CONVERGENCE IN HOMOGENEOUS RANDOM GRAPHS

TOMASZ ŁUCZAK† AND SAHARON SHELAH‡

Abstract. For a sequence $\overline{p} = (p(1), p(2), \dots)$ let $G(n, \overline{p})$ denote the random graph with vertex set $\{1, 2, \dots, n\}$ in which two vertices i, j are adjacent with probability $p(|i - j|)$, independently for each pair. We study how the convergence of probabilities of first order properties of $G(n, \overline{p})$, can be affected by the behaviour of $\overline{p}$ and the strength of the language we use.

1. Introduction.

Random graph theory studies how probabilities of properties of random graphs change when the size of the problem, typically the number of vertices of the random graph, approaches infinity. The most commonly used random graph model is $G(n, p)$ the graph with vertex set $[n] = \{1, 2, \dots, n\}$, in which two vertices are joined by an edge independently with probability p. It was shown by Glebskii, Kogan, Liogonkii and Talanov [GKLT 69] and, independently, by Fagin [Fa 76], that in $G(n, p)$, the probability of every property which can be expressed by a first order sentence ψ tends to 0 or 1 as $n \to \infty$. Lynch [Ly 80] proved that even if we add to the language the successor predicate the probability of each first order sentence still converges to a limit. (Here and below the probability of a sentence ψ means the probability that ψ is satisfied.) However it is no longer true when we enrich the language further. Kaufmann and Shelah [KS 85] showed the existence of a monadic second order sentence ϕ, which uses only the relation of adjacency in $G(n, p)$, whose probability does not converge as $n \to \infty$. Furthermore, Compton, Henson and Shelah [CHS 87] gave an example of a first order sentence ψ containing predicate "$\leq$" such that the probability of ψ does not converge – in fact in both these cases the probability of sentences ϕ and ψ approaches both 0 and 1 infinitely many times.

One may ask whether analogous results remain valid when the probability p_{ij} that vertices i and j are connected by an edge varies with i and j. Is it still true that a zero-one law holds for every first order property which uses only the adjacency relation? Or, maybe, is it possible to put some restrictions on the set $\{p_{ij} : i, j \in [n]\}$ such that the convergence of the probability of each first order sentence is preserved even in the case of a linear order? The purpose of this paper is to shed some light on problems of this type in a model of the random graph in which the probability that two vertices are adjacent may depend on their distance.

For a sequence $\overline{p} = \{p(i)\}_{i=1}^{\infty}$, where $0 \leq p(i) \leq 1$, let $G(n, \overline{p})$ be a graph with vertex set $[n]$, in which a pair of vertices $v, w \in [n]$ appears as an edge with probability $p(|v - w|)$,

independently for each pair (this is a finite version of the probabilistic model introduced by Grimmett, Keane and Marstrand [GKM 84]). Furthermore, let us define L as the first order logic whose vocabulary contains the binary predicate of adjacency relation, whereas in language L_+ the successor predicate is also available and in $L_\leq$ one may say that $x \leq y$. We study how the behaviour of sequence $\overline{p}$ could affect the convergence of sequence $\{\text{Prob}(n, \overline{p}; \psi)\}_{n=1}^\infty$, where ψ is a sentence from L, L_+ or $L_\leq$ and $\text{Prob}(n, \overline{p}; \psi)$ denotes the probability that ψ is satisfied in a model with universe $[n]$, adjacency relation determined by $G(n, \overline{p})$, and, in the case of languages L_+ and $L_\leq$, additional binary predicates "$x = y + 1$" and "$x \leq y$" (here and below $x, y \in [n]$ are treated as natural numbers).

The structure of the paper goes as follows. We start with the short list of basic notions and results useful in the study of first order theories. Then, in the next three sections, we study the convergence of sequence $\text{Prob}(n, \overline{p}; \psi)$, where ψ is a first order sentence from languages L, L_+ and $L_\leq$ respectively. It turns out that differences between those three languages are quite significant. Our first result gives a sufficient and necessary condition which, imposed on $\overline{p}$, assures convergence of $\text{Prob}(n, \overline{p}; \psi)$ for every ψ from L. In particular, we show that each sequence $\overline{a}$ can be "diluted" by adding some additional zero terms in such a way that for the resulting sequence $\overline{p}$ and every ψ from L the probability $\text{Prob}(n, \overline{p}; \psi)$ tends either to 0 or to 1. It is no longer true for sentences ψ from L_+. In this case the condition $\prod_{i=1}^\infty (1 - p(i)) > 0$ turns out to be sufficient (and, in a way, necessary) for convergence of $\text{Prob}(n, \overline{p}; \psi)$ for every ψ from L_+. Thus, the convergence of $\text{Prob}(n, \overline{p}; \psi)$ depends mainly on the positive terms of $\overline{p}$ and adding zeros to $\overline{p}$, in principle, does not improve convergence properties of $G(n, \overline{p})$. On the contrary, we give an example of a property ψ from L_+ and a sequence $\overline{a}$ such that for every $\overline{p}$ obtained from $\overline{a}$ by adding enough zeros $\liminf_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 0$ whereas $\limsup_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 1$.

The fact that, unlike in the case of L, additional zeros in $\overline{p}$ might spoil the convergence properties of $G(n, \overline{p})$ becomes even more evident in the language $L_\leq$. We show that there exists a property ψ from $L_\leq$ such that every infinite sequence of positive numbers $\overline{a}$ can be diluted by adding zeros such that for the resulting sequence $\overline{p}$ we have $\liminf_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 0$ but $\limsup_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 1$. Furthermore, it turns out that in this case we can distinguish two types of the limit behaviour of $\text{Prob}(n, \overline{p}; \psi)$. If $\prod_{i=1}^\infty (1 - p(i))^i > 0$ then for every ψ from $L_\leq$ the probability $\text{Prob}(n, \overline{p}; \psi)$ converges. However, if we assume only that $\prod_{i=1}^\infty (1 - p(i)) > 0$, then, although, as we mentioned above, we can not assure the convergence of $\text{Prob}(n, \overline{p}; \psi)$ for every ψ from $L_\leq$, a kind of a "weak convergence" takes place, namely, for every ψ from $L_\leq$ we have $\limsup_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) - \liminf_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) < 1$.

We also study convergence properties of a similar random graph model $C(n, \overline{p})$ which uses as the universe a circuit of n points instead of interval $[n]$. It appears that in this case convergence does not depend very much on the strength of the language. We show that there is a first order sentence ψ which uses only adjacency relation such that for every infinite sequence $\overline{a}$ of positive numbers, there exists a sequence $\overline{p}$, obtained from $\overline{a}$ by

adding enough zero terms, for which the probability that $C(n, \bar{p})$ has ψ does not converge. On the other hand, under some rather natural constraints imposed on $\bar{p}$, one can show that a zero-one law holds for a large class of first order properties provided $\bar{p}$ is such that $\log(\prod_{i=1}^{n}(1 - p(i)))/\log n \to 0$.

We conclude the paper with some additional remarks concerning presented results and their possible generalizations. Here we show also that even in the case when a zero-one law holds the probability $\mathrm{Prob}(n, \bar{p}; \psi)$ can tend to the limit very slowly and no decision procedure can determine the limit $\lim_{n\to\infty} \mathrm{Prob}(n, \bar{p}; \psi)$ for every first order sentence ψ.

2. First order logic – useful tools and basic facts.

In this part of the paper we gather basic notions and facts concerning the first order logic which shall be used later on. Throughout this section $\bar{L}$ denotes a first order logic whose vocabulary contains a finite number of predicates $P_1, P_2, \ldots, P_m$, where the i-th predicate P_i has j_i arguments. More formally, for a vocabulary τ let L^τ be the first order logic (i.e the set of first order formulas) in the vocabulary τ. A τ-model M, called also a L^τ-model, is defined in the ordinary way. In the paper we use four vocabularies:

1) τ_0 such that the τ_0-models are just graphs; we write L instead L^{τ_0}.

2) τ_1 such that the τ_1-models are, up to isomorphism, quintuples $([n], S, c, d, R)$, where $[n] = \{1, 2 \ldots, n\}$, S is the successor relation, c, an individual constant, is 1, the other individual constant d is n and $([n], R)$ is a graph; we write L_+ instead L^{τ_1}.

3) τ_2 such that the τ_2-models are, up to isomorphism, triples $([n], \leq, R)$, where $\leq$ is the usual order on $[n]$ and $([n], R)$ is a graph; we write $L_\leq$ instead L^{τ_2}.

4) τ_3 such that the τ_3-models are, up to isomorphism, triples $([n], C, R)$, where C is the ternary relation of between in clockwise order (i.e. $C(v_1, v_2, v_3)$ means that $v_{\sigma(1)} \leq v_{\sigma(2)} \leq v_{\sigma(3)}$ for some cyclic permutation σ of set $\{1, 2, 3\}$) and $([n], R)$ is a graph; we write $L_\leq^c$ instead L^{τ_3}.

For every natural number k and $\bar{L}$-model $M = (U^M; P_1^M, P_2^M, \ldots, P_m^M)$ we set

$$\mathrm{Th}_k(M) = \{\phi : M \models \phi, \phi \text{ is a first order sentence from } \bar{L} \text{ of quantifier depth } \leq k\}.$$

The *Ehrenfeucht game* of length k on two $\bar{L}$-models M^1 and M^2 is the game between two players, where in the i-th step of the game, $i = 1, 2, \ldots, k$, the first player chooses a point v_i^1 from U^{M^1} or v_i^2 from U^{M^2} and the second player must answer by picking a point from the universe of the other model. The second player wins such a game when the structures induced by points $v_1^1, v_2^1, \ldots, v_k^1$ and $v_1^2, v_2^2, \ldots, v_k^2$ are isomorphic, i.e.

$$P_i^{M^1}(v_{l_1}^1, v_{l_2}^1, \ldots, v_{l_{j_i}}^1) = P_i^{M^2}(v_{l_1}^2, v_{l_2}^2, \ldots, v_{l_{j_i}}^2)$$

for every $i = 1, 2, \ldots, m$ and $l_1, l_2, \ldots, l_{j_i} \in [k]$. The following well known fact (see for example Gurevich [Gu 85]) makes the Ehrenfeucht game a useful tool in studies of first order properties.

FACT 1. *Let M_1, M_2 be τ-models. Then the second player has a winning strategy for the Ehrenfeucht game of length k played on M^1 and M^2 if and only if $\mathrm{Th}_k(M^1) = \mathrm{Th}_k(M^2)$.* ☐

In the paper we shall use also a "local" version of the above fact. For a $\bar{L}$-model and a point $v \in U^M$ the *neighbourhood* $N(v)$ of v is the set of all points w from U^M such that either $v = w$ or there exists i, $i = 1, 2, \ldots, m$, and $v_1, v_2, \ldots, v_{j_i} \in U^M$ for which $P_i^M(v_1, v_2, \ldots, v_{j_i})$, where $v = v_{i_1}$, $w = v_{i_2}$ for some i_1, i_2. Set $N_1(v) = N(v)$ and $N_{i+1}(v) = \bigcup_{w \in N_i(v)} N(w)$ for $i = 1, 2, \ldots$, and define the distance between two points $v, w \in U^M$ as the smallest i for which $v \in N_i(w)$. Clearly, the distance defined in such a way is a symmetric function for which the triangle inequality holds. Now let M^1 and M^2 be two $\bar{L}$-models, $v^1 \in U^{M^1}$ and $v^2 \in U^{M^2}$. We say that pair (M^1, v^1) is *k-equivalent* to (M^2, v^2) when the second player has a winning strategy in the "restricted" Ehrenfeucht game of length k in which, in the first step players must choose vertices $v^1 = v_1^1$ and $v^2 = v_2^2$ and in the i-th step, $i = 1, 2, \ldots, k$ they are forced to pick vertices v_i^1 and v_i^2 from sets $\bigcup_{j=1}^{i-1} N_{3^{k-i}}(v_j^1)$ and $\bigcup_{j=1}^{i-1} N_{3^{k-i}}(v_j^2)$. The following result (which, in fact, is a version of a special case of Gaifman's result from [Ga 82]) is an easy consequence of Fact 1.

FACT 2. *Let M^1 and M^2 be two $\bar{L}$-models such that for $l = 1, 2, \ldots, k$, $i = 1, 2$, every choice of points $v_1^i, v_2^i, \ldots, v_l^i \in U^{M^i}$ and $v_1^{3-i}, v_2^{3-i}, \ldots, v_{l-1}^{3-i} \in U^{M^{3-i}}$, such that no two of the v_j^i and v_j^{3-i} are at a distance less than 3^{k-l+1} from each other and (M^i, v_j^i) is $(k-l+1)$-equivalent to (M^{3-i}, v_j^{3-i}) for $j = 1, 2, \ldots, l-1$, there exists $v_l^{3-i} \in U^{M^{3-i}}$ such that (M^i, v_l^i) is $(k-l)$-equivalent to (M^{3-i}, v_l^{3-i}).*
Then $\mathrm{Th}_k(M^1) = \mathrm{Th}_k(M^2)$. ☐

Finally, we need some results from the theory of additivity of models. Call Σ a *scheme of a generalized sum* with respect to vocabularies $\hat{\tau}$, τ and τ' if for each predicate $P(\bar{x})$ of τ' and breaking $<\bar{x}_i>_{i \leq k}$ of $\bar{x}$, Σ gives a first order, quantifier free formula $\phi_P(z_1, \ldots, z_k)$ in vocabulary

$$\hat{\tau} \cup \{R_i^\psi : \psi \text{ is a quantifier free formula in } L^\tau \text{ with free variables } \bar{x}_i\},$$

where R_i^ψ denotes a zero-place predicate, i.e. a truth value.

DEFINITION. *Let Σ be a scheme of a generalized sum with respect to vocabularies $\hat{\tau}$, τ and τ', I be a $\hat{\tau}$-model and $\{M_i\}_{i \in I}$ be a family of τ-models. We shall say that a τ'-model N is a (I, Σ)-sum of $\{M_i\}_{i \in I}$ if the universe of N is the disjoint sum of the universes of $\{M_i\}_{i \in I}$ and for each τ'-predicate $P(\bar{x})$ relation P^N is the set of $\bar{a}$ such that for some breaking $<\bar{x}_i>_{i \leq k}$ of $\bar{x}$ there are distinct members $t_1, t_2, \ldots, t_k$ of I and sequences $\bar{a}_i$ of members of M_{t_i} of the same length as $\bar{x}_i$, $i \leq k$, such that $\bar{a}$ is a concatenation of $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ and, if for each $i \leq k$ we interpret R_i^ψ as the truth whenever $M_{t_i} \models \psi(\bar{a}_i)$, in the model I the formula $\phi_P(t_1, t_2, \ldots, t_k)$ is satisfied.*

Example 1.

(i) Let $G_1, G_2, \ldots, G_k$ be graphs, treated as models of language L, whose vocabulary contains only a binary predicate interpreted as the adjacency relation. Then the graph

$$G = G_1 \oplus G_2 \oplus \ldots G_k \,,$$

defined as the sum of disjoint copies of $G_1, G_2, \ldots, G_k$, is a (Σ, I)-sum of these graphs, for $I = \{1, 2, \ldots, k\}$ and empty vocabulary $\hat{\tau}$.

(ii) For $i = 1, 2, \ldots, m$, let G_i be a graph with vertex set $\{1, 2, \ldots, n_i\}$ and

$$G = G_1 \bar{\oplus} G_2 \bar{\oplus} \ldots \bar{\oplus} G_m$$

denote a graph with vertex set $\{1, 2, \ldots, \sum_{i=1}^{m} n_i\}$ such that vertices v and w are adjacent in G if and only if for some $j = 1, 2, \ldots, m$,

$$\sum_{i=1}^{j-1} n_i < v < w \leq \sum_{i=1}^{j} n_i$$

and vertices $v - \sum_{i=1}^{j-1} n_i$ and $w - \sum_{i=1}^{j-1} n_i$ are adjacent in G_j.

Let us view graphs $G_1, G_2, \ldots, G_m, G$ as models of language L_+, which contains the adjacency relation and the successor predicate and two individual constants which represent the first and the last elements of a graph. Then, G can be treated as a (Σ, I)-sum of $G_1, G_2, \ldots, G_m$, for $I = \{1, 2, \ldots, m\}$ and $\hat{\tau} = L_+$.

(iii) Let graphs $G_1, G_2, \ldots, G_m, G$ be defined as in the previous case. Then, if these graphs are treated as models of language $L_\leq$, which contains the adjacency relation and the predicate "$\leq$", G can be viewed as a (Σ, I)-sum of $G_1, G_2, \ldots, G_m$, where $I = \{1, 2, \ldots, m\}$ is the model of linear order.

(iv) Let graphs $G_1, G_2, \ldots, G_m, G$ be defined as in (ii) and $L_\leq^c$ be the language which contains predicate $C(v_1, v_2, v_3)$ which means that for some cyclic permutation σ of indices 1,2,3 we have $v_{\sigma(1)} \leq v_{\sigma(2)} \leq v_{\sigma(3)}$. Then, if we treat $G_1, G_2, \ldots, G_m$ as $L_\leq$-models and G as a $L_\leq^c$-model, G can be viewed as (Σ, I)-sum of $G_1, G_2, \ldots, G_m$ with $I = \{1, 2, \ldots, m\}$ treated as a $L_\leq^c$-model.

Remark. Note that in the definition of a scheme of generalized sum the formula ϕ_P which corresponds to predicate P must be quantifier free. This is the reason why we need two individual constants in the language L_+.

The main theorem about (Σ, I)-sums we shall use can be stated as follows.

FACT 3. *Let Σ be a fixed scheme of addition with respect to some fixed vocabularies. Then, for every k and N, a (Σ, I)-sum of $\{M_i\}_{i \in I}$, $\mathrm{Th}_k(N)$ can be computed from*

$\{\mathrm{Th}_k(I, R_s) : s \in S\}$, where $S = \{\mathrm{Th}_k(M) : M \text{ is a } \tau\text{-model}\}$ and $R_s = \{i \in I : \mathrm{Th}_k(M_i) = s\}$. □

We apply this result to (Σ, I)-sums of graphs described in Example 1.

FACT 4. *Let operations "$\oplus$" and "$\bar{\oplus}$" and languages L, L_+, $L_\leq$ and $L_\leq^c$ be defined in the same way as in the Example 1. Furthermore, let $\mathcal{G}$ and $\bar{\mathcal{G}}$ be families of graphs closed under $\oplus$ and $\bar{\oplus}$ respectively, and let k be a natural number. Then*

(i) *there exists a graph $G \in \mathcal{G}$ such that for every $H \in \mathcal{G}$*

$$\mathrm{Th}_k(G) = \mathrm{Th}_k(G \oplus H),$$

where in the above equation all graphs are treated as L-models;

(ii) *there exists a graph $\bar{G} \in \bar{\mathcal{G}}$ such that for every $\bar{H} \in \bar{\mathcal{G}}$*

$$\mathrm{Th}_k(\bar{G}) = \mathrm{Th}_k(\bar{G}\bar{\oplus}\bar{H}\bar{\oplus}\bar{G}),$$

where either all graphs are treated as L_+-models or all of them are viewed as $L_\leq$-models;

(iii) *there exists a graph $\bar{G} \in \bar{\mathcal{G}}$ such that for every $\bar{H} \in \bar{\mathcal{G}}$*

$$\mathrm{Th}_k(\bar{G}) = \mathrm{Th}_k(\bar{G}\bar{\oplus}\bar{H}),$$

where both $\bar{G}$ and $\bar{G}\bar{\oplus}\bar{H}$ are treated as $L_\leq^c$-models.

Proof. Let $\mathcal{U}$ be a set of all finite words over finite alphabet S. Words from $\mathcal{U}$ can be viewed as models of a language $L(S)$, whose vocabulary consists of unary predicates P_s, for $s \in S$. Then, for any word α which contains k copies of each letter of the alphabet and any other word β we have

(1)
$$\mathrm{Th}_k(\alpha) = \mathrm{Th}_k(\alpha \circ \beta),$$

where $\alpha \circ \beta$ denotes concatenation of α and β.

Let us set $S = \{\mathrm{Th}_k(G) : G \text{ is a } L\text{-model}\}$, and choose $\{G_s\}_{s \in S}$ such that $\mathrm{Th}_k(G_s) = s$. Furthermore, let

$$G' = \bigoplus_{s \in S} G_s \quad \text{and} \quad G = \underbrace{G' \oplus \ldots \oplus G'}_{k \text{ times}}.$$

Then, from Fact 3 and (1), for every H, treated as a L-model, we have $\mathrm{Th}_k(G) = \mathrm{Th}_k(G \oplus H)$.

Now, treat words from $\mathcal{U}$ as $L_+(S)[L_\leq(S)]$-models for a language $L_+(S)$ $[L_\leq(S)]$ which, in addition to the predicates P_s, contains also the successor predicate [the predicate "$\leq$"].

6

It is not hard to show (see, for instance, Shelah and Spencer [SS 94]) that for every k there exists a word α such that for every β we have $\text{Th}_k(\alpha) = \text{Th}_k(\alpha \circ \beta \circ \alpha)$ for both $L_+(S)$ and $L_{\leq}(S)$. Thus, similarly as in the case of (i), the second part of the assertion follows from Fact 3.

Finally, let $L_{\leq}^c(S)$ be a language which contains the predicates P_s and the ternary predicate C denoting clockwise order. It is known (see again [SS 94]) that for every k there exists a word α such that for every other word β we have $\text{Th}_k(\alpha) = \text{Th}_k(\alpha \circ \beta)$, where this time both α and $\alpha \circ \beta$ are treated as $L_{\leq}^c(S)$-models. Hence, using Fact 3 once again, we get the last part of Fact 4. □

3. Zero-one laws for language L.

In this section we characterize sequences $\overline{p}$ for which the probability $\text{Prob}(n, \overline{p}; \psi)$ converges for every sentence ψ from L, or, more precisely, for which a zero-one law holds, i.e. it converges to either 0 or 1. One could easily see that the proof of either Glebskii, Kogan, Liogonkii and Talanov [GKLT 69], or Fagin [Fa 76], can be mimicked whenever a sequence $\overline{p}$ is such that

$$0 < \liminf_{i \to \infty} p(i) \leq \limsup_{i \to \infty} p(i) < 1 \,,$$

so it is enough to consider only the case when $\liminf_{i \to \infty} p(i) = 0$ (if $\limsup_{i \to \infty} p(i) = 1$ one can instead consider properties of the complement of $G(n, \overline{p})$). The main result of this section describes rather precisely how convergence properties of $G(n, \overline{p})$ depend on the fact how fast the product $\prod_{i=1}^{\infty}(1 - p(i))$ tends to 0.

THEOREM 1.

    (i)   *For every sequence* $\overline{p} = (p(1), p(2), \dots,)$ *such that* $p(i) < 1$ *for all* i *and*

$$(2) \qquad \log\left(\prod_{i=1}^{n}(1 - p(i))\right)\Big/\log n \to 0$$

        and every sentence ψ *from* L *a zero-one law holds.*

    (ii) *For every positive constant* ϵ *there exists a sequence* $\overline{p}$ *and a sentence* ψ *from* L *such that*

$$-\log\left(\prod_{i=1}^{n}(1 - p(i))\right)\Big/\log n < \epsilon$$

        but $\liminf_{n \to \infty} \text{Prob}(n, \overline{p}; \psi) = 0$ *while* $\limsup_{n \to \infty} \text{Prob}(n, \overline{p}; \psi) = 1$.

In order to show Theorem 1 we need some information about the structure of $G(n, \overline{p})$. A subgraph H' of a graph G with the vertex set $[n]$ is the *exact copy* of a graph H with the vertex set $[l]$, if for some i, $0 \leq i \leq n - l$, and every $j, k \in [l]$ the pair $\{j, k\}$ is an edge of H if and only if $\{i + j, i + k\}$ appears as an edge of H' and G contains no edges with precisely one end in $\{i + 1, i + 2, \dots, i + l\}$. Furthermore, call a graph H on the vertex set

[l] *admissible* by a sequence $\overline{p}$ if the probability that $H = G(l, \overline{p})$ is positive. We shall show first that, with probability tending to 1 as $n \to \infty$, $G(n, \overline{p})$ contains many disjoint exact copies of every finite admissible graph, provided $\prod_{i=1}^{n}(1 - p(i))$ tends to infinity slowly enough.

LEMMA. *For $k \geq 1$ let $\overline{p}$ be a sequence such that*

$$\prod_{i=1}^{n}(1 - p(i)) \geq n^{-1/(10k)},$$

and let H be an admissible graph with vertex set $[k]$. Then, the probability that in $G(n, \overline{p})$ there exist at least $n^{0.1}$ vertex disjoint exact copies of H, none of them containing vertices which are either less than $\log n$ or larger than $n - \log n$, tends to 1 as $n \to \infty$.

Proof. Let $\mathcal{A}$ denote the family of disjoint sets $A_i = \{\lceil \log n \rceil + ik + 1, \lceil \log n \rceil + ik + 2, \dots, \lceil \log n \rceil + (i+1)k\}$, where $i = 0, 1, 2, \dots, i_0 - 1$, $i_0 = \lfloor (n - 2 \log n)/k \rfloor$. For every set $A_i \in \mathcal{A}$ the probability that the subgraph induced in $G(n, \overline{p})$ by A_i is an exact copy of a graph H with edge set $E(H)$ equals $P(A_i) = P(H)P'(A_i)$, where the factor

$$P(H) = \prod_{e=\{i,j\}\in E(H)} p(|i - j|) \prod_{e'=\{i',j'\}\notin E(H)} (1 - p(|i' - j'|)) > 0$$

remains the same for all sets A_i, whereas the probability $P'(A_i)$ that no vertices of A_i are adjacent to vertices outside A_i, given by

$$P'(A_i) = \prod_{r=1}^{k} \prod_{s=1}^{\lceil \log n \rceil + ik} (1 - p(\lceil \log n \rceil + ik + r - s)) \prod_{t=\lceil \log n \rceil + (i+1)k+1}^{n} (1 - p(t - \lceil \log n \rceil - ik - r)),$$

may vary with i. Nevertheless, for a sequence $\overline{p}$ which fulfills the assumptions of the Lemma, we have always

$$P'(A_i) \geq \left(\prod_{r=1}^{n}(1 - p(r)) \right)^{2k} \geq n^{-0.2}.$$

(Here and below we assume that all inequalities hold only for n large enough.) Thus, there exists a subfamily $\mathcal{A}'$ of $\mathcal{A}$ with $\lfloor \sqrt{n} \rfloor$ elements, such that for every $A \in \mathcal{A}'$ the probability $P(A)$ is roughly the same, i.e. for some function $f(n)$, where $n^{-0.2} \leq f(n) \leq 1$,

$$f(n)(1 - o(n^{-0.1})) \leq P(A) \leq f(n)(1 + o(n^{-0.1}))$$

for all $A \in \mathcal{A}'$.

8

Now, let X denote the number of sets from $\mathcal{A}'$ which are exact copies of H. For the expectation of X we get

$$\mathrm{E}\,X = \sum_{A \in \mathcal{A}'} P(A) = (1 + o(n^{-0.1}))f(n)\lfloor\sqrt{n}\rfloor \geq n^{0.2}\,.$$

To estimate the variance of X we need to find an upper bound for

$$\mathrm{E}\,X(X-1) = \sum_{\substack{A,B \in \mathcal{A}' \\ A \neq B}} P(A)P(B) / \left(\prod_{r \in A}\prod_{s \in B}(1 - p(|r-s|))\right)\,.$$

Note first that $(1 - n^{-0.2})^{n^{0.3}} \leq n^{-1/(10k)}$, so in every sequence $\overline{p}$ for which the assumption holds at most $n^{0.3}$ of the first n terms are larger than $n^{-0.2}$. Hence, for all, except for at most $n^{0.8}$, pairs $A, B \in \mathcal{A}'$ we have

$$\prod_{r \in A}\prod_{s \in B}(1 - p(|r-s|)) \geq (1 - n^{-0.2})^{k^2} \geq 1 - n^{-0.1}\,.$$

Moreover, for every $A, B \in \mathcal{A}'$

$$\prod_{r \in A}\prod_{s \in B}(1 - p(|r-s|)) \geq \prod_{i=1}^{n}(1 - p(i))^2 \geq n^{-1/(5k)} \geq n^{-0.1}\,.$$

Thus,

$$\mathrm{E}\,X(X-1) \leq nf^2(n)(1 + O(n^{-0.1})) + n^{0.8}f^2(n)n^{0.1} \leq nf^2(n)(1 + O(n^{-0.1}))\,,$$

the variance of X is $o((\mathrm{E}\,X)^2)$, and from Chebyshev's inequality with probability tending to 1 as $n \to \infty$ we have $X > \mathrm{E}\,X/2 > n^{0.1}$. □

Proof of Theorem 1. Let ψ be a first order sentence of quantifier depth k. For two graphs G_1 and G_2 define graph $G_1 \oplus G_2$ as the disjoint sum of G_1 and G_2. Since all $p(i)$ are less than 1 the family of admissible graphs is closed under the operation "$\oplus$". Thus, from Fact 4, there exists an admissible graph G such that for every admissible graph H we have $\mathrm{Th}_k(G \oplus H) = \mathrm{Th}_k(G)$ (Let us recall that all graphs are treated here as models of language L which contains one binary predicate interpreted as the adjacency relation.) From the Lemma we know that, for every sequence $\overline{p}$ for which (2) holds, the probability that $G(n, \overline{p})$ contains an exact copy of G tends to 1 as $n \to \infty$. Thus, with probability $1 - o(1)$, $\mathrm{Th}_k(G(n, \overline{p})) = \mathrm{Th}_k(G)$ and the first part of Theorem 1 follows.

Now fix $k \geq \lceil 1/\epsilon \rceil$ and let $\overline{b}$ be a sequence of natural numbers such that $b(1) > 6k$ and $b(m+1) \geq (b(m))^{50}$ (e.g. $b(m) = (2k)^{50^m}$). Let us define a sequence $\overline{p}$ setting

$$p(i) = \begin{cases} 1/2 & \text{for} \quad i \leq b(1) \\ \frac{1}{3ik} & \text{for} \quad b(2m-1) < i \leq b(2m), \quad \text{where } m = 1, 2, \ldots \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, using the fact that for every $x \in (0, 2/3)$

$$\exp(-2x) < 1 - x < \exp(-x) \, ,$$

we get

$$\prod_{i=1}^{n}(1 - p(i)) \geq 2^{-b(1)} \prod_{i=1}^{n} \left(1 - \frac{1}{3ki}\right) \geq n^{-1/k} \geq n^{-\epsilon} \, .$$

for every sufficiently large n. We shall show that the probability that $G(n, \overline{p})$ contains an exact copy of the complete graph K_l on $l = 6k$ vertices approaches both 0 and 1 infinitely many times.

Indeed, for $n = b(2m + 1)$ and m large enough we have

$$\prod_{i=1}^{n}(1 - p(i)) \geq 2^{-b(1)} \prod_{i=1}^{b(2m)} \left(1 - \frac{1}{3ik}\right) \geq O(1)(b(2m))^{-2/(3k)}$$

$$\geq (b(2m + 1))^{-1/(70k)} = n^{-1/(70k)} \, .$$

Thus, from the Lemma, the probability that $G(b(2m + 1), \overline{p})$ contains an exact copy of K_l tends to 1 as $m \to \infty$. On the other hand, the expected number of exact copies of K_l in $G(b(2m + 2), \overline{p})$ is, for m large enough, bounded from above by

$$b(2m + 2)\left(\prod_{i=6k}^{b(2m+2)/2}(1 - p(i))\right)^{6k} \leq b(2m + 2)\left(\prod_{i=b(2m+1)+1}^{b(2m+2)/2} \left(1 - \frac{1}{3ik}\right)\right)^{6k}$$

$$\leq (1 + o(1))b(2m + 2)\left(\frac{b(2m + 2)}{2b(2m + 1)}\right)^{-2} \leq (b(2m + 2))^{-1/3} \, ,$$

and tends to 0 as $m \to \infty$. $\square$

Note that Theorem 1 implies that each sequence $\overline{a}$ can be "diluted" by adding zeros so that for the resulting sequence $\overline{p}$, graph $G(n, \overline{p})$ has good convergence properties.

COROLLARY. *For every sequence $\overline{a}$, where $0 \leq a(i) < 1$, there is a sequence $f(i)$ such that every sequence $\overline{p}$ obtained from $\overline{a}$ by the addition of more than $f(i)$ zeros after the i-th term of $\overline{a}$ and each sentence ψ from L a zero-one law holds.* $\square$

4. Convergence for language L_+.

In [Ly 80] Lynch showed that if $p(i)$ does not depend on i, i.e. when $p(i) = p_0$ for some constant $0 < p_0 < 1$ and $i = 1, 2, \ldots$, then $\text{Prob}(n, \overline{p}; \psi)$ converges for every sentence ψ from L_+. In fact, his argument guarantees the existence of $\lim_{n \to \infty} \text{Prob}(n, \overline{p}; \psi)$ for each sentence ψ from L_+ and every sequence $\overline{p}$ which tends to a positive limit strictly smaller than one. Furthermore, if $\liminf_{i \to \infty} p(i) < \limsup_{i \to \infty} p(i)$ the probability of the property that vertices 1 and n are adjacent does not converge, so, it is enough to consider the case when $p(i) \to 0$. Our first result says that the condition that $\prod_{i=1}^{\infty}(1 - p(i)) > 0$, or, equivalently, $\sum_{i=1}^{\infty} p(i) < \infty$, is sufficient and, in a way, necessary, for the convergence of $\text{Prob}(n, \overline{p}; \psi)$ for every sentence ψ from L_+.

THEOREM 2.

(i) *If $\overline{p}$ is a sequence such that $p(i) < 1$ for all i and*

$$(3) \qquad \prod_{i=1}^{\infty}(1 - p(i)) > 0 \,,$$

then for every sentence ψ from L_+ the limit $\lim_{n \to \infty} \mathrm{Prob}(n, \overline{p}; \psi)$ exists.

(ii) *For every function $\omega(n)$ which tends to infinity as $n \to \infty$ there exist a sequence $\overline{p}$ and sentence ψ from L_+ such that*

$$(4) \qquad \omega(n) \prod_{i=1}^{n}(1 - p(i)) \to \infty$$

but $\liminf_{n \to \infty} \mathrm{Prob}(n, \overline{p}; \psi) = 0$ whereas $\limsup_{n \to \infty} \mathrm{Prob}(n, \overline{p}; \psi) = 1$.

Proof. We shall deduce the first part of Theorem 2 from Fact 2. Since our language contains the successor predicate, in this section the distance between two vertices $v, w \in [n]$ of a graph G will be defined as the length of the shortest path joining v to w in the graph $\hat{G}$ obtained from G by adding to the set of edges of G all pairs $\{i, i + 1\}$, where $i = 1, 2, \ldots, n - 1$, and the neighbourhood of a vertex v will mean always neighbourhood in $\hat{G}$.

Let $\overline{p}$ be a sequence for which (3) holds and ψ be a sentence from L_+ of quantifier depth k. Call a pair (H, v) *safe* if H is an admissible graph on $[l]$ and v is a vertex of H which lies at a distance at least 3^k from both 1 and l. Since there are only finite number of k-equivalence classes we can find a finite family $\mathcal{H}$ of safe pairs such that every safe pair (H', v') is k-equivalent to some pair (H, v) from $\mathcal{H}$. Now, due to the Lemma, with probability tending to 1 as $n \to \infty$, $G(n, \overline{p})$ contains at least k exact copies of every safe pair (H, v) from $\mathcal{H}$. Thus, roughly speaking, the "local" properties of the "internal" vertices are roughly the same for all graphs $G(n, \overline{p})$, provided n is large enough.

In order to deal with vertices lying near 1 and n we need to "classify" graphs with vertex set $[n]$ according to their "boundary" regions. More specifically, let $H_{n,k}(1)$ and $H_{n,k}(n)$ denote subgraphs induced in $G(n, \overline{p})$ by all vertices which lie within the distance 3^{k+2} from 1 and n respectively. We show that the probability that $(H_{n,k}(1), 1)$ $[(H_{n,k}(n), n)]$ is $(k + 1)$-equivalent to some (H, v) converges as $n \to \infty$.

Let ϵ be any positive constant. Note first that the expected number of neighbours of a given vertex in $G(n, \overline{p})$ is bounded from above by $C_1 = 2 + \sum_{i=2}^{\infty} p(i)$. Hence, the expected number of vertices in $H_{n,k}(1)$ is less than $C_2 = \sum_{i=0}^{3^{k+2}} C_1^i$ and, from Markov inequality, the probability that $H_{n,k}(1)$ contains more than $C_3 = C_2/\epsilon$ vertices is less than ϵ. Moreover, choose C_4 in such a way that $\sum_{i \geq C_4} p(i) \leq \epsilon/C_3$. Then, the conditional probability that some $v \in H_{n,k}(1)$ has a neighbour w such that $|v - w| \geq C_4$, provided the size of $H_{n,k}(1)$ is

11

less than C_3, is bounded from above by ϵ. Hence, with probability at least $1 - 2\epsilon$, $H_{n,k}(1)$ contains no vertices v for which $v \geq C_5 = 3^{k+2}C_4 + 1$. Thus, for every $n, m \geq C_5$, $H_{n,k}(1)$ and $H_{m,k}(1)$ are "isomorphic" with probability at least $1 - 4\epsilon$, or, more precisely, for every property ϕ

$$|P(H_{n,k}(1) \text{ has } \phi) - P(H_{m,k}(1) \text{ has } \phi)| \leq 4\epsilon .$$

In particular, for every graph H on $[l]$ vertices the probability that $(H_{n,k}(1), 1)$ and $(H, 1)$ are $(k+1)$-equivalent converges as $n \to \infty$. Clearly, the analogous result holds for $H_{n,k}(n)$.

To complete the proof note that the fact that (H, v) and (H', v') are $(k+1)$-equivalent implies that for every vertex w in (H, v), lying within a distance 3^k from v, there exists a vertex w' in (H', v') within a distance 3^k of v' such that (H, w) and (H', w') are k-equivalent. Thus, if a graph G with vertex set $[n]$ contains k exact copies of every safe pairs from $\mathcal{H}$, $\mathrm{Th}_k(G)$ can be computed from the $(k+1)$-equivalence classes of its 3^{k+1} neighbourhoods of 1 and n and the assertion follows.

Now let $\omega(n)$ be a function which tends to infinity as $n \to \infty$. We may assume that $\omega(n)$ is non-decreasing, and, say, $\omega(n) \leq n/100$. Let $f(2) = 1$ and for $m \geq 3$

$$f(m) = \min\{l : \omega(l) \geq 2^{f(m-1)}\} + 4m^3 .$$

Define a sequence $\overline{p}$ setting

$$p(i) = \begin{cases} 1/m & \text{for} \quad f(m) - m^3 \leq i \leq f(m), \quad m = 2, 3, \dots \\ 0 & \text{otherwise} , \end{cases}$$

and let $m = \max\{l : f(l) \leq n\}$. Note that for every j we have $f(j) > f(j-1) + j^3$ so $p(i)$ is correctly defined. Furthermore, $f(j) \geq 2^{j-2}$ for all j, so $\omega(n) \geq 2^{2^{m-2}}$ and the sequence $p(1), p(2), \dots, p(n)$ contains at most m^4 non-zero terms. Consequently,

$$\omega(n) \prod_{i=1}^{n} (1 - p(i)) \geq 2^{2^{m-2}} (1 - 1/2)^{m^4} = 2^{2^{m-2} - m^4} \to \infty$$

as $m \to \infty$. Furthermore, let ψ be the property that vertices 1 and n are joined by a path of length two. Then $P(2f(m) - 2m^3, \overline{p}; \psi) = 0$ while

$$P(2f(m) - m^3, \overline{p}; \psi) \geq 1 - \left(1 - 1/m^2\right)^{m^3 - 1} \geq 1 - 3e^{-m} = 1 - o(1) . \quad \square$$

Remark. Note that in fact we have shown that if ψ belongs to L_+ and a sequence $\overline{p}$ fulfills (3) then $\lim_{n \to \infty} \mathrm{Prob}(n, \overline{p}; \psi)$ is equal to the probability that $G(\infty, \overline{p})$ has ψ, where $G(\infty, \overline{p})$ is a graph with vertex set $V = V_1 \cup V_2 = \{1, 2, \dots\} \cup \{\dots, -2, -1, \}$ which contains no edges between sets V_1 and V_2 and vertices $v_i, w_i \in V_i$, $i = 1, 2$, are adjacent with probability $p(|v_i - w_i|)$.

The second part of Theorem 2 suggests that the analog of the Corollary derived from Theorem 1 is not valid for language L_+. The following example shows that, in fact, much more is true – there exist a sentence ψ from L_+ and a sequence $\overline{a}$ such that for *each* sequence $\overline{p}$ obtained from $\overline{a}$ by adding enough zero terms the probability $\mathrm{Prob}(n, \overline{p}; \psi)$ approaches both 0 and 1 infinitely many times.

EXAMPLE 2. *Let $\bar{b}$ be a sequence such that $b(j+1) \geq (b(j))^{10}$ (e.g. $b(j) = \exp(10^j)$) and*

$$a(i) = \begin{cases} i^{-0.2} & for \quad b(2j) < i \leq b(2j+1) \\ i^{-0.95} & for \quad b(2j+1) < i \leq b(2j+2) \,. \end{cases}$$

Furthermore, let $\bar{f}$ be any sequence such that for every $i \geq 2$ we have $f(i) > 10 \sum_{j=1}^{i-1} f(j)$,

$$p(j) = \begin{cases} a(i) & if \quad j = f(i) \\ 0 & otherwise \,. \end{cases}$$

and ψ be a sentence from L_+ saying that any two neighbours of vertex 1 are connected by a path of length four not containing vertex 1.

Then $\liminf_{n\to\infty} \mathrm{Prob}(n, \bar{p}; \psi) = 0$ and $\limsup_{n\to\infty} \mathrm{Prob}(n, \bar{p}; \psi) = 1$.

Validation. Let $n = f(m)$, where $m = b(2j+1)$, and let $v = f(i)+1$ and $w = f(j)+1$ be neighbours of 1 in $G(n, \bar{p})$. Note that $f(m) > 4f(m-1)$, so for every edge $\{s,t\}$ of $G(n, \bar{p})$ we have $|s - t| < n/4$, in particular $v, w \leq n/4$. The probability that v and w are *not* connected by a path of type $v(v+f(k))(v+f(k)+f(j))(w+f(j))w$ is bounded from above by

$$\prod_{k=1}^{m-1} (1 - a(k)a(j)a(k)a(j)) \leq \prod_{r=m^{0.1}}^{m-1} (1 - r^{-0.4}a^2(j))$$

$$\leq \exp\left(-a^2(j) \sum_{r=m^{0.1}}^{m/2} r^{-0.4}\right) \leq \exp\left(-a^2(j)m^{0.5}\right) \leq \exp\left(-m^{0.1}\right)$$

since for a vertex $u = f(l)$ either $l \geq m^{0.1} > b(2j)$ and then $a(l) = l^{-0.2} \geq m^{-0.2}$, or $l \leq m^{0.1}$ and so $a(l) \geq l^{-0.95} \geq m^{-0.2}$. In $G(n, \bar{p})$ vertex 1 has at most m neighbours, so the expected number of pairs v and w such that both v and w are adjacent to 1 but they are not connected by a path of length 4 is bounded from above by $m^2 \exp\left(-m^{0.1}\right)$ and tends to 0 as $m \to \infty$.

Now let $n = f(m)$, where $m = b(2j+2)$, and let $v = f(i)+1$ and $w = f(j)+1$ denote the largest and the second largest neighbours of 1 in $G(n, \bar{p})$, respectively. Note that, since the sequence $\bar{f}$ grows very quickly, every path $vv_1v_2v_3w$ of length four joining v and w has the property that among $|v - v_1|$, $|v_1 - v_2|$, $|v_2 - v_3|$ and $|v_3 - w|$ each from distances $f(i)$ and $f(j)$ appears once and some distance $f(k)$ appears twice. Since, for given k, there are at most eight possible paths of length four joining v and w whose "edge lengths" are $f(i), f(j), f(k), f(k)$, the probability that v and w are joined by a path of length four is bounded from above by

$$8 \sum_{k=1}^{m} a(i)a(j)a(k)a(k) \,.$$

13

But, since v and w are largest neighbours of 1 in $G(n, \overline{p})$, with probability tending to 1 as $m \to \infty$ we have $i, j \geq m/2$ and, consequently, $a(i), a(j) \leq 2m^{-0.95}$. Thus, the probability that v and w are connected by a path of length four is less than

$$o(1) + 32m^{-1.9} \sum_{k=1}^{m} (a(k))^2 \leq o(1) + 32m^{-1.9} \sum_{k=1}^{m} m^{-0.4} \leq o(1) + 1/m = o(1) . \quad \square$$

In the last two sections we assumed that $p(i) < 1$ for all i. Now we discuss briefly the situation when we allow the sequence $\overline{p}$ to contain terms which are equal to 1. If we are dealing with L_+, any finite number of ones in $\overline{p}$ is not a problem at all. Indeed, for each formula ψ one can easily find ψ' such that ψ holds in G if and only if ψ' holds for G', where G' is obtained from G by deleting all edges $\{v, w\}$ for which $p(|v - w|) = 1$.

When we use language L even a finite number of ones in $\overline{p}$ can cause some troubles. For example, if we set $p(1) = p(2) = 1$ and the sequence $\overline{p}$ is such that for every $i < j < k$ we have $p(j - i)p(k - j)p(k - i) = 0$ unless $k = j + 1 = i + 2$, then one can easily "identify" 1 and n in $G(n, \overline{p})$ as those vertices which are contained in precisely one triangle. Thus, in this case, convergence properties for language L become very similar to those of L_+, in particular, the assumption (2) in Theorem 1i should be replaced by (3) (where products in (2) and (3) are taken over all i such that $p(i) < 1$).

When $\overline{p}$ contains an infinite number of ones the probability of some simple properties of $G(n, \overline{p})$ may oscillate between 0 and 1. To see it set, for instance,

$$p(i) = \begin{cases} 1 & \text{when } i = 4^j \text{ and } j = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

and let ϕ be the property that each edge of a graph is contained in a cycle of length 4. Then, clearly,

$$\mathrm{Prob}(n, \overline{p}; \phi) = \begin{cases} 0 & \text{when } n = 4^j + 1 \\ 1 & \text{otherwise} . \end{cases}$$

On the other hand we should mention that there are sequences with unbounded number of zeros and ones for which $\mathrm{Prob}(n, \overline{p}; \psi)$ converges. Let us take for example the random sequence $\overline{p}_{\mathrm{rand}}$ of zeros and ones such that

$$P(p_{\mathrm{rand}}(i) = 0) = P(p_{\mathrm{rand}}(i) = 1) = 1/2 ,$$

independently for each $i = 1, 2, \ldots$. Furthermore, for a given k, say that a graph G has property $\mathcal{A}_k$ if for every subset A of the vertices of G with precisely k elements and every $A' \subseteq A$ there exists a vertex v of G such that v is adjacent to all vertices from A' and not adjacent to all vertices from $A \setminus A'$. It is not hard to prove that with probability 1 sequence $\overline{p}_{\mathrm{rand}}$ has the property that for every k there exists $n_0 = n_0(k)$ such that $G(n, \overline{p}_{\mathrm{rand}})$ has $\mathcal{A}_k$ for $n \geq n_0$ (note that the probability space in this case is related only to the random construction of $\overline{p}_{\mathrm{rand}}$ – since this sequence contains only zeros and ones once it is chosen graph $G(n, \overline{p}_{\mathrm{rand}})$ is uniquely determined). Thus, the second player can easily win the Ehrenfeucht game of length k on $G(n, \overline{p}_{\mathrm{rand}})$ and $G(m, \overline{p}_{\mathrm{rand}})$, provided $n, m > n_0(k)$, and so for every sentence ψ from L a zero-one law holds.

5. Linear order case.

As one might expect, conditions which were sufficient for convergence of $\mathrm{Prob}(n, \overline{p}; \psi)$ for ψ from L_+ are too weak to assure convergence for every ψ from $L_\leq$. Our first result states that, although for a sequence $\overline{p}$ with a finite number of non-zero terms and every ψ from $L_\leq$ the probability $\mathrm{Prob}(n, \overline{p}; \psi)$ tends to a limit as $n \to \infty$, the assumption of finiteness could not be replaced by any convergence condition imposed on positive terms of the sequence $p(i)$.

THEOREM 3.

(i) *If $\overline{p}$ contains only finitely many non-zero terms then the probability $\mathrm{Prob}(n, \overline{p}; \psi)$ converges for every first order sentence ψ from $L_\leq$.*

(ii) *For every infinite sequence $\overline{a}$, where $a(i) > 0$ for $i = 1, 2, \ldots$, and every positive constant $\epsilon > 0$, there exist a sequence $\overline{p}$ obtained from $\overline{a}$ by addition of some zero terms and a first order sentence ψ from $L_\leq$ such that $\limsup_{n\to\infty} \mathrm{Prob}(n, \overline{p}; \psi) = 1$ and $\liminf_{n\to\infty} \mathrm{Prob}(n, \overline{p}; \psi) \leq \epsilon$.*

Proof of Theorem 3. Let $\overline{p}$ be a sequence with finitely many non-zero terms and let $\overline{p}'$ be obtained from $\overline{p}$ by replacing all terms equal to one by zeros. Since the successor relation can be expressed in $L_\leq$, for every sentence ψ from $L_\leq$ there exists a sentence ψ' in $L_\leq$ such that $\mathrm{Prob}(n, \overline{p}; \psi) = \mathrm{Prob}(n, \overline{p}'; \psi')$. Thus, we may assume that all terms of $\overline{p}$ are strictly less than one.

Now, let ψ be a sentence of $L_\leq$ of quantifier depth k and let $C = C(k) > 3^{k+1}$ be a constant such that no two vertices v, v' of $G(n, \overline{p})$ with $|v - v'| \leq C - 3^{k+1}$ are, with positive probability, joined by a path of length less than 3^{k+1} in $G(n, \overline{p})$. Now, in order to show the first part of Theorem 3 it is enough to classify all graphs according to the structure of the finite subgraphs induced by subsets of vertices $\{v : v < C\}$ and $\{v : v > n - C\}$ and observe that the Lemma implies that the probability that $G(n, \overline{p})$ belongs to a given class converges as $n \to \infty$. (Since Theorem 3i follows from much stronger Theorem 6 proven below we omit details.)

Now let $\overline{a}$ be an infinite sequence of numbers such that $0 < a(i) < 1$ for $i = 1, 2, \ldots$. Set $f(1) = 1$ and, for $i = 2, 3, \ldots$,

$$(5) \qquad f(i) = \lceil \max\{(i+1)/a(i+1), 4if(i-1)[1 - \max\{a(j) : j \leq i-1\}]^{-(f(i-1))^2}\} \rceil.$$

Moreover let

$$p(j) = \begin{cases} a(i) & \text{if } j = f(i) \\ 0 & \text{otherwise}. \end{cases}$$

Call a vertex v of a graph a *cutpoint* if a graph contains no edges $\{w', w''\}$ such that $w' \leq v$ and $w'' > v$ and let $\psi(r)$ be the property that a graph $G(n, \overline{p})$ contains a cutpoint v such that

$$f(r) + r \leq 2f(r) \leq v \leq n - 2f(r) \leq n - f(r) - r$$

Note first that the probability $\mathrm{Prob}(f(i),\overline{p},\psi(r))$ tends to 1 as $i \to \infty$. Indeed, since $G(f(i),\overline{p})$ contains no edges joining vertices which are at a distance larger than $f(i-1)$, the probability that a vertex $3kf(i-1)$ is a cutpoint for some $k = 1, 2, \ldots, f(i)/4f(i-1)$ is larger than $[1 - \max\{a(j) : j \le i-1\}]^{(f(i-1))^2}$ and all such events are independent. Hence, from (5), the number of cutpoints in $G(f(i),\overline{p})$ is bounded from below by the binomially distributed random variable with expectation i.

On the other hand, the probability $\mathrm{Prob}(n(i),\overline{p},\neg\psi(r))$, where $n(i) = f(i) + i/a(i)$, is bounded from below by some constant independent of i, which quickly tends to 1 as r grows. Indeed, call an edge k-small [k-large] if it is of the type $\{j, j+f(k)\}$ [$\{n(i)-j, n(i)-j-f(k)\}$] for some $j \le k/a(k)$. Since $f(k) \ge (k+1)/a(k+1)$, the existence of at least one k-small and k-large edge for each $k = r, r+1, \ldots, i$, implies $\neg\psi(r)$. The probability that none of the k-small [k-large] edges appears in $G(n(i),\overline{p})$ equals $(1 - a(k))^{k/a(k)}$ and so the probability that it happens for some $k = r, r+1, \ldots, i$ is bounded from above by

$$2\sum_{k=r}^{i}(1 - a(k))^{k/a(k)} \le 2\sum_{k=r}^{\infty}\exp(-k) < 4e^{-r}\,.$$

To complete the proof of Theorem 3 it is enough to observe that when the sequence $\overline{a}$ contains a finite number of ones we may ignore them and repeat the above argument, whereas in the case when in $\overline{a}$ an infinite number of ones appear one may just consider the property that vertex 1 is adjacent to n. $\square$

Typically, when for some probabilistic model of a finite structure and sentence ψ from language $\overline{L}$ convergence does not hold, it is possible to find another sentence ϕ in $\overline{L}$ such that the probability of ϕ has both 0 and 1 as the limiting points. Our next result says that it is not the case in $G(n,\overline{p})$, provided $\prod_{i=1}^{\infty}(1 - p(i)) > 0$.

THEOREM 4. *Let $\prod_{i=1}^{\infty}(1 - p(i)) > 0$ and let $\{\psi_\alpha\}_{\alpha \in A}$ be a finite set of sentences from $L_\le$. Then there exist a subset A' of A, a positive constant $\epsilon > 0$ and a natural number N such that for all $n > N$ the probability that $G(n,\overline{p})$ has the property $\forall_{\alpha \in A'}\psi_\alpha \wedge \forall_{\alpha \notin A'}\neg\psi_\alpha$ is larger than ϵ.*

Proof. Let k bound from above quantifier depth of sentences from A and "$\overline{\oplus}$" be the operation in a family of graphs defined in Example 1iii. Fact 4 guarantees the existence of an admissible graph G, such that $\mathrm{Th}_k(G\overline{\oplus}H\overline{\oplus}G) = \mathrm{Th}_k(G)$ for every admissible H (let us recall that all graphs are treated here as $L_\le$-models). One can easily see that if $\prod_{i=1}^{\infty}(1 - p(i)) > 0$ then the probability that $G(n,\overline{p}) = G\overline{\oplus}H\overline{\oplus}G$ for some H is bounded from below by some positive constant independent of n. Thus, the assertion follows with $A' = A \cap \mathrm{Th}_k(G)$. $\square$

COROLLARY. *If $\prod_{i=1}^{\infty}(1 - p(i)) > 0$ then for every ψ from $L_\le$*

$$\limsup_{n\to\infty}\mathrm{Prob}(n,\overline{p};\psi) - \liminf_{n\to\infty}\mathrm{Prob}(n,\overline{p};\psi) < 1. \quad \square$$

In order to get the convergence of $\mathrm{Prob}(n,\overline{p};\psi)$ the condition $\prod_{i=1}^{\infty}(1 - p(i)) > 0$ must be replaced by a significantly stronger one.

THEOREM 5. *If $\prod_{i=1}^{\infty}(1-p(i))^{i} > 0$ then for every ψ from $L_{\leq}$ the probability $\mathrm{Prob}(n, p; \psi)$ converges.*

Proof. Let ψ be a sentence from $L_{\leq}$ of quantifier depth k and $\overline{p}$ be a sequence for which $\prod_{i=1}^{\infty}(1-p(i))^{i} > 0$. We shall show that the sequence $\{\mathrm{Prob}(n, \overline{p}; \psi)\}_{n=1}^{\infty}$ is Cauchy.

Let $\{\hat{G}(n, \overline{p})\}_{n=1}^{\infty}$ be a Markov process such that $\hat{G}(n, p)$ is a graph with vertex set $[n] = \{1, 2, \ldots, n\}$, and for $n \geq 2$ graph $\hat{G}(n+1, \overline{p})$ is such that

(i)   if $1 \leq v \leq w < \lfloor n/2 \rfloor$ then the pair $\{v, w\}$ is an edge of $\hat{G}(n+1, p)$ if and only if $\{v, w\}$ is an edge of $\hat{G}(n, p)$;

(ii) if $\lfloor n/2 \rfloor < v \leq w \leq n+1$ then the pair $\{v, w\}$ is an edge of $\hat{G}(n+1, p)$ if and only if $\{v-1, w-1\}$ is an edge of $\hat{G}(n, p)$;

(iii) if $1 \leq v \leq \lfloor n/2 \rfloor \leq w \leq n+1$ and $v \neq w$ then $\{v, w\}$ is an edge of $\hat{G}(n+1, p)$ with probability $p(|v - w|)$, independently for each such pair.

Thus, roughly speaking, graph $\hat{G}(n+1, \overline{p})$ is obtained from $\hat{G}(n, \overline{p})$ by adding a new vertex in the middle of the set $[n]$. Clearly, we may (and will) identify $\hat{G}(n, \overline{p})$ with $G(n, \overline{p})$.

Now let G be an admissible graph such that for every other admissible H we have $\mathrm{Th}_k(G) = \mathrm{Th}_k(G \overline{\oplus} H \overline{\oplus} G)$, where $\overline{\oplus}$ is the operation defined in the Example 1iii. We show first that the probability that for some H_1, H_2, H_3

$$(6) \qquad\qquad \hat{G}(n, \overline{p}) = H_1 \overline{\oplus} G \overline{\oplus} H_2 \overline{\oplus} G \overline{\oplus} H_3$$

tends to 1 as $n \to \infty$.

Let k denote the number of vertices in G and $p(G)$ be the probability that $G = G(k, \overline{p})$. Moreover set $l = l(n) = \lceil \log n \rceil$ and for $i = 0, 1, \ldots, l-1$, let X_i be a random variable equal to 1 when

$$(7) \qquad\qquad \hat{G}(n, \overline{p}) = H' \overline{\oplus} G \overline{\oplus} H''$$

for some H' with il vertices and 0 otherwise. Then, for the expectation of X_i we have

$$\mathrm{E}\, X_i = p(G) \prod_{s=1}^{n-1}(1-p(s))^{\min\{s, k+il, n-s\}} \prod_{s=1}^{k+il-1}(1-p(s))^{\min\{s, k, k+il-s\}} > 0$$

and for $1 \leq i < j \leq l$,

$$\mathrm{E}\, X_i X_j = \mathrm{E}\, X_i \, \mathrm{E}\, X_j \prod_{s=(j-i)l-k+1}^{n-1}(1-p(s))^{-\min\{s-(j-i)l+k, k+il, n-s\}}$$

$$\leq \mathrm{E}\, X_i \, \mathrm{E}\, X_j \prod_{s=l}^{\infty}(1-p(s))^{-s},$$

17

where here and below we assume that n is large enough to have $l > k$. Thus, the expectation of the random variable $X = \sum_{i=0}^{l-1} X_i$ is of the order $\log n$ and

$$\operatorname{Var} X \le (\operatorname{E} X)^2 \left(\prod_{s=l}^{\infty} (1 - p(s))^{-s} - 1 \right) + O(\log n) ,$$

so, due to Chebyshev's inequality, (7) holds for some H' with at most $l(l-1)$ vertices with probability at least $1 - O(1/\log n) - O\left(\prod_{s=l}^{\infty}(1 - p(s))^{-s} - 1\right)$. Clearly, an analogous argument shows that (7) remains valid for H'' of size not larger than $l(l-1)$ so with probability at least $1 - O(1/\log n) - O\left(\prod_{s=l}^{\infty}(1 - p(s))^{-s} - 1\right)$ (6) holds for some H_1 and H_3, both of them with not more than $l(l-1)$ vertices.

Now assume that (6) is valid and let $m > n$. Then, the probability that for some H_2' we have

(6′) $$\hat{G}(m, \overline{p}) = H_1 \bar{\oplus} G \bar{\oplus} H_2' \bar{\oplus} G \bar{\oplus} H_3$$

with the same H_1 and H_3 as in (6) is at least

$$1 - \left(\prod_{s=n/2-l^2}^{m-1} (1 - p(s))^{\min\{s-n/2+l^2, l^2, m-s\}} \right)^2 \ge 1 - \prod_{s=n/3}^{\infty} (1 - p(s))^{2s} .$$

But, provided (6) and (6′) holds, $\operatorname{Th}_k(\hat{G}(n, \overline{p})) = \operatorname{Th}_k(\hat{G}(m, \overline{p}))$. Hence, for every n and m such that $m > n$ we have

$$|\operatorname{Prob}(n, \overline{p}, \psi) - \operatorname{Prob}(m, \overline{p}, \psi)| \le \epsilon(n) ,$$

where

$$\epsilon(n) = O(1/\log n) + O\left(\prod_{s=(\lceil \log n \rceil)^2}^{\infty} (1 - p(s))^{-s} - 1 \right) + 1 - \prod_{s=n/3}^{\infty} (1 - p(s))^{2s} \to 0 .$$

Thus, sequence $\{\operatorname{Prob}(n, \overline{p}, \psi)\}_{n=1}^{\infty}$, being Cauchy, must converge. □

6. First order properties of $C(n, \overline{p})$.

Let $C(n, \overline{p})$ denote a graph with vertex set $[n]$ in which a pair of vertices v, w, are joined by an edge with probability $p(\min\{|v-w|, n-|v-w|\})$. Let L^c be the first order logic which uses only the adjacency predicate, in L_+^c one may say also $v = w + 1 \,(\operatorname{mod}(n))$, whereas the vocabulary of $L_{\le}^c$ contains the predicate $C(v_1, v_2, v_3)$ which means that starting from v_1 and moving clockwise v_2 is met before v_3, i.e. for some cyclic permutation σ of indices $v_{\sigma(1)} \le v_{\sigma(2)} \le v_{\sigma(3)}$. It turns out that differences between L^c, L_+^c and $L_{\le}^c$ are not so substantial as those between L, L_+ and $L_{\le}$.

18

THEOREM 6.

(i) *If a sequence $\overline{p}$ contains only finitely many non-zero terms then for every sentence ψ from $L^c_{\leq}$ a zero-one law holds.*

(ii) *For every infinite sequence $\overline{a}$ such that $0 < a(i) < 1$, $i = 1, 2, \ldots$, there exist a sequence $\overline{p}$ obtained from $\overline{a}$ by the addition of some number of zero terms and a first order sentence ψ from L^c such that $\liminf_{n\to\infty} \mathrm{Prob}(n, \overline{p}; \psi) = 0$ but $\limsup_{n\to\infty} \mathrm{Prob}(n, \overline{p}; \psi) = 1$.*

Proof. The first part of Theorem 6 follows from the Lemma in a similar way as in the proof of Theorem 3i. To show (ii) assume, for simplicity, that the sequence $a(i)$ decreases, define $\overline{p}$ setting

$$p(j) = \begin{cases} a(i) & \text{if } j = \lfloor 3^i/a(i) \rfloor \\ 0 & \text{otherwise}, \end{cases}$$

and let ψ be a sentence that $C(n, \overline{p})$ contains a cycle of length 3. It is not hard to see that $C(3\lfloor 3^i/a(i) \rfloor - 1, \overline{p})$ contains no cycles of length 3 whereas the number of such cycles in $C(3\lfloor 3^i/a(i) \rfloor, \overline{p})$ is binomially distributed with parameters $\lfloor 3^i/a(i) \rfloor$ and $a(i)$. ☐

Clearly, the proof of Theorem 6ii is based on the fact that, unlike in the case of $G(n, \overline{p})$, for some subgraphs H the probability that H is contained in $C(n, \overline{p})$ might be smaller than the probability that H is contained in $C(n+1, \overline{p})$. To eliminate such a pathological situation let us call a subgraph H of $C(n, \overline{p})$ with vertices $v_1, v_2, \ldots, v_k$ L^c-flat [L^c_+-flat] if there is a sequence $w_1, w_2, \ldots, w_k$ of vertices of $G(n, \overline{p})$ such that for every $i, j \in [k]$ the probability $p(|w_j - w_i|)$ is positive whenever $\{v_i, v_j\}$ is an edge of H [and, moreover, $w_j = w_i + 1$ if and only if $v_j = v_i + 1 \,(\mathrm{mod}(n))$]. Furthermore, such a subgraph H is $L^c_{\leq}$-flat if there exists i, $1 \leq i \leq k$, such that for all i', i'', $i' \neq i''$, $1 \leq i', i'' \leq k$, such that $C(v_i, v_{i'}, v_{i''})$ and $|v_{i'} - v_i| + |v_i - v_{i''}| \leq n/2$, the pair $\{v_{i'}, v_{i''}\}$ is not an edge of H. Finally call a sequence $\overline{p}$ flat [asymptotically flat] with respect to language $L_\bullet$, $L_\bullet = L^c, L^c_+, L^c_{\leq}$, if the probability that $C(n, \overline{p})$ contains a subgraph which is not $L_\bullet$-flat is 0 [tends to 0].

THEOREM 7. *If a sequence $\overline{p}$ fulfilling the assertion of Theorem 1i is asymptotically flat with respect to $L_\bullet$, where $L_\bullet = L^c, L^c_+$ or $L^c_{\leq}$, then for every ψ from $L_\bullet$ a zero-one law holds.*

Proof. Let ψ be a sentence of quantifier depth k. From Fact 4, there exists a $L_\bullet$-flat graph G such that for every $L_\bullet$-flat H we have $\mathrm{Th}_k(G) = \mathrm{Th}_k(G \bar{\oplus} H)$, where both graphs G and $G \bar{\oplus} H$ are treated as $L_\bullet$-models. Now it is enough to observe that from the Lemma the probability that $C(n, \overline{p})$ contains an exact copy of G tends to 1 as $n \to \infty$, provided the assertion of Theorem 1 holds. ☐

COROLLARY. *If for each k there exists m such that $p(im) > 0$ for $i = 1, 2, \ldots, k$, and if the sequence $\overline{p}$ fulfills condition (2) then for every ψ from L^c_+ a zero-one law holds.*

Proof. It is enough to note that each sequence $\overline{p}$ for which the assumption of the Corollary remains valid is L^c_+-flat. ☐

7. Final remarks and comments.

One may ask whether additional restrictions imposed on the sequence $\overline{p}$ like non-negativity of all terms or monotonicity could lead to other interesting results concerning convergence properties of $G(n, \overline{p})$. However, all sequences $\overline{p}$ which appeared in all our counterexamples (with the single exception of Theorem 6ii, where non-negativity plays an important role) could be modified in such a way that they become both non-negative and monotonically decreasing, so no new sufficient conditions for convergence can be shown under these new assumptions.

In the paper we have studied properties of a random graph $G(n, \overline{p})$ which is a generalization of $G(n, p)$ when the probability p does not depend on n. The problem of characterizing convergence properties in the case when $\overline{p}$ varies with n seems to be a much more challenging problem – we recall only that if $p(n) \to 0$ then convergence properties of $G(n, p)$ become quite involved and strongly depend on the limit behaviour of function $p = p(n)$ (for details see papers of Shelah and Spencer [SS 88] and Łuczak and Spencer [ŁS 91]).

It is not hard to observe that if we are interested only in properties described by sentences of quantifier depth bounded by k then Theorems 1 and 2 remain valid also when zeros are replaced by very small constants $\epsilon(k)$ i.e. (2) could be replaced by

$$(2') \qquad -\log\Big(\prod_{i=1}^{n}(1 - p(i))\Big)/(\log n) < \epsilon.$$

On the other hand, even if $\overline{p}$ is such that the probability $\mathrm{Prob}(n, \overline{p}; \psi)$ converges for every sentence ψ from L the rate of this convergence may be very slow for sentences of large quantifier depth.

THEOREM 8. *There exists a sequence* $\{\psi_k\}_{k=1}^{\infty}$ *of first order sentences from* L, ψ_k *of depth* k *for every* $k = 1, 2, \ldots$, *such that for every sequence* $\overline{p}$, $0 < p(i) < 1$, *for which* (2) *holds we have*

$$\lim_{n \to \infty} \mathrm{Prob}(n, \overline{p}; \psi_k) = 1 \qquad \text{for every} \quad k = 1, 2, \ldots$$

but the function $m(k) = \min\{i : \mathrm{Prob}(i, \overline{p}; \psi_k) > 0\}$ *grows faster than any recursive function of* k.

Proof. It is well known (see, for instance, [Tr 50]) that there exists a sequence $\{\phi_k\}_{k=2}^{\infty}$ such that for $k = 2, 3, \ldots$, ϕ_k is a first order sentence of depth k from L and a function $m'(k) = \min\{\mathrm{card}(M) : M$ is a model of $\phi_{k-1}\}$ grows faster than any recursive function of k. Let ψ_k be the sentence which states that a graph contains a vertex v such that for a subgraph induced by all neighbours of v ϕ_{k-1} holds. Clearly, ψ_k has depth k. Now let $\overline{p}$ be a sequence such that $0 < p(i) < 1$ for all $i = 1, 2, \ldots$ for which (2) holds. For such a sequence all graphs are admissible, so the Lemma implies that $\mathrm{Prob}(n, \overline{p}; \psi_k)$ tends to 1

as $n \to \infty$. On the other hand,

$$m(k) = \min\{i : \text{Prob}(i, \overline{p}; \psi_k) > 0\} \geq m'(k) \,. \quad \square$$

Thus, the behaviour of $\text{Prob}(n, \overline{p}; \psi)$ for small n does not tell us very much about the asymptotic behaviour of $\text{Prob}(n, \overline{p}; \psi)$. It is not hard to show that even if $\overline{p}$ is such that for every first order sentence a zero-one law holds, there is no procedure which, applied to ψ, could decide whether $\lim_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 0$ or $\lim_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 1$.

THEOREM 9. *Let $\overline{p}$ be a sequence such that $0 < p(i) < 1$ for all i, for which (2) holds. Then there exists no procedure which can decide for each first order sentence ψ from L, whether $\lim_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 0$ or $\lim_{n\to\infty} \text{Prob}(n, \overline{p}; \psi) = 1$.*

Proof. Let ϕ be a first order sentence from L and ψ_ϕ denote the sentence that for some vertex v in a graph the subgraph which is induced in a graph by all neighbours of v has property ϕ. Since $p(i) > 0$ for all i, every graph is admissible for $G(n, \overline{p})$ and the Lemma implies that, with probability tending to 1 as $n \to \infty$, every finite graph appears in $G(n, \overline{p})$ as a component. Thus, $\lim_{n\to\infty} \text{Prob}(n, \overline{p}; \psi_\phi) = 1$ if and only if ϕ is satisfied for some finite graph. Now the assertion follows from the fact that, due to the Traktenbrot-Vought Theorem [Tr 50], there is no decision procedure to determine whether a first order sentence ϕ from L has a finite model. $\square$

REFERENCES

[CHS 87]   K. J. Compton, C. W. Henson and S. Shelah, *Nonconvergence, undecidability, and intractability in asymptotics problems*, Annals Pure Appl. Logic, **36** (1987), pp. 207–224.

[Fa 76]   R. Fagin, *Probabilities on finite models*, J. Symbolic Logic, **41** (1976), pp. 50–58.

[Ga 82]   H. Gaifman, *On local and non-local properties*, Proc. Herbrand Symp. Logic Colloquium '81, ed. J. Stern, Studies in Logic Found. Math. vol.107, North-Holland, 1982, pp. 105–135.

[Gu 85]   Y. Gurevich, *Monadic second-order theories*, Chapter XIII of "Model-Theoretic Logics", ed. J. Barwise and S. Feferman, Springer-Verlag, 1985, pp. 479–506.

[GKLT 69]   Y. V. Glebskii, D. I. Kogan, M. I. Liogonkii and Talanov, *Range and degree of realizability of formulas in the restricted predicate calculus*, Cybernetics, **5** (1969), pp. 142–154.

[GKM 84]   G. R. Grimmett, M Keane and J. M. Marstrand, *On the connectedness of a random graph*, Math. Proc. Camb. Phil. Soc., **96** (1984), pp. 151–166.

[KS 85]   M. Kaufmann and S. Shelah, *On random models of finite power and monadic logic*, Discrete Math., **54** (1985), pp. 285–293.

[Ly 80]   J. F. Lynch, *Almost sure theories*, Ann. Math. Logic, **18** (1980), pp. 91–135.

[ŁS 91]   T. Łuczak and J. Spencer, *When does the zero-one law holds*, J. Amer. Math. Soc., **4** (1991), pp. 451–468.

[SS 88]   S. Shelah and J. Spencer, *Zero-one laws for sparse random graphs*, J. Amer. Math. Soc., **1** (1988), pp. 97–115.

[SS 94]   S. Shelah and J. Spencer, *Random sparse unary predicates*, Random Structures & Algorithms, **5** (1994), pp. 375–394.

[Tr 50]   B. A. Traktenbrot, *Impossibility of an algorithm for the decision problem on finite classes*, Doklady Akademii Nauk S.S.R., **70** (1950), pp. 569–572.