[\mathbf{z}'_{\text{load}\,N}, -\mathbf{z}''_{\text{load}\,N}\right]$ and $\mathbf{a}_2 = \left[\mathbf{z}''_{\text{load}\,N}, \mathbf{z}'_{\text{load}\,N}\right]$. The second pair of constraints are

$$c_N = \mathbf{a}_3\mathbf{c} = b_3 = 1/\sqrt{R_L} \tag{22a}$$
$$c_{2N} = \mathbf{a}_4\mathbf{c} = b_4 = 0 \,, \tag{22b}$$

where $\mathbf{a}_3$ and $\mathbf{a}_4$ are row vectors selecting the corresponding Nth and $2N$th (last) entry, respectively, from the (real) current vector $\mathbf{c}$.

4) Partially convex optimization formulation: Using the objective function, Eqn. (16), and the constraints, Eqn. (20), a quadratic programming (QP) problem [17] is obtained:

$$\begin{aligned}\eta_{\max}(x) &= \max_{\mathbf{c}} \eta \\ &= \Big[\min_{\mathbf{c}} \underbrace{\mathbf{c}^T \tilde{\mathbf{R}}_{\text{load}} \mathbf{c}}_{\text{convex in } \mathbf{c}} \ \text{ s.t. } \ \underbrace{\mathbf{A}_{\text{eq}}(x)\mathbf{c} = \mathbf{b}_{\text{eq}}}_{\text{affine in } \mathbf{c}}\Big]^{-1} .\end{aligned} \tag{23}$$

Even though the reactance values x_n ($n = 1, ..., N_{\text{tx}}$) are not part of the convex optimization problem, the receiver reactance x (the last entry in $\mathbf{x}_c$) cannot be optimized this way, since it is part of the equality constraints, $\mathbf{A}_{\text{eq}} = \mathbf{A}_{\text{eq}}(x)$. Thus, the result is the maximum transfer efficiency given the receiver reactance x, $\eta_{\max}(x)$.

The problem of finding the maximum achievable transfer efficiency can be solved by choosing some receiver reactance x, optimizing η by finding the best currents $\mathbf{c}$ and then going back and choosing a better x, resulting in an outer optimization loop for x:

$$\eta_{\max} = \max_x \eta_{\max}(x) = \max_x \left\{\max_{\mathbf{c}} \eta(x)\right\} \,. \tag{24}$$

5) Fully convex optimization formulation: Using the known receiver current components $i_N = c_N + jc_{2N}$, given from Eqns. (19a) and (19b), explicitly (thereby reducing $\mathbf{c}$ by two entries, leading to the transmitter current vector $\mathbf{c}_{\text{tx}}$), the constraints become

$$\underbrace{\begin{bmatrix} z'_{N1}, \dots, z'_{NN_{\text{tx}}}, -z''_{N1}, \dots, -z''_{NN_{\text{tx}}} \\ z''_{N1}, \dots, z''_{NN_{\text{tx}}}, z'_{N1}, \dots, z'_{NN_{\text{tx}}} \end{bmatrix}}_{\begin{bmatrix}\mathbf{z}'_{\text{tx}}, -\mathbf{z}''_{\text{tx}}\\ \mathbf{z}''_{\text{tx}}, \mathbf{z}'_{\text{tx}}\end{bmatrix}} \underbrace{\begin{bmatrix} c_1\\ \vdots\\ c_{N-1}\\ c_{N+1}\\ \vdots\\ c_{2N-1}\end{bmatrix}}_{\mathbf{c}_{\text{tx}}} + \frac{1}{\sqrt{R_L}} \begin{bmatrix} z'_{NN} + R_L\\ z''_{NN} - x\end{bmatrix} = 0 \,, \tag{25}$$

where as always $N_{\text{tx}} = N - 1$, for MISO WPT systems. The receiver reactance x is an unknown and is, therefore, added to the unknown vector, whereby $\mathbf{c}_{\text{tx}}$ becomes $\hat{\mathbf{c}}$, the modified currents vector (losing some of its physical meaning, since it contains not just currents anymore):

$$\underbrace{\begin{bmatrix}\mathbf{z}'_{\text{tx}}, -\mathbf{z}''_{\text{tx}}, 0\\ \mathbf{z}''_{\text{tx}}, \mathbf{z}'_{\text{tx}}, -1/\sqrt{R_L}\end{bmatrix}}_{\hat{\mathbf{A}}_{\text{eq}}} \underbrace{\begin{bmatrix} c_1\\ \vdots\\ c_{N_{\text{rx}}}\\ c_{N+1}\\ \vdots\\ c_{N+N_{\text{rx}}}\\ x\end{bmatrix}}_{\hat{\mathbf{c}}} = \underbrace{-\frac{1}{\sqrt{R_L}} \begin{bmatrix} z'_{NN} + R_L\\ z''_{NN}\end{bmatrix}}_{\hat{\mathbf{b}}_{\text{eq}}} . \tag{26}$$

From here on, hats mark all "modified" vectors and matrices (i.e. such with components/conditions of the reactance added to the ones for the currents and not including the known receiver currents) and $\mathbf{z}'_{\text{tx}}$ and $\mathbf{z}''_{\text{tx}}$ refer to the real and imaginary parts, respectively of the unloaded impedance matrix $\mathbf{Z}$, coupling each of the transmitters to the receiver (the N_{tx} first elements of the Nth (last) row of $\mathbf{Z}$).

The matrix of the modified affine constraints is then of dimensions $2 \times (2N - 1)$, since it only has to account for the transmitter voltage to be zero. This way, the receiver reactance can be optimized together with the transmitter currents, simultaneously, in a single optimization. The problem is still of the quadratic programming type, but it has to be adjusted somewhat, since the receiver power and mutual parts of the receiver and transmitters are missing in the quadratic term:

$$\mathbf{c}^T \tilde{\mathbf{R}}_{\text{load}} \mathbf{c} = \hat{\mathbf{c}}^T \hat{\tilde{\mathbf{R}}}_{\text{tx}} \hat{\mathbf{c}} + 2\hat{\mathbf{q}}^T \hat{\mathbf{c}} + p \,, \tag{27}$$

where the modified matrix of the quadratic term is

$$\hat{\tilde{\mathbf{R}}}_{\text{tx}} = \begin{bmatrix}\tilde{\mathbf{R}}_{\text{tx}} & \mathbf{0}\\ \mathbf{0} & 0\end{bmatrix} = \begin{bmatrix}\mathbf{R}_{\text{tx}} & \mathbf{0} & 0\\ \mathbf{0} & \mathbf{R}_{\text{tx}} & \vdots\\ 0 & \cdots & 0\end{bmatrix} \geq 0 \,, \tag{28}$$

with $\mathbf{R}_{\text{tx}}$ being the part of the resistance matrix $\mathbf{R}$ corresponding to the transmitters (i.e. $\mathbf{R}$ reduced by the last column and row). Note that this part of the matrix is the same for both $\mathbf{R}$ and $\mathbf{R}_{\rm load}$ (since $\mathbf{R}_{\rm load} = \mathbf{R} + \mathbf{R}_L$).

The linear term is (since all resistance matrices are symmetric)

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{R_L}} \begin{bmatrix} \mathbf{z}'^{T}_{\rm tx} \\ \mathbf{0} \end{bmatrix} \tag{29}$$

and the power in the receiver loop is $p = (z'_{NN} + R_L)/R_L$ (both loss and power dissipated in the load).

Thus, the final single quadratic programming problem to optimize both currents and the receiver reactance such as to maximize the transfer efficiency (or rather minimize its inverse) is

$$\eta_{\max} = \left\{ \begin{array}{l} \min\limits_{\hat{\mathbf{c}}} \hat{\mathbf{c}}^T \hat{\hat{\mathbf{R}}}_{\rm tx} \hat{\mathbf{c}} + 2\hat{\mathbf{q}}^T \hat{\mathbf{c}} + p \\ \text{s.t. } \hat{\mathbf{A}}_{\rm eq} \hat{\mathbf{c}} = \hat{\mathbf{b}}_{\rm eq} \ . \end{array} \right\}^{-1} , \tag{30}$$

where, again, the objective is convex in $\hat{\mathbf{c}}$ since $\hat{\hat{\mathbf{R}}}_{\rm tx} \geq 0$ and the equality constraints are affine. Clearly, the last added scalar p, the power in the receiver loop and load, is not necessary for optimizing the parameters, but is added to get the optimal value to be $1/\eta_{\max}$, directly.

Note that $\hat{\hat{\mathbf{R}}}_{\rm tx}$ is only positive semi-definite (≥ 0) and not positive definite (> 0) anymore, since one of its eigenvalues is zero at all time (due to the row and column full of zeros). Further, note that $\hat{\mathbf{A}}_{\rm eq}$ is obviously not square, thus the system of constraints can not just be solved linearly for $\hat{\mathbf{c}}$ and be put into the minimization to find the solution (trivial case).

B. Transmitter capacitors

As derived, the maximum transfer efficiency, Eqn. (24), is independent of the transmitter capacitors C_n (or rather their reactances x_n), $n = 1, \ldots, N_{\rm tx}$, since it depends on the currents in the loops (and the receiver reactance x) directly. However, the transmitter capacitors are essential to ensure that these optimal currents actually result from the voltages, by which the whole system is excited. Thus, in this second step, the correct transmitter capacitor values to accomplish that task have to be found.

Recall that a MISO-$N_{\rm tx}$1 setup (total number of loops $N = N_{\rm tx} + 1$) can be described as

$$\underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_{N_{\rm tx}} \\ v_N = 0 \end{bmatrix}}_{\mathbf{v}} = \left(\mathbf{Z} - j \underbrace{\begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & 0 & x_{N_{\rm tx}} & 0 \\ 0 & \cdots & 0 & x \end{bmatrix}}_{\mathbf{X}_c} + \mathbf{R}_L \right) \underbrace{\begin{bmatrix} i_1 \\ \vdots \\ i_{N_{\rm tx}} \\ i_N \end{bmatrix}}_{\mathbf{i}} . \tag{31}$$

Separating the voltages v_n into real and imaginary parts and putting it into vector form leads to

$$\mathbf{v}_n = \begin{bmatrix} v'_n \\ v''_n \end{bmatrix} = x_n \begin{bmatrix} i''_n \\ -i'_n \end{bmatrix} + \begin{bmatrix} \mathbf{z}'_n & -\mathbf{z}''_n \\ \mathbf{z}''_n & \mathbf{z}'_n \end{bmatrix} \begin{bmatrix} \mathbf{i}' \\ \mathbf{i}'' \end{bmatrix} \tag{32a}$$

$$= x_n \mathbf{s}_n + \mathbf{t}_n \ , \tag{32b}$$

for $n = 1, \ldots, N_{\rm tx}$ (but not v_N, since the receiver voltage is already contained in the constraints of the convex optimization process to get the efficiency), where $\mathbf{z}'_n$ and $\mathbf{z}''_n$ are the real and imaginary parts, respectively, of the nth row of the unloaded impedance matrix $\mathbf{Z}$.

There is no requirement as to how the voltage and capacitor values should be chosen (other than practical realizability). From common (SISO-type) WPT systems, it is known that both the transmitter as well the receiver should be driven at resonance, to maximize both the efficiency as well as the actual power transferred to the load [7]. Since both input and output of the transfer system are of the series resonant circuit type, they are tuned to series resonance (in the coupled case, not separately), thus to the point where the impedance has its minimum and the amplitude of the voltage to produce a certain current is minimal.

To obtain similar results and because it is well-known that unconstrained least-square problems have analytical solutions, the capacitive reactance values x_n (for $n = 1, \ldots, N-1$) are chosen so as to minimize the voltages (i.e. thereby tuning the coupled transmitters to resonance)

$$x_n^\star = \arg\min_{x_n} |\mathbf{v}_n|^2 = -\frac{\mathbf{s}_n^T \mathbf{t}_n}{\mathbf{s}_n^T \mathbf{s}_n} \ . \tag{33}$$

Finally, the capacitor values are found using $C_n = 1/(\omega x_n^\star)$.

C. Summary of the convex optimization formulation

Two algorithms to optimize both the currents and receiver capacitance such as to maximize the transfer efficiency of WPT with multiple transmitters have been presented: The first is in a physically intuitive inner/outer optimization loop form (where only the inner loop is a standard convex optimization form), the second is a fully convex optimization formulation.

The optimum transmitter capacitors are not essential for maximizing the achievable power transfer efficiency. However, they are responsible to generate the optimal currents from given excitation voltages. Here, they are chosen by analytically minimizing the excitation amplitudes for the required currents using Eqns. (32b) and (33). Minimizing the excitation amplitudes relates the MISO solution to the maximized output power condition (transmitter resonance) for SISO WPT systems.

IV. Closed Form Results for the Maximum Transfer Efficiency of MISO WPT Systems

The previously discussed optimization problem can be efficiently solved using well-known quadratic programming algorithms and the solutions are perfectly smooth (no optimization issues as experienced with the global methods). However, as shown next, this optimization problem actually has analytical solutions.

As will be derived step by step, for WPT systems with multiple transmitters, all parameters, i.e. all currents and capacitors can be found analytically such as to maximize the transfer efficiency, which has a closed-form solution as well. Finally, using that formulation, the load resistance can be optimized as well, so as to find the maximum achievable transfer efficiency of a given WPT system directly from its (unloaded) impedance matrix.

A. Dual formulation

The basic idea of Lagrangian duality is to account for the objective and the constraints in one function, by adding a weighted sum of the constraints to the objective function [17]. If certain conditions apply, then, by optimizing these weights (called dual variables), the problem can be solved using this dual formulation, which can be advantageous in some cases.

The Lagrangian function [17] follows straightforwardly from the convex optimization formulation, Eqn. (23), by adding a weighted sum of the constraints to the objective function

$$L(\hat{\mathbf{c}},\boldsymbol{\nu}) = \hat{\mathbf{c}}^T\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}\hat{\mathbf{c}} + 2\hat{\mathbf{q}}^T\hat{\mathbf{c}} + p + \boldsymbol{\nu}^T(\hat{\mathbf{A}}_{\mathrm{eq}}\hat{\mathbf{c}} - \hat{\mathbf{b}}_{\mathrm{eq}}) \;, \quad (34)$$

where the dual variables (also called Lagrangian multipliers) for equality constraints are $\boldsymbol{\nu} = \left[\nu_1,\nu_2\right]^T$. Note that these variables are unconstrained, i.e. $\nu_1,\nu_2 \in \mathbb{R}$. Further, note that if these variables are maximized (go to infinity), the original problem results:

$$\min_{\hat{\mathbf{c}}}\max_{\boldsymbol{\nu}} L(\hat{\mathbf{c}},\boldsymbol{\nu}) = \left\{\begin{array}{l}\min\limits_{\hat{\mathbf{c}}}\ \hat{\mathbf{c}}^T\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}\hat{\mathbf{c}} + 2\hat{\mathbf{q}}^T\hat{\mathbf{c}} + p \\ \text{s.t. } \hat{\mathbf{A}}_{\mathrm{eq}}\hat{\mathbf{c}} = \hat{\mathbf{b}}_{\mathrm{eq}}\end{array}\right\} . \quad (35)$$

Thus, the original problem is the minimum of the upper bound of the Lagrangian function.

The dual function [17] is defined as the lower bound of the Lagrangian given by

$$g(\boldsymbol{\nu}) = \min_{\hat{\mathbf{c}}} L(\hat{\mathbf{c}},\boldsymbol{\nu}) \,, \quad (36)$$

which in this case can be found analytically, at the point where the derivative with respect to the modified current vector $\hat{\mathbf{c}}$ (gradient) vanishes:

$$\boldsymbol{\nabla}_{\hat{\mathbf{c}}}\, L(\hat{\mathbf{c}},\boldsymbol{\nu}) = 0 = 2\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}\hat{\mathbf{c}} + 2\hat{\mathbf{q}} + \hat{\mathbf{A}}_{\mathrm{eq}}^T\boldsymbol{\nu} \quad (37a)$$

$$\hat{\mathbf{c}} = -\frac{1}{2}\,\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}^{-1}\left(2\hat{\mathbf{q}} + \hat{\mathbf{A}}_{\mathrm{eq}}^T\boldsymbol{\nu}\right) . \quad (37b)$$

However, $\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}$, as given in Eqn. (28) is obviously non-invertible. A closer look at this problem reveals

$$0 = 2\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}\hat{\mathbf{c}} + 2\hat{\mathbf{q}} + \hat{\mathbf{A}}_{\mathrm{eq}}^T\boldsymbol{\nu} \quad (38a)$$

$$= 2\begin{bmatrix}\tilde{\mathbf{R}}_{\mathrm{tx}} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{c}_{\mathrm{tx}}\\ x\end{bmatrix} + 2\begin{bmatrix}\mathbf{q}\\ 0\end{bmatrix} + \begin{bmatrix}\mathbf{a}_1^T & \mathbf{a}_2^T\\ 0 & R_L^{-1/2}\end{bmatrix}\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix} . \quad (38b)$$

The last element/row is only zero if $-\nu_2/R_L = 0$, thus the second dual variable has to be $\nu_2 = 0$, for the Lagrangian to have a minimum.

A dual variable equal to zero means there is slack in that constraint or, in other words, that constraint is "not essential" to solve the whole problem. Here, the optimal x has no influence on the rest of the problem (since the gradient of the Lagrangian does not depend on it) and can be obtained at a later point, using that particular constraint and the optimized currents.

The transmitter currents that minimize the Lagrangian are, therefore,

$$\mathbf{c}_{\mathrm{tx}} = -\frac{1}{2}\,\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\left(2\mathbf{q} + \nu_1\mathbf{a}_1^T\right) \quad (39)$$

and the dual function Eqn. (36) reduces to

$$g(\nu_1) = \mathbf{c}_{\mathrm{tx}}^T\tilde{\mathbf{R}}_{\mathrm{tx}}\mathbf{c}_{\mathrm{tx}} + 2\mathbf{q}^T\mathbf{c}_{\mathrm{tx}} + p + \nu_1(\mathbf{a}_1\mathbf{c}_{\mathrm{tx}} - b_1) \quad (40a)$$

$$= -\frac{1}{4}\,(2\mathbf{q} + \nu_1\mathbf{a}_1^T)^T\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\left(2\mathbf{q} + \nu_1\mathbf{a}_1^T\right) + p - \nu_1 b_1 \,. \quad (40b)$$

This dual function is concave in ν_1 if

$$-\nu_1\mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\mathbf{a}_1^T\nu_1 = -\mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\mathbf{a}_1^T\nu_1^2 = -\alpha\nu_1^2 \quad (41)$$

is concave in ν_1, meaning $\alpha = \mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\mathbf{a}_1 \geq 0$. Substituting the first constraint vector $\mathbf{a}_1 = \left[\mathbf{z}_{\mathrm{tx}}', -\mathbf{z}_{\mathrm{tx}}''\right]$ in and using $\tilde{\mathbf{R}}_{\mathrm{tx}}$ from Eqn. (28), it can be seen that

$$\mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\mathbf{a}_1 = \left[\mathbf{z}_{\mathrm{tx}}', -\mathbf{z}_{\mathrm{tx}}''\right]\begin{bmatrix}\mathbf{R}_{\mathrm{tx}} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_{\mathrm{tx}}\end{bmatrix}\left[\mathbf{z}_{\mathrm{tx}}', -\mathbf{z}_{\mathrm{tx}}''\right]^T \quad (42a)$$

$$= \mathbf{z}_{\mathrm{tx}}\mathbf{R}_{\mathrm{tx}}\mathbf{z}_{\mathrm{tx}}^H > 0 \quad (42b)$$

since $\mathbf{R}_{\mathrm{tx}} > 0$, for real passive impedance matrices, because of radiation loss. Thus, the dual function is indeed concave in ν_1 and, therefore, has a single global maximum.

The theorem of strong duality (SD) [17] states that, if the original problem is differentiable and convex in $\hat{\mathbf{c}}$ (which is true in this case, as stated before), then the minimum of the upper bound (the original problem) and the maximum of the lower bound (dual problem) are equivalent:

$$\underbrace{\max_{\boldsymbol{\nu}}\min_{\hat{\mathbf{c}}} L(\hat{\mathbf{c}},\boldsymbol{\nu})}_{\text{original problem}} \overset{\mathrm{SD}}{=} \underbrace{\min_{\hat{\mathbf{c}}}\max_{\boldsymbol{\nu}} L(\hat{\mathbf{c}},\boldsymbol{\nu})}_{\text{dual problem}} \,, \quad (43a)$$

which translates to

$$\max_{\boldsymbol{\nu}} g(\boldsymbol{\nu}) \overset{\mathrm{SD}}{=} \left\{\begin{array}{l}\min\limits_{\hat{\mathbf{c}}}\ \hat{\mathbf{c}}^T\hat{\tilde{\mathbf{R}}}_{\mathrm{tx}}\hat{\mathbf{c}} + 2\hat{\mathbf{q}}^T\hat{\mathbf{c}} + p \\ \text{s.t. } \hat{\mathbf{A}}_{\mathrm{eq}}\hat{\mathbf{c}} = \hat{\mathbf{b}}_{\mathrm{eq}}\end{array}\right\} . \quad (43b)$$

Thus, solving (minimizing) the dual problem is equivalent to solving (maximizing) the original problem, in this case.

Note that maximizing Eqn. (40b) is still a quadratic programming problem, as it should be, since QPs are self-dual. However, unlike the original problem, on the right-hand side of Eqn. (43b), the dual formulation (left-hand side) is unconstrained, since $\nu_1 \in \mathbb{R}$.

B. Analytical solution of the dual problem

Quadratic Programming (QP) problems are known to have analytic solutions in some cases, especially unconstrained. In this case, the dual formulation of the QP problem has an analytical solution. At the maximum of $g(\boldsymbol{\nu})$, the gradient with respect to the dual variable $\boldsymbol{\nu}$, which here reduces to just the partial derivative with respect to ν_1, vanishes:

$$\boldsymbol{\nabla}_{\nu_1} g(\nu_1) = 0 = -\frac{1}{2}\,\mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\left(2\mathbf{q} + \nu_1\mathbf{a}_1^T\right) - b_1 \quad (44a)$$

$$\nu_1 = -2\,\frac{b_1 + \mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\mathbf{q}}{\mathbf{a}_1\tilde{\mathbf{R}}_{\mathrm{tx}}^{-1}\mathbf{a}_1^T} \,. \quad (44b)$$

As previously stated, $\nu_2 = 0$ and thus all dual variables are defined and the dual problem is solved. Using these solutions, the quantities actually looked for, the optimal transmitter currents $\mathbf{c}_{\mathrm{tx}}$ and the receiver capacitance (via its reactance) as well as the transfer efficiency are derived in the following.

1) Transmitter currents: Substituting the optimal dual variable ν_1 from Eqn. (44b) as well as the (impedance) vectors $\mathbf{q} = 1/\sqrt{R_L}\left[\mathbf{z}'_{\text{tx}}, \mathbf{0}\right]^T$, $\mathbf{a}_1 = \left[\mathbf{z}'_{\text{tx}}, -\mathbf{z}''_{\text{tx}}\right]$ and $\mathbf{a}_2 = \left[\mathbf{z}''_{\text{tx}}, \mathbf{z}'_{\text{tx}}\right]$ back into Eqn. (39) the transmitter currents are obtained as follows:

$$\mathbf{c}_{\text{tx}} = \frac{1}{\sqrt{R_L}\,\mathbf{z}_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}_{\text{tx}}^H}\begin{bmatrix}\beta_1\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}'^T_{\text{tx}}\\ \beta_2\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}''^T_{\text{tx}}\end{bmatrix}, \tag{45}$$

where the coefficients are given by

$$\beta_1 = -\mathbf{z}''_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}''^T_{\text{tx}} - z'_{NN} - R_L \tag{46a}$$
$$\beta_2 = -\mathbf{z}'_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}'^T_{\text{tx}} + z'_{NN} + R_L\,. \tag{46b}$$

2) Receiver reactance and capacitor: The optimal receiver reactance is obtained from the (up to this point still unused) second constraint:

$$\hat{\mathbf{a}}_2\hat{\mathbf{c}} = \mathbf{a}_2\mathbf{c}_{\text{tx}} - \frac{x}{\sqrt{R_L}} = b_2 = -\frac{z''_{NN}}{\sqrt{R_L}}\,. \tag{47}$$

Rearranging and substituting the transmitter currents $\mathbf{c}_{\text{tx}}$, Eqn. (45), and the dual variable ν_1, Eqn. (44b), in leads to

$$x = z''_{NN} - \sqrt{R_L}\,\mathbf{a}_2\tilde{\mathbf{R}}_{\text{tx}}^{-1}\left(\mathbf{q} - \frac{b_1 + \mathbf{a}_1\tilde{\mathbf{R}}_{\text{tx}}^{-1}\mathbf{q}}{\mathbf{a}_1\tilde{\mathbf{R}}_{\text{tx}}^{-1}\mathbf{a}_1^T}\,\mathbf{a}_1^T\right). \tag{48a}$$

Finally, by replacing the vectors by their respective definitions $\mathbf{q} = 1/\sqrt{R_L}\left[\mathbf{z}'_{\text{tx}}, \mathbf{0}\right]^T$, $\mathbf{a}_1 = \left[\mathbf{z}'_{\text{tx}}, -\mathbf{z}''_{\text{tx}}\right]$ and $\mathbf{a}_2 = \left[\mathbf{z}''_{\text{tx}}, \mathbf{z}'_{\text{tx}}\right]$ as well as using the fact that $\mathbf{a}_2\tilde{\mathbf{R}}_{\text{tx}}^{-1}\mathbf{a}_1^T = 0$, the optimal receiver reactance is given by

$$x = z''_{NN} - \mathbf{z}''_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}'^T_{\text{tx}} \tag{48b}$$

and the capacitor value is obtained using $C_N = 1/(\omega x)$.

3) Optimized and maximum achievable transfer efficiencies: Starting from the dual function Eqn. (40b) and substituting in the transmitter currents Eqn. (39), the transfer efficiency (to a specific load R_L) can be expressed by

$$\eta_{\max} = \frac{1}{g(\nu_1)} = \frac{R_L\,\mathbf{z}_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}_{\text{tx}}^H}{(z'_{NN}+R_L-\mathbf{z}'_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}'^T_{\text{tx}})(z'_{NN}+R_L+\mathbf{z}''_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}''^T_{\text{tx}})}\,. \tag{49}$$

It might seem like the efficiency could turn negative, because of the negative sign in the first factor in the denominator. However, it can be shown that $z'_{NN} - \mathbf{z}'_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}'^T_{\text{tx}} > 0$, because it is a Schur complement of the (symmetric) resistance matrix $\mathbf{R} = \operatorname{Re}\mathbf{Z}$. As such, it has to be positive definite (or in this case just positive, since it is a scalar in this case) together with the submatrix $\mathbf{R}_{\text{tx}} > 0$ being positive definite, for the resistance matrix to be positive definite, $\mathbf{R} > 0$. Both of these last conditions are a given for lossy passive networks. Thus, the transfer efficiency can never become negative.

It can easily be seen that the transfer efficiency $\eta_{\max}$ given in Eqn. (49) is concave in R_L. Its maximum is found at

$$\left.\frac{\partial g(\nu_1)}{\partial R_L}\right|_{R_L = R_L^\star} = 0 \tag{50}$$

and, thus, the optimal load resistance is given by

$$R_L^\star = \sqrt{(z'_{NN} - \mathbf{z}'_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}'^T_{\text{tx}})(z'_{NN} + \mathbf{z}''_{\text{tx}}\mathbf{R}_{\text{tx}}^{-1}\mathbf{z}''^T_{\text{tx}})}\,. \tag{51}$$

Note that, for the same reason as before (Schur complement), the optimal $R_L^\star$ exists for all real and passive WPT systems (and cannot turn complex). Further, in the SISO case with identical antennas ($z_{11} = z_{22}$), Eqns. (51) and (49) lead to an $\eta_{\max}(R_L^\star)$ identical to the one given in [5].

Using the optimal load Eqn. (51) in Eqn. (49), the maximum achievable transfer efficiency of *any* MISO WPT system can be obtained. As expected (since the impedance matrix actually already contains all information about the WPT system), the maximum performance only depends on the systems (unloaded) impedance matrix.

V. Results

A. Problem setups

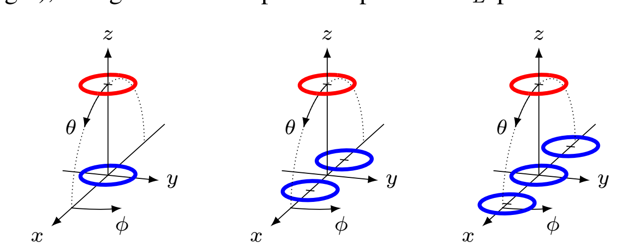
For this paper, the unloaded impedance matrix $\mathbf{Z}$ was obtained using the *Double Multiradius Bridge-Current* (DMBC) method, [18], [19], implementing a thin-wire, piecewise sinusoidal Galerkin moment method in 64-bit precision. DMBC is known to be powerful and accurate, [20], [21].

For verification, the optimized values were included in loaded models and simulated using DMBC again. The results were compared and agreed very well. Differences are mainly due to the fact that for DMBC the capacitances and excitation voltages have to be handed over in a text file and, thus, the values are subject to round-off errors.

Depending on the method or software used to obtain the unloaded impedance matrix, and for some models, non-passive impedance matrices might be received. Using DMBC, this was the case for a few angles, when considering lossless or very highly conductive (but finite) loops (e.g. $\sigma = 10^6\,\sigma_{\text{Cu}}$) and particularly for WPT systems with a higher number of ports, e.g. 7 loops (i.e. MISO-61) or more. Since in these cases the received impedance matrix does not represent the physics of the model, passivity has been enforced as described in the Appendix A. The change in the norm of the impedance matrix always remained negligible.

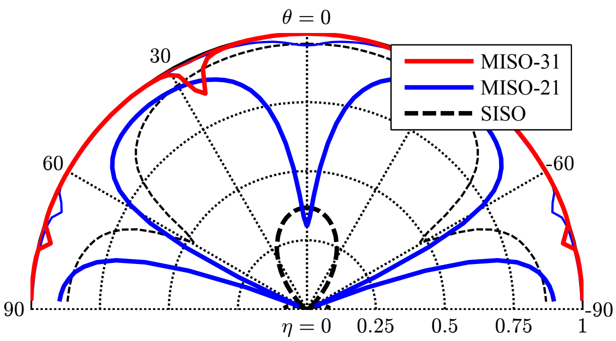
B. Planar setups

1) One-dimensional models: First, the transfer efficiency of WPT systems with multiple (up to three) transmitters is investigated. Fig. 3 shows the setups in question: the MISO-21 and MISO-31 as well as the SISO reference. The transfer efficiency patterns for $R_L = 1\,\Omega$ as well as the optimum load resistance $R_L^\star$ are shown in Fig. 4, for loops with infinite conductance (σ_∞, left) and copper conductance ($\sigma_{\text{Cu}} = 58 \times 10^6$ S/m, right), along with the respective optimum R_L patterns.

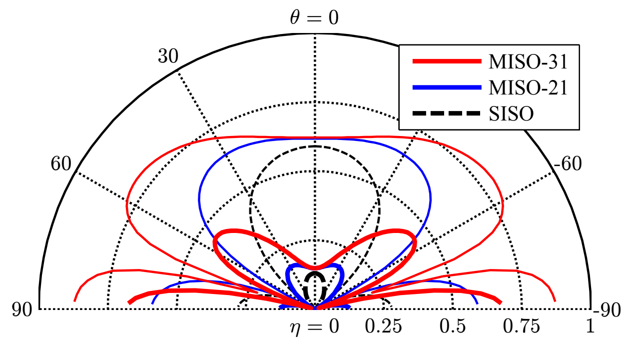


(a) SISO (reference) (b) MISO-21 (c) MISO-31

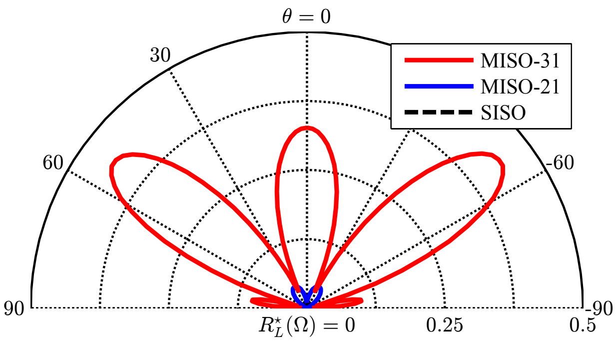
Fig. 3. Illustrations of the planar MISO WPT setups: multiple transmitters, with the position $(x_n, y_n, z_n = 0)$, $i = 1, ..., N_{\text{tx}}$ and a receiver at (d, θ, ϕ).



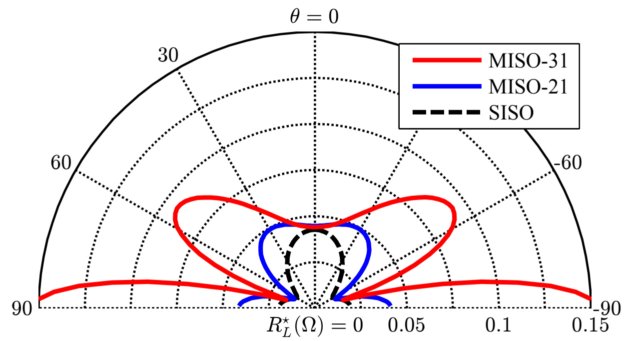
(a) Transfer efficiency $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_∞.



(b) Transfer efficiency $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_{Cu}.



(c) Optimum load resistance $R_L^\star$ (in Ω), for σ_∞



(d) Optimum load resistance $R_L^\star$ (in Ω), for σ_{Cu}

Fig. 4. Planar MISO-21, MISO-31 vs. SISO: maximum transfer efficiency $\eta_{\max}$ for $R_L = 1\,\Omega$ and $R_L^\star$ vs. θ and optimum load resistance $R_L^\star$, for $\phi = 0, \pi$, $R_L = 1\,\Omega$ and lossless, σ_∞, (a,c) and lossy, σ_{Cu}, (b,d) loops at a distance $d = 0.1\lambda$.

The geometrical parameters involved are:

- separation between the transmitter loops $\Delta x = 0.05\,\lambda$,
- loop radii $r_{\mathrm{loop}} = 0.01\,\lambda$,
- wire thickness $r_{\mathrm{wire}} = 0.2\,r_{\mathrm{loop}}$,

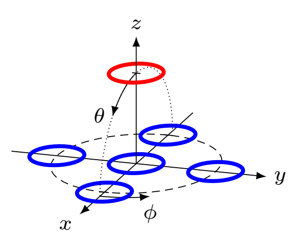
where $\lambda = c_0/f$ and $f = 40\,\mathrm{MHz}$ (the results are scalable).

As expected, the maximum achievable power transfer efficiency is higher in the cases with lossless loops, than in those with lossy loops. However, it is more than just an overall drop; for cases with multiple loops the overall form of pattern changes. As is well-known, impedance matching, meaning using the optimum load $R_L^\star$ has a strong impact on the achieved transfer efficiency, particularly at broadside.

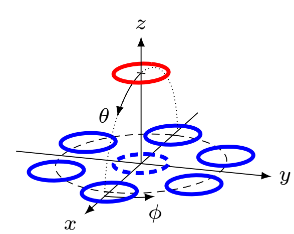
Note that, for lossless loops, the MISO-21 setup can perform somewhat worse than the SISO case at broadside, since it is missing a center element, whereas the MISO-31 always has to perform equally well or better than the SISO reference.

2) Two-dimensional models: Fig. 5 shows two-dimensional setups with loops arranged the xy-plane (at a constant distance to the center element of 0.05λ). Their transfer efficiency patterns for are shown in Fig. 6. The results seem physically reasonable, by comparison, since the resulting efficiency pattern for the MISO-71 setup is the sum ("overlay") of the patterns of the MISO-61 (no center element) and MISO-51 (center element) setups.

While the patterns in Fig. 6a reveal substantial transfer efficiency enhancement using multiple loops even at a distance of $d = \lambda/4$, in the lossy case, see Fig. 6b, all efficiency patterns collapse more or less to the one of the MISO-31 setup, no matter if the optimum load $R_L^\star$ is present or not.



(a) MISO-51 setup

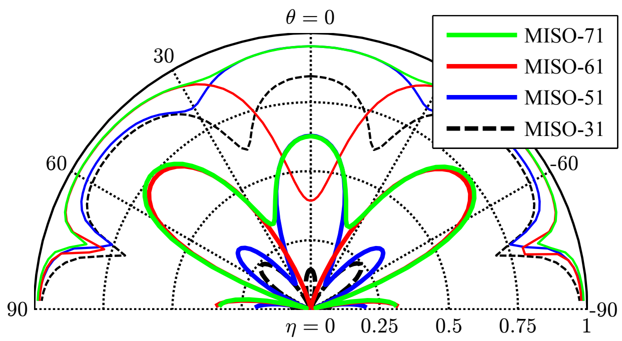


(b) MISO-61 and -71 (including the dashed center element) setups

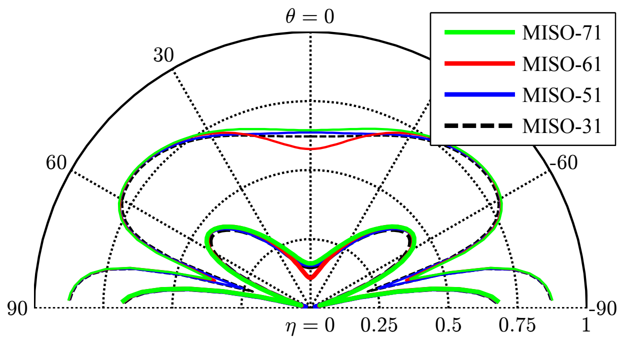
Fig. 5. Illustrations of the larger MISO WPT setups; the satellite transmitters are arranged symmetrically in the xy-plane, at a constant distance of 0.05λ around the center transmitter position.

3) Comparison & Interpretation: Comparison of the lossless and lossy cases reveals that the overall working principles are different, for the two cases. Without conduction loss in the loops, the only loss to minimize (to maximize transfer efficiency), is radiation. Minimizing radiation is achieved by minimizing far-field effects using (almost) canceling primary fields due to currents of opposing direction and almost equal amplitudes, see Fig. 8a. Since the amount of transferred power is constant $1\,\mathrm{W}$, and the currents almost cancel each others fields, the primary currents are very high. Additionally, these currents and fields do not change substantially, for different receiver positions, shown in Fig. 7.

In cases with conduction loss, such high currents are not feasible and the far-field cancellation cannot be achieved, at least not to the same degree, as shown in Fig. 8b. The optimal currents in the transmitter loops are very different for different

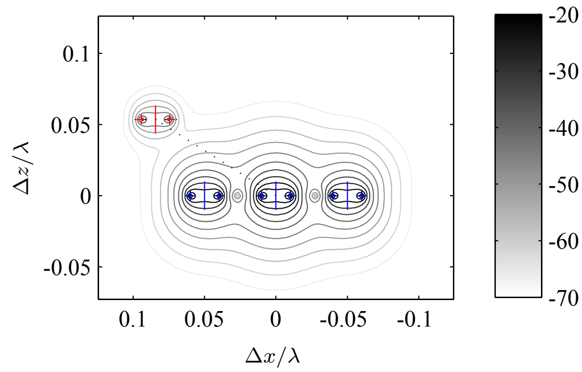


(a) $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_∞, and $d = 0.25\lambda$

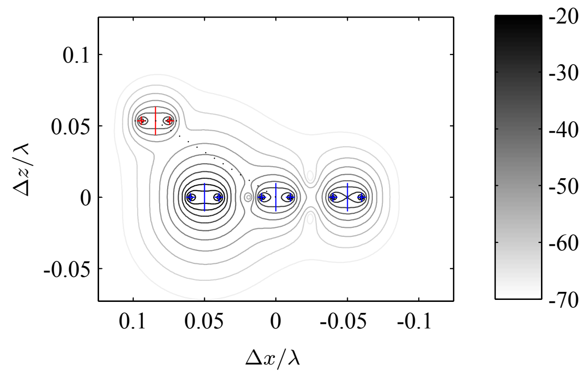


(b) $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_{Cu}, and $d = 0.1\lambda$

Fig. 6. Transfer efficiencies for MISO-51, -61 and -71 setups vs. MISO-31, without and with conduction loss: while in the lossless cases there is substantial efficiency gain from using this many loops, in the lossy case all patterns collapse to the MISO-31 pattern.



(a) lossless loops, σ_∞

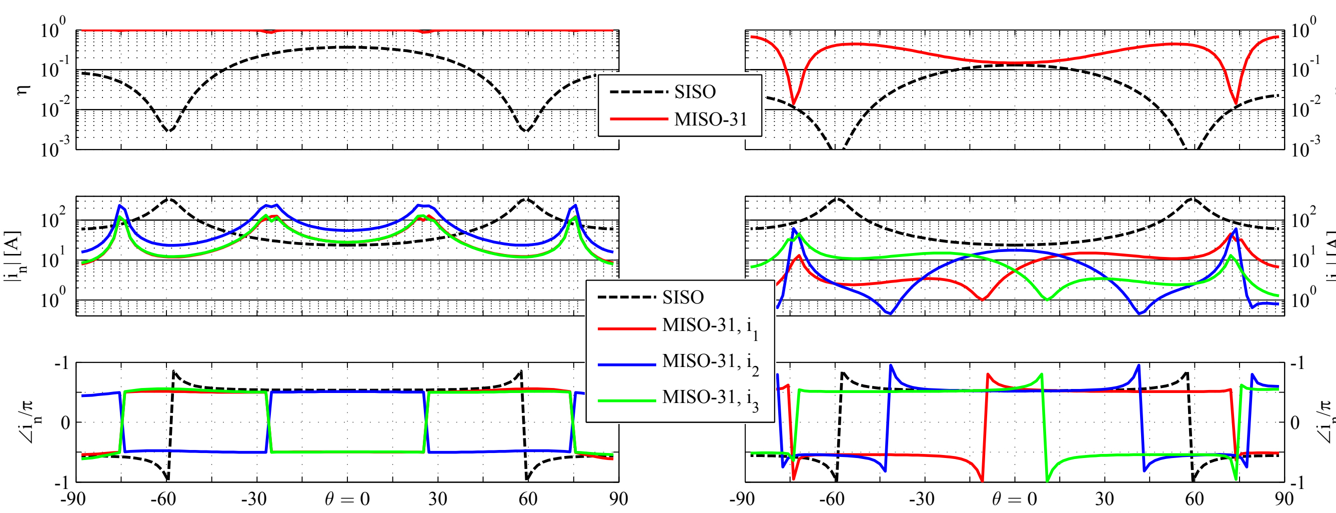


(b) lossy loops, σ_{Cu}

Fig. 8. Contours of the magnetic field $|B_x|^2 + |B_z|^2$ (in dB) in the $y = 0$ plane for MISO-31 setups with the receiver at 60 off broadside, $d = 0.1\lambda$.

receiver positions — essentially all the energy is put into the element closest to the receiver, shown in Fig. 7b. The primary reason for higher transfer efficiency is closer proximity; the transmitter closest to the receiver is favored, while the others carry much lower currents.

This is the reason why there is no (substantial) performance enhancement for lossy setups with more than three transmitters, as shown in Fig. 6b: Since the distance to the closest transmitter remains the same (for receivers in the xz-plane) for all these setups, so does the achieved transfer efficiency. On the other side, in the lossless cases, the far-field cancellation can be achieved more efficiently (particularly in some directions) with a larger number of transmitters, leading to an increased transfer efficiency, as can be seen in Fig. 6a.



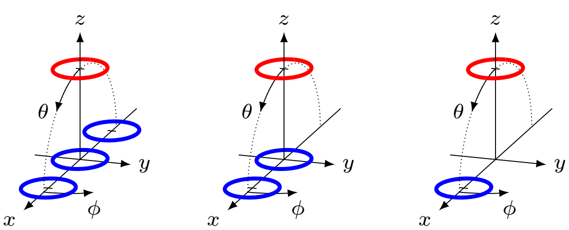
(a) lossless, σ_∞

(b) lossy, σ_{Cu}

Fig. 7. Optimized currents: MISO-31 and SISO vs. angle of elevation θ, at distance $d = 0.1\lambda$, for lossless and lossy loop conductors.

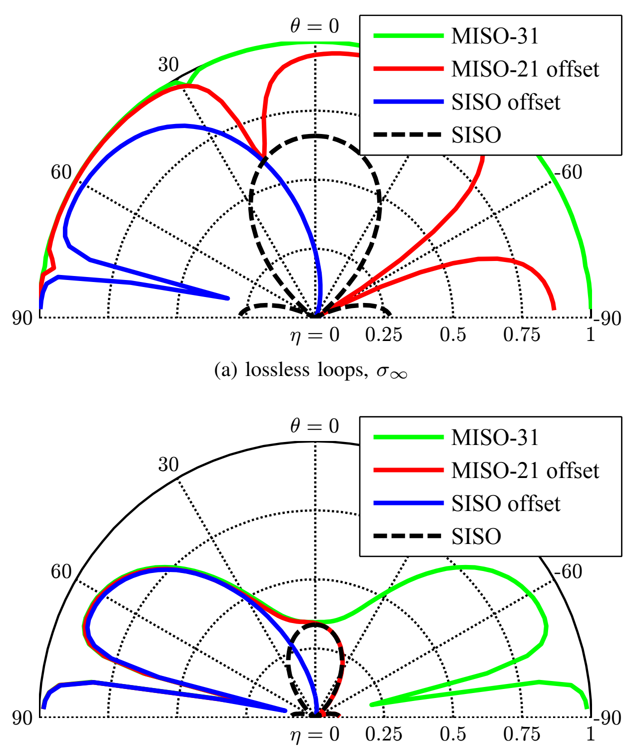
4) Verification: This is verified using the models shown in Fig. 9, where one or two loops are positioned off-center, so as to represent part of the (centered) MISO-31 setup. As can be seen from the transfer efficiency patterns in Fig. 10, in the lossy case (b) the final MISO-31 pattern can be obtained by "adding" the patterns from the offset cases, thus confirming that the setup with multiple transmitters is essentially the same as a single loop at the position closest to the receiver.

Note that this is not the case for lossless loops: due to the fact that the fields can be canceled more efficiently with more loops, the transfer efficiency pattern of the MISO constellation cannot be obtained by overlaying the separated patterns.



(a) MISO-31 (b) MISO-21 offset (c) SISO offset

Fig. 9. Illustrations of the MISO-31 WPT setup and its decompositions, offset MISO-21 and offset SISO, to separate their effects on the efficiency.



(a) lossless loops, σ_∞

(b) lossy loops, σ_{Cu}

Fig. 10. Comparison of the separated transfer efficiency effects and the total transfer efficiency pattern, at distance $d = 0.08\lambda$.

5) Realistic capacitor values: It is worth noticing that the obtained optimal capacitor values are not unrealistic for a practical realization of these MISO WPT systems.

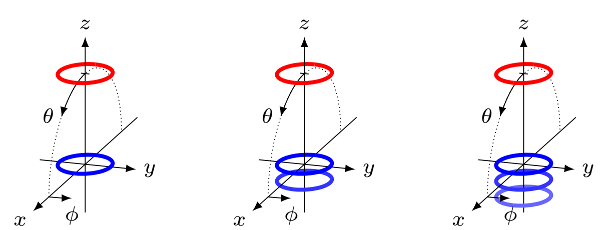
Tab. I compares the obtained capacitor values for lossless and lossy MISO-31 WPT setups, with the receiver loop at three different angles, at a distance of $d = 0.1\lambda$. As can be seen, the values seem not to change at all for the lossless setup (changes appear at the fourth and fifth digit behind the comma; thus, high sensitivity on actual values), since that optimized solution essentially remains the same, independent of the receiver position, see Fig. 7a. The values are different overall and do change for the lossy setup, however, particularly on the transmitter side (C_1, C_2 and C_3).

	MISO-31 lossless			**MISO-31 lossy**		
	$\theta = 0°$	30°	60°	$\theta = 0°$	30°	60°
C_1	35.22 pF	35.22 pF	35.22 pF	98.65 pF	97.47 pF	97.31 pF
C_2	35.26 pF	35.26 pF	35.26 pF	97.62 pF	98.57 pF	96.06 pF
C_3	35.22 pF	35.22 pF	35.22 pF	98.65 pF	96.86 pF	97.57 pF
C_4	35.33 pF	35.33 pF	35.33 pF	97.43 pF	97.43 pF	97.43 pF

TABLE I
OPTIMIZED CAPACITOR VALUES AT THE DIFFERENT ANGLES AT DISTANCE $d = 0.1\lambda$, FOR LOSSLESS AND LOSSY MISO-31 WPT SETUPS.

C. Coaxial setups

1) Non-planar setups: The only geometry which never profits from proximity effects is a coaxial (but not necessarily planar) arrangement of the transmitter loops, as illustrated in Fig. 11; for any elevation angle θ (and azimuth ϕ), the minimum distance between any of the transmitters and the receiver is always the same or more as in the SISO case.



(a) SISO (b) MISO-21 coaxial (c) MISO-31 coaxial

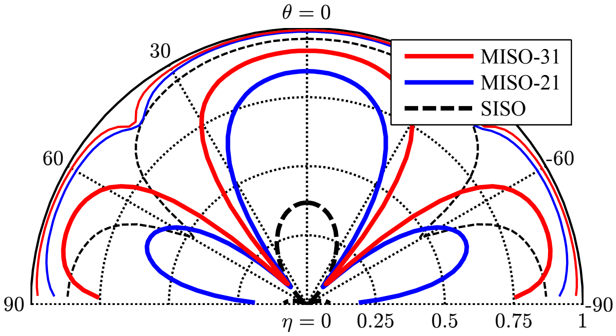
Fig. 11. The coaxial MISO WPT setups: multiple transmitters, with the positions $(0, 0, z_n)$, $i = 1, ..., N_{\mathrm{tx}}$, where $z_n \leq 0$ and a receiver at (d, θ, ϕ).

Fig. 12 shows the optimized results, again for the distance of $d = 0.1\lambda$. As can be seen, even in lossy cases, there is a slight increase in transfer efficiency, when using multiple loops. Clearly, since the added transmitters are further away from the receiver, this cannot be due to the same proximity effect as before, in the planar case.

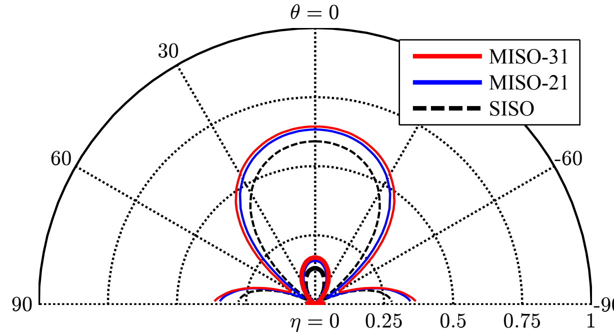
However, also in this case the working principles of the lossless and lossy cases are different, as Figs. 13 and 15 reveal: While in the lossless case again far-field radiation is minimized by almost canceling currents, in the lossy case the loops are used more or less in parallel, thereby reducing conduction loss.

2) Planar coaxial setup: The combination of both ideas, to have a planar setup but multiple loops that are all at the same distance (or further) from the receiver, leads to the planar coaxial (concentric) setup, known from metasurface and subwavelength focusing applications [12], depicted in Fig. 14.

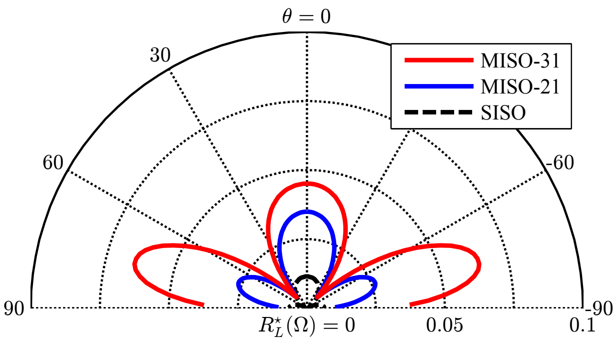
Since, as is well known, larger transmitter loops lead to higher flux and thus stronger coupling and higher transfer efficiency, for a fair comparison, only loops of equal or smaller



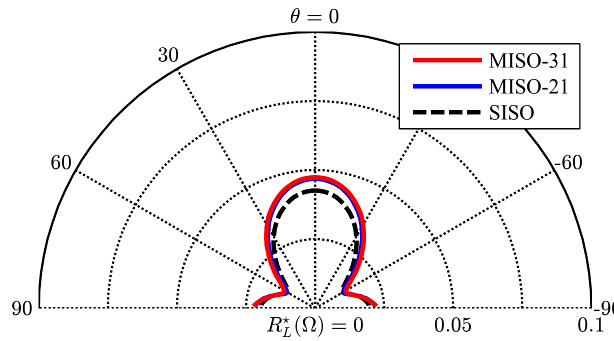
(a) Transfer efficiency $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_∞



(b) Transfer efficiency $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_{Cu}

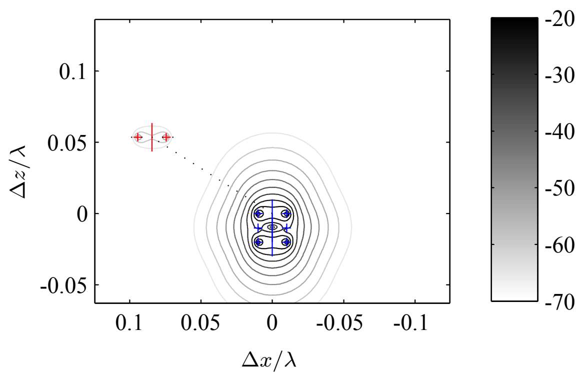


(c) Optimum load resistance $R_L^\star$ (in Ω), for σ_∞

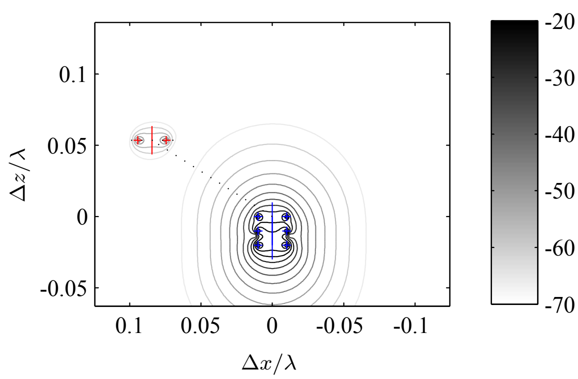


(d) Optimum load resistance $R_L^\star$ (in Ω), for σ_{Cu}

Fig. 12. Coaxial MISO-21, MISO-31 vs. SISO: maximum transfer efficiency $\eta_{\max}$ for $R_L = 1\,\Omega$ and $R_L^\star$ vs. θ and optimum load resistance $R_L^\star$, for $\phi = 0, \pi$, $R_L = 1\,\Omega$ and lossless, σ_∞, (a,c) and lossy, σ_{Cu}, (b,d) loops at a distance $d = 0.1\lambda$.

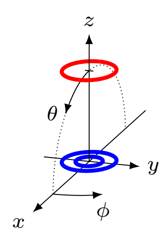


(a) lossless, σ_∞

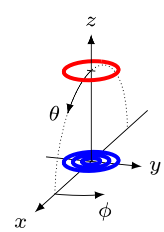


(b) lossy, σ_{Cu}

Fig. 13. Contours of the magnetic field $|B_x|^2 + |B_z|^2$ (in dB) in the $y = 0$ plane for MISO-31 setups with the receiver at 60 off broadside, $d = 0.1\lambda$.



(a) planar coaxial MISO-21



(b) planar coaxial MISO-31

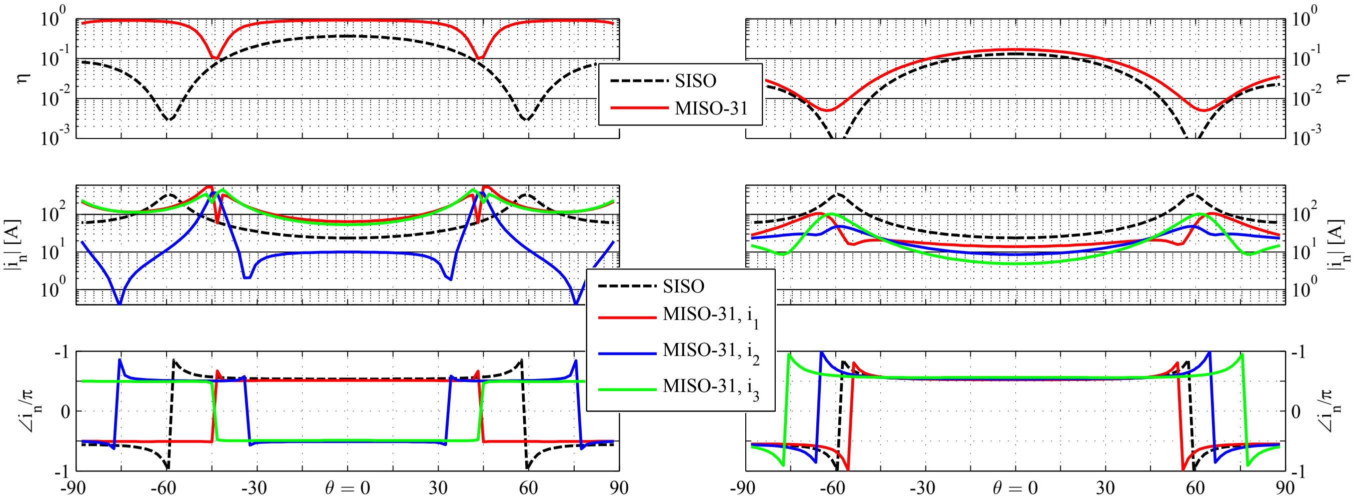
Fig. 14. Planar coaxial (concentric) MISO WPT setups.

radius than the receiver loop are considered: The loop radii are $[1, 0.5]\, r_{\mathrm{loop}}$ for the MISO-21 and $[1, 0.66, 0.33]\, r_{\mathrm{loop}}$.

The currents, capacitors and load resistance have been optimized using the analytical formulations. Fig. 16 shows the resulting efficiency patterns for lossless and lossy loops. They look very similar to the one in Fig. 12b and also the working principle is quite similar to the coaxial cases: While in the lossless cases, the inner loops are used to cancel far-field effects (radiation) of the outer loop, in the lossy case all loops are used in parallel, to minimize conduction loss. Plots of the optimal receiver resistance $R_L^\star$ are omitted since they look very similar to Fig. 12d.

Thus, solutions with highly oscillating currents, as received for subwavelength beamforming [12], are not feasible when transfer (or coupling) efficiency is of the main interest.

Coaxial MISO WPT setups have shown that there is potential transfer efficiency enhancement even without closer proximity to the receiver. However, as can be seen from Figs. 12b and 16b, the potential efficiency enhancement due to



(a) lossless, σ_∞ (b) lossy, σ_{Cu}

Fig. 15. Optimized currents: Coaxial MISO-31 and SISO setups vs. elevation angle θ, at distance $d = 0.1\lambda$, for lossless and lossy conductors.

multiple transmitters is smaller than the one due to impedance matching: choosing the optimal load resistance R_L is essential to achieve maximum power transfer.

VI. Summary & Conclusions

The transfer efficiency enhancement potential of wireless power transfer systems with multiple transmitters (MISO WPT systems) was analyzed using convex optimization and Lagrangian duality theory.

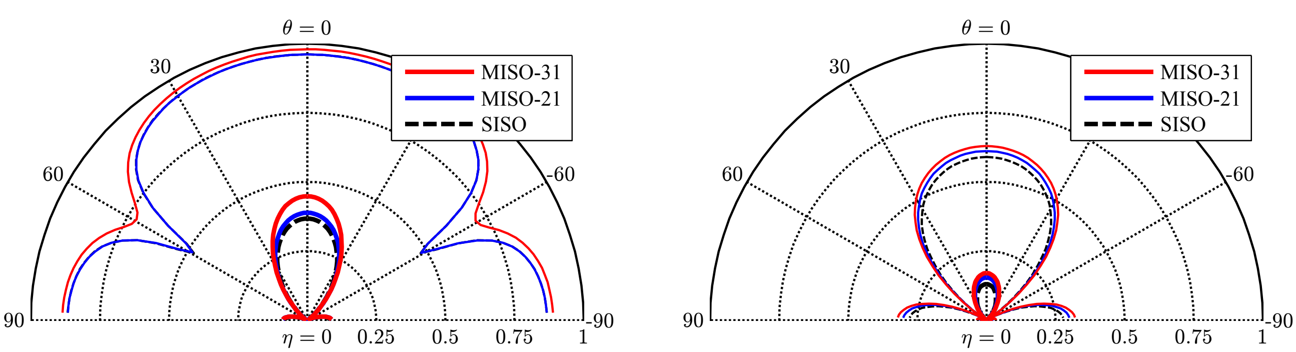
A convex optimization algorithm (quadratic programming) to maximize the transfer efficiency by optimizing the required currents as well as the receiver capacitor was presented. The excitation voltages together with the transmitter capacitors were obtained by least-square minimization of the amplitudes.

Further, using Lagrangian duality theory, analytical solutions of the dual optimization problem were found, with which the maximum transfer efficiency of a given arrangement and load R_L as well as all required parameters could be obtained, directly. From these formulations, also the optimum load $R_L^\star$ could be obtained, to assess the maximum achievable transfer efficiency of a given system.

For problems of non-passivity, originating from the numerical code(s) used to obtain the (unloaded) impedance matrix of the WPT setup in the first place, a simple passivity enforcement was applied. The negative eigenvalues of the eigenvalue decomposition were replaced by positive ones, whereby both passivity and physical meaning of the model was recovered.

Using these methods, performances and working principles of planar and coaxial MISO WTP systems were analyzed and compared for cases with lossless and lossy conductors. Lossless cases lead to solutions similar to the ones obtained for near-field focusing [11], [13], with almost canceling currents to minimize far-field radiation. This working principle is not feasible under lossy conditions, however, if (transfer) efficiency is the main interest.

Since the impedance matrix and reactances vary with frequency, the optimum is only achieved at the particular frequency of operation; how fast the transfer efficiency drops depends on the distance, geometry and quality factor of the



(a) Transfer efficiency $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_∞ (b) Transfer efficiency $\eta_{\max}(R_L)$, thick, $\eta_{\max}(R_L^\star)$, thin, for σ_{Cu}

Fig. 16. Concentric MISO-21, MISO-31 vs. SISO: maximum transfer efficiency $\eta_{\max}$ for $R_L = 1\,\Omega$ and $R_L^\star$ vs. θ for $\phi = 0, \pi$, lossless, σ_∞, (a) and lossy, σ_{Cu}, (b) loops at a distance $d = 0.1\lambda$. (Graphs for $R_L^\star$ are omitted, since they look very similar to Fig. 12d.)

whole system. Over larger frequency spans, resonance splitting effects are obtained, similar in nature to those reported for SISO systems [14], but with more resonances (due to the higher number of resonators involved).

Appendix A
Treatment of Non-Passive Models

In some cases, numerical solvers such as DMBC or HFSS return non-passive (unloaded) impedance matrices, especially when using perfect conductors or very low-loss materials. In these cases, the unloaded impedance matrix does not satisfy the passivity condition [22]

$$\mathbf{Z} + \mathbf{Z}^H \geq 0 \tag{52}$$

which with $\mathbf{Z}' = \mathbf{Z}'^T$ reduces to

$$\mathbf{Z}' \geq 0 \,, \tag{53}$$

the (unloaded) resistance matrix being positive definite. Due to the fact that there is at least some radiation loss, Eqns. (52) and (53) actually reduce to strict inequalities.

Therefore, for non-positive semi-definite matrices, the eigenvalue decomposition returns at least one negative eigenvalue. It is evident that the same is true for the (real) matrices $\tilde{\mathbf{R}}$ and $\tilde{\mathbf{R}}_{\text{tx}}$. Even if that eigenvalue is small, it will be picked up during the optimization process by the quadratic programming algorithm (the minimization leads to $\mathbf{c}^H\tilde{\mathbf{R}}\mathbf{c} \to -\infty$) and the solution will be physically meaningless.

There are a variety of elaborate methods to enforce passivity available. However, most of them rely on the impedance matrix as a function of frequency (e.g. adequately sampled over a certain frequency range) [23], [24]. While the impedance matrix could be sampled over a certain frequency range, using elaborate methods of enforcing passivity is beyond the scope of this discussion. As numerical studies have shown, a much more simple method suffices in these cases.

The general idea is to make the matrix positive definite by replacing the negative eigenvalue(s) in the (diagonal) matrix $\mathbf{\Lambda}$ obtained from the eigenvalue decomposition by a positive one, making it purely positive $\mathbf{\Lambda}^+$ and resulting in a passive approximation:

$$\mathbf{Z} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} \quad \xrightarrow{\text{enforce passivity}} \quad \mathbf{Z}^+ = \mathbf{Q}\mathbf{\Lambda}^+\mathbf{Q}^{-1} \,. \tag{54}$$

By comparing the norms of the matrices, it can be assessed how much of the matrix changed and by comparing the optimal solution to the corresponding eigenvector it can be assessed of how much importance that particular eigenvalue/-vector actually is.

The negative eigenvalue is replaced according to

$$\lambda_n \to \epsilon \,, \tag{55}$$

where $\epsilon > 0$ is a very small positive number, chosen smaller than the smallest of all other eigenvalues, but large enough as not to produce numerical problems. If the amplitude of the negative eigenvalue is already (much) smaller than all the other (positive) eigenvalues, then $\epsilon = |\lambda_n|$ could be an option, which, as tests have shown, does not lead to numerical issues.

In the case of DMBC, non-passive matrices were only obtained for larger models (7 loops and more) or for very high (but finite) conductivity, e.g. $10^6 \times \sigma_{\text{Cu}}$. For these models most of the time only one, in rare cases two of the eigenvalues turn out negative and their amplitude is usually very small in amplitude, compared to all others. Thus, the replacement $\lambda_n \to |\lambda_n|$ was used.

Since those non-passive impedance matrices do not represent the physics of the model anyway, it stands to reason to perturb them just enough so as to both be able to use it for the optimization procedure as well as to get most of the physical meaning back. The graphs in the result section seem physically reasonable and serve, therefore, as evidence of successful passivity enforcement.

Acknowledgments

The authors would like to thank Prof. Piero Triverio for helpful discussions on the conditions for passive impedance matrices and passivity enforcement techniques as well as pointing to Schur's complement as means to validate the non-negative nature of the efficiency in Eqn. (49).

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