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Analysis of Multiple Antenna Systems With Finite-Rate Channel Information Feedback Over Spatially Correlated Fading Channels

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Abstract—This paper employs a high resolution quantization framework to study the effects of finite-rate quantization of the channel state information (CSI) on the performance of MISO systems over correlated fading channels. The contributions of this paper are twofold. First, as an application of the general distortion analysis, tight lower bounds on the capacity loss of correlated MISO systems due to the finite-rate channel quantization are provided. Closed-form expressions for the capacity loss in high-signal-to-noise ratio (SNR) and low-SNR regimes are also provided, and their analysis reveals that the capacity loss of correlated MISO channels is related to that of i.i.d. fading channels by a simple multiplicative factor which is given by the ratio of the geometric mean to the arithmetic mean of the eigenvalues of the channel covariance matrix. Second, this paper extends the general asymptotic distortion analysis to the important practical problem of suboptimal quantizers resulting from mismatches in the distortion functions, source statistics, and quantization criteria. As a specific application, two types of mismatched MISO CSI quantizers are investigated: quantizers whose codebooks are designed with minimum mean square error (MMSE) criterion but the distortion measure is the ergodic capacity loss (i.e., mismatched design criterion), and quantizers with codebook designed with a mismatched channel covariance matrix (i.e., mismatched statistics). Bounds on the channel capacity loss of the mismatched codebooks are provided and compared to that of the optimal quantizers. Finally, numerical and simulation results are presented and they confirm the tightness of theoretical distortion bounds.

Index Terms—Bennett's integral, capacity analysis, channel quantization, constrained source, CSI feedback, distortion analysis, encoder side information, finite-rate feedback, high-resolution quantization theory, imperfect CSIT, mismatched channel quantizer, multiple-input multiple-output (MIMO), spatially correlated fading, suboptimal channel quantizer, transmit precoding, vector quantization.

I. INTRODUCTION

THIS paper considers multiple antenna systems with partial channel state information (CSI) available at the transmitter, which is conveyed by the receiver through a finite-rate feedback link. Recently, several interesting papers have appeared proposing design algorithms as well as analytically quantifying the performance of finite-rate feedback multiple antenna systems [1]–[22]. The analysis is quite involved and several approaches have been developed for this purpose in these papers.

Mukkavilli *et al.* [1] approximated the channel quantization region corresponding to each code point based on the channel geometric property. They derived a universal lower bound on the outage probability of quantized MISO beamforming systems with arbitrary number of transmit antennas t over i.i.d. Rayleigh fading channels. Based on the geometric interpretation of the beamforming vectors in the codebook, the authors also proposed a codebook design criterion which is based on minimizing the maximum inner product between any two distinct beamforming vectors in the codebook. The approach taken by Love and Heath in [2] and [3] is based on relating the problem to that of Grassmannian line packing [4]. They derived the same min-max criterion in a i.i.d. Rayleigh fading multiple-input multiple-output (MIMO) channel setting, and proposed a random computer search algorithm to generate the codebook that optimizes the proposed criterion. Results on the density of Grassmannian line packings were derived and used to develop bounds on the codebook size given a capacity or signal-to-noise ratio (SNR) loss. The authors also investigated in [5] the problem of quantizing the beamforming vector under a per-antenna power constraint, which is referred to as quantized equal gain transmission. The problem of quantized equal gain transmission was recently revisited by Murthy *et al.* wherein a vector quantization (VQ) approach was suggested for codebook design [6] and a closed-form capacity loss analysis was conducted.

Another approach is based on approximating the statistical distribution of the key random variable that characterizes the system performance. This approach was used by Xia *et al.* in [7], [8], Zhou *et al.* in [9], and Roh *et al.* in [10] and [11], where the authors first derived an (weighted) inner product criterion and used the Lloyd algorithm [12] to generate the codebook. These works analyzed the performance of MISO systems with limited rate-feedback in the case of i.i.d. Rayleigh fading channels, and obtained closed-form expressions of the capacity loss (or SNR loss) in terms of feedback rate B and the number of antennas t. In [13] and [14], Roh *et al.* extended the results from MISO channels to the case of MIMO systems with quantized feedback. Other interesting work in the CSI-feedback context can be found in [8] and [16], where the authors extended the

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beamforming codebook design algorithms to correlated MISO (and MIMO) fading channels by introducing a rotation-based transformation on the i.i.d. codebooks of the beamforming vectors according to the channel statistical information. Another analytical approach adopted by Narula *et al.* in [15] is based on relating the quantization problem to rate distortion theory. They derived an approximation to the expected loss of the received SNR due to finite-rate quantization of the beamforming vectors in an MISO system with a large number of antennas t. More recent results on this problem can be found in [16]–[22].

Despite recent progress, the analysis of finite-rate feedback systems has proven to be difficult. All the works aforementioned are case specific, limited to i.i.d. channels, mainly MISO channels, and are difficult to extend to more general scenarios. Recently, in our work [23], [24], a general framework for the analysis of quantized feedback multiple antenna systems was developed using a source coding perspective thereby leveraging of the vast body of source coding theory, particularly high resolution quantization theory. Specifically, the channel quantization was formulated as a general finite-rate vector quantization problem with attributes tailored to meet the general issues that arise in feedback based communication systems, including encoder side information, source vectors with constrained parameterizations, and general non-mean-squared distortion functions. Asymptotic (high quantization rate) distortion analysis of the proposed general quantization problem was provided by extending Bennett's classic analysis [25] as well as its corresponding vector extensions [26], [27]. By using the proposed general framework, performance analysis of a finite-rate feedback MISO beamforming system over i.i.d. Rayleigh flat fading channels was also provided. We build upon these results in this paper.

The contributions of this paper are twofold. First, as an extended application of the general distortion analysis provided in [24], this paper investigates the effects of finite-rate CSI quantization on MISO systems over correlated fading channels.[1] More specifically, tight lower bounds of the average asymptotic distortion, which is defined as the system capacity loss due to the finite-rate channel quantization, are provided. Closed-form analysis of the capacity loss in high-SNR and low-SNR regimes are also provided revealing the interesting fact that the capacity loss of correlated MISO channels is related to that of the i.i.d. channels by a simple multiplicative factor. This factor is the ratio of the geometric mean to the arithmetic mean of the eigenvalues of the channel covariance matrix. Second, capitalizing on the generality of the framework developed in [24], this paper extends the asymptotic analysis to the important problem of suboptimal quantizers with mismatched distortion functions, source statistics, and quantization criteria. Bounds on the average distortion of these different mismatched quantizers are provided. As a specific application of the developed analytical results, two types of mismatched MISO CSI quantizers are investigated. These include quantizers that are designed with minimum mean square error (MMSE) criterion but the desired measure is ergodic capacity loss (i.e., mismatched design criterion), and quantizers whose codebooks are designed with

a mismatched channel covariance matrix (i.e., mismatched statistics). Bounds on the system capacity loss of MISO feedback systems with these two types of mismatched CSI quantizers are provided and compared to that of the optimal quantizers. Numerical and simulation results are presented which confirm the tightness of the theoretical asymptotic distortion bounds.

II. BACKGROUND INFORMATION OF THE GENERALIZED VECTOR QUANTIZER

It was shown in [24] that the problem of analyzing various CSI-feedback-based multiple antenna systems can be characterized by a general vector quantization framework. Moreover, high resolution distortion analysis used in classical vector quantizations were extended to deal with this generalized problem. In this paper, we take a similar approach as [24] by first extending the proposed distortion analysis and further utilizing it to investigate a MISO system with finite-rate CSI feedback. For the sake of background information, we briefly summarize in this section some important results of the general distortion analysis. For interested readers, please refer to [24], [30] for more details.

A. Problem Formulation

It is assumed that the source variable $\mathbf{x}$ is a two-vector tuple denoted as $(\mathbf{y}, \mathbf{z})$, where vector $\mathbf{y} \in \mathbb{Q}$ represents the actual variable to be quantized (*quantization objective*) of dimension k_{q}, and $\mathbf{z} \in \mathbb{Z}$ is the additional side information of dimension k_{z}. The *side information* $\mathbf{z}$ is available at the encoder (receiver) but not at the decoder (transmitter). Quantization objective $\mathbf{y}$ and side information $\mathbf{z}$ have joint probability density function given by $p(\mathbf{y}, \mathbf{z})$ and a fixed-rate (B bits per channel update) quantizer with $N = 2^B$ quantization levels is considered. Based on a particular source realization $\mathbf{x}$, the encoder (or the quantizer) represents vector $\mathbf{y}$ by one of the N vectors $\widehat{\mathbf{y}}_1, \widehat{\mathbf{y}}_2, \cdots, \widehat{\mathbf{y}}_N$, which form the codebook. The encoding or the quantization process is denoted as $\widehat{\mathbf{y}} = \mathcal{Q}(\mathbf{y}, \mathbf{z})$. The distortion of a finite-rate quantizer is defined as

$$D = E_{\mathbf{x}}[D_{\mathrm{Q}}(\mathbf{y}, \widehat{\mathbf{y}}; \mathbf{z})] \qquad (1)$$

where $D_{\mathrm{Q}}(\mathbf{y}, \widehat{\mathbf{y}}; \mathbf{z})$ is a general *non-mean-squared distortion* function between $\mathbf{y}$ and $\widehat{\mathbf{y}}$ that is parameterized by $\mathbf{z}$. It is further assumed that function D_{Q} has a continuous second order derivative[2] (or Hessian matrix with respect to (w.r.t.) $\mathbf{y}$) $\mathbf{W_z}(\widehat{\mathbf{y}})$ with the (i, j)th element given by

$$w_{i,j} = \frac{1}{2} \cdot \frac{\partial^2}{\partial y_i \partial y_j} D_{\mathrm{Q}}(\mathbf{y}, \widehat{\mathbf{y}}; \mathbf{z}) \Big|_{\mathbf{y}=\widehat{\mathbf{y}}}. \qquad (2)$$

B. Asymptotic Distortion Integral

Under high resolution assumptions (large N), the asymptotic distortion of a finite-rate feedback system has been shown to

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[1] The material in this paper was presented in part at the following conferences [28], [29].

[2] By viewing $D_{\mathrm{Q}}(\mathbf{y}, \widehat{\mathbf{y}}; \mathbf{z})$ as scaler function of vector $\mathbf{y}$, and simply treating $\widehat{\mathbf{y}}$ and $\mathbf{z}$ as constants, matrix $\mathbf{W_z}(\widehat{\mathbf{y}})$ is its

have the following form [24]:

$$
\begin{aligned}
D &= E[D_Q(\mathbf{y}, Q(\mathbf{y}, \mathbf{z}); \mathbf{z})] \\
&= 2^{-\frac{2B}{k_q}} \int_{\mathbb{Z}} \int_{\mathbb{Q}} m(\mathbf{y}; \mathbf{z}; \mathbb{E}_{\mathbf{z}}(\mathbf{y})) p(\mathbf{y}, \mathbf{z}) \lambda(\mathbf{y})^{-\frac{2}{k_q}} d\mathbf{y} \, d\mathbf{z}
\end{aligned}
\tag{3}
$$

where $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$ denotes the asymptotic projected Voronoi cell that contains $\mathbf{y}$ with side information $\mathbf{z}$ and captures the shape attribute of the quantization cell in the asymptotic sense ($N \to \infty$). In (3), $\lambda(\mathbf{y})$ is a function representing the relative density of the codepoints, also referred to as the point density, such that $\lambda(\mathbf{y}) \, d\mathbf{y}$ is approximately the fraction of quantization points in a small neighborhood of $\mathbf{y}$. Function $m(\mathbf{y}; \mathbf{z}; \mathbb{E})$ is the normalized inertial profile that represents the asymptotic normalized distortion or the relative distortion of the quantizer $\mathcal{Q}$ at position $\mathbf{y}$ conditioned on side information $\mathbf{z}$ with Voronoi shape $\mathbb{E}$. It is defined as

$$
\begin{aligned}
m(\mathbf{y}; \mathbf{z}; \mathbb{E}) &\triangleq \left(\int_{\mathbf{y}' \in \mathbb{E}} d\mathbf{y}' \right)^{-\frac{2+k_q}{k_q}} \\
&\quad \cdot \left(\int_{\mathbf{y}' \in \mathbb{E}} (\mathbf{y}' - \mathbf{y})^{\top} \cdot \mathbf{W}_{\mathbf{z}}(\mathbf{y}) \cdot (\mathbf{y}' - \mathbf{y}) \, d\mathbf{y}' \right).
\end{aligned}
\tag{4}
$$

The point density function $\lambda(\mathbf{y})$ and the normalized inertial profile $m(\mathbf{y}; \mathbf{z}; \mathbb{E})$ are the key functions that describe the behavior of a specific quantizer. Hence, given a vector quantizer, the problem reduces to finding these two functions and the average system distortion can be obtained by substituting them into the distortion integral given by (3), [24]. Note that the integral given by (3) is similar to Bennett's integral provided in [25] and its vector extension provided in [26]. In fact, it can be viewed as a further extension of Bennett's results to a generalized fixed-rate vector quantization problem with encoder side information and general distortion metric function. It describes the dependency between the average system distortion and the properties of the source variable (through $p(\mathbf{y}, \mathbf{z})$ and $D_Q(\mathbf{y}, \hat{\mathbf{y}}; \mathbf{z})$) as well as the characteristics of the vector quantizer used (through $\lambda(\mathbf{y})$ and $m(\mathbf{y}; \mathbf{z}; \mathbb{E})$).

C. Characterizing the Normalized Inertial Profile of an Optimal Quantizer

The normalized inertial profile of an optimal quantizer is defined as the minimum inertia of all admissible regions $\mathbb{E}_{\mathbf{z}}(\mathbf{y})$, i.e.,

$$
m_{\mathrm{opt}}(\mathbf{y}; \mathbf{z}) \triangleq \min_{\mathbb{E}_{\mathbf{z}}(\mathbf{y}) \in \mathcal{H}_Q} m(\mathbf{y}; \mathbf{z}; \mathbb{E}_{\mathbf{z}}(\mathbf{y}))
\tag{5}
$$

where $\mathcal{H}_Q$ represents the set of all admissible tessellating polytopes that can tile the quantization space. It is known that finding the optimal Voronoi region as well as characterizing the exact optimal inertial profile is hard. However, the inertial profile of

any Voronoi shape, including the optimal inertial profile, can be tightly lower bounded by that of an "M-shaped" hyper-ellipsoid, i.e.,

$$
\begin{aligned}
m(\mathbf{y}; \mathbf{z}; \mathbb{E}) &\geq m_{\mathrm{opt}}(\mathbf{y}; \mathbf{z}) \gtrapprox \tilde{m}_{\mathrm{opt}}(\mathbf{y}; \mathbf{z}) \\
&= m(\mathbf{y}; \mathbf{z}; \mathcal{T}(\mathbf{y}, \mathbf{W}_{\mathbf{z}}(\mathbf{y}), v)) \\
&= \frac{k_q}{k_q + 2} \cdot \left(\frac{|\mathbf{W}_{\mathbf{z}}(\mathbf{y})|}{\kappa_{k_q}^2} \right)^{\frac{1}{k_q}}, \quad \kappa_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.
\end{aligned}
\tag{6}
$$

where $|\cdot|$ represents matrix determinant, and $\mathcal{T}(\mathbf{y}, \mathbf{M}, v)$ is the hyper-ellipsoidal set centered at $\mathbf{y}$ with volume v, defined as

$$
\begin{aligned}
&\mathcal{T}(\mathbf{y}, \mathbf{M}, v) \\
&= \left\{ \mathbf{x} \,\middle|\, \left(\frac{\kappa_{k_q}^2}{v^2 |\mathbf{M}|} \right)^{\frac{1}{k_q}} (\mathbf{x} - \mathbf{y})^{\top} \mathbf{M} (\mathbf{x} - \mathbf{y}) \leq 1 \right\}.
\end{aligned}
\tag{7}
$$

D. Asymptotic Distortion Bounds

Under high resolution assumptions (B or N large), the asymptotic distortion of the generalized finite-rate quantization system can be lower bounded by the following two bounds:

$$
D_{\mathrm{Opt}} \geq D_{\mathrm{Low},1} \geq D_{\mathrm{Low},2}
\tag{8}
$$

where D_{Opt} represents the distortion of an optimal quantizer, and the lower bound $D_{\mathrm{Low},1}$ is given by

$$
D_{\mathrm{Low},1} = 2^{-\frac{2B}{k_q}} \cdot \left(\int_{\mathbb{Q}} \left(m_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_q}{2+k_q}} d\mathbf{y} \right)^{\frac{2+k_q}{k_q}}
\tag{9}
$$

where $m_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{y})$ is the average optimal inertial profile defined as

$$
m_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{y}) = \int_{\mathbb{Z}} m_{\mathrm{opt}}(\mathbf{y}; \mathbf{z}) \cdot p(\mathbf{z} | \mathbf{y}) \, d\mathbf{z}.
\tag{10}
$$

Equation (9) can be obtained from (3) by using $m_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{y})$ given above and selecting the point density optimally to minimize the asymptotic system distortion, i.e., [24]

$$
\begin{aligned}
\lambda^*(\mathbf{y}) &= \left(m_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_q}{2+k_q}} \\
&\quad \cdot \left(\int_{\mathbb{Q}} \left(m_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_q}{2+k_q}} d\mathbf{y} \right)^{-1}.
\end{aligned}
\tag{11}
$$

Distortion lower bound $D_{\mathrm{Low},2}$, though being a loosen lower bound than $D_{\mathrm{Low},1}$ but can be evaluated much easier under certain situations (such as the MISO example in Section III-C), is given by (12), shown at the bottom of the page. It represents the average system distortion of a quantizer when the encoder side information is also available at the decoder.

$$
D_{\mathrm{Low},2} = 2^{-\frac{2B}{k_q}} \left(\int_{\mathbb{Z}} p(\mathbf{z}) \left(\int_{\mathbb{Q}} \left(m_{\mathrm{opt}}(\mathbf{y}; \mathbf{z}) \cdot p(\mathbf{y} | \mathbf{z}) \right)^{\frac{k_q}{2+k_q}} d\mathbf{y} \right)^{\frac{2+k_q}{k_q}} d\mathbf{z} \right).
\tag{12}
$$

When the sensitivity matrix $\mathbf{W}_{\mathbf{z}}(\mathbf{y})$ can be factored into the following form:

$$\mathbf{W}_{\mathbf{z}}(\mathbf{y}) = f(\mathbf{z}) \cdot \mathbf{W}(\mathbf{y}) \qquad (13)$$

the asymptotic distortion lower bound $D_{\text{Low},1}$ is actually achievable, i.e., $D_{\text{Opt}} = D_{\text{Low},1}$. If the source vector $\mathbf{y}$ and the side information $\mathbf{z}$ are further statistically independent, then both lower bounds are achievable, i.e., $D_{\text{Opt}} = D_{\text{Low},1} = D_{\text{Low},2}$. In this case, the encoder side information $\mathbf{z}$ is irrelevant to the quantization process, i.e., $\mathcal{Q}(\mathbf{y};\mathbf{z}) = \mathcal{Q}(\mathbf{y})$. By assuming $\mathbf{z}$ is also available at the decoder does not improve system performance or reduce system distortion. Therefore, the problem can be viewed as a multi-component mixed source with the component index (or the side information) $\mathbf{z}$ available at both sides. The distortion of such a system is equivalent to $D_{\text{Low},2}$ given by (12), which intuitively explains the achievability of the two lower bounds. By substituting the tight lower bound (6) of the inertial profile into (9)–(12), one can obtain corresponding tight lower bounds of the average inertial profile $\widetilde{m}_{\text{opt}}^{\text{w}}(\mathbf{y})$, point density $\widetilde{\lambda}^*(\mathbf{y})$, as well as asymptotic distortion bounds $\widetilde{D}_{\text{Low},1}$ and $\widetilde{D}_{\text{Low},2}$, respectively.[3]

E. Distortion Analysis of Constrained Source

In feedback wireless systems, the quantized variable is often constrained, e.g., unit norm beamforming vectors. Hence, it is also of interest to quantize the k_{q}-dimensional source vector $\mathbf{y} \in \mathbb{Q}$ subject to a multidimensional constraint function $\mathbf{g}(\mathbf{y}) = \mathbf{0}$ of size $k_{\text{c}} \times 1$, e.g., scalar function $g(\mathbf{y}) = (\|\mathbf{y}\|^2 - 1)$ represents the unit norm constraint. In this case, the distortion analysis discussed above has been shown to still be valid with the following modification. First, the degrees of freedom in $\mathbf{y}$ reduce from k_{q} to $k_{\text{q}}' = k_{\text{q}} - k_{\text{c}}$. Next, the sensitivity matrix is replaced by its constrained version $\mathbf{W}_{\text{c},\mathbf{z}}(\mathbf{y})$ given by

$$\mathbf{W}_{\text{c},\mathbf{z}}(\mathbf{y}) = \mathbf{V}_2^{\mathsf{T}} \cdot \mathbf{W}_{\mathbf{z}}(\mathbf{y}) \cdot \mathbf{V}_2 \qquad (14)$$

where $\mathbf{V}_2 \in \mathbb{R}^{k_{\text{q}} \times k_{\text{q}}'}$ is an orthonormal matrix with its columns constituting an orthonormal basis for the null space

$$\mathcal{N}\left(\frac{\partial}{\partial \mathbf{y}}\,\mathbf{g}(\mathbf{y})\right).$$

Last, the multidimensional integrations used in evaluating the average distortions are over the constrained space $\mathbf{g}(\mathbf{y}) = \mathbf{0}$.

III. CSI-Quantized Beamforming in Correlated MISO Systems

Due to the complexity of the analysis, past work has mainly dealt with CSI-Quantized beamforming in i.i.d. Rayleigh fading channels. By utilizing the high-rate distortion analysis described in Section II, we investigate the capacity loss of a finite-rate CSI-Quantized MISO beamforming system over correlated fading

channels. The results provide interesting insights and demonstrate the general nature and utility of the high resolution framework.

A. System Model of MISO Systems With Finite-Rate Feedback

This section considers a MISO system, with t transmit antennas and one receive antenna, signaling through a frequency flat block fading channel. For the sake of simplicity, the time index is omitted, and hence the channel model can be represented as the following form:

$$y = \mathbf{h}^{\mathsf{H}} \cdot \mathbf{x} + n \qquad (15)$$

where y is the received signal (scalar), n is the additive complex Gaussian noise with zero mean and unit variance, and $\mathbf{h}^{\mathsf{H}} \in \mathbb{C}^{1 \times t}$ is the correlated[4] MISO channel response with distribution given by $\mathbf{h} \sim \mathcal{N}_{\text{c}}(\mathbf{0}, \boldsymbol{\Sigma}_{\text{h}})$. The transmitted signal vector $\mathbf{x}$ is normalized to have a power constraint given by $E[\|\mathbf{x}\|^2] = \rho$, with ρ representing the average receiver SNR.

In this paper, the channel state information $\mathbf{h}$ is assumed to be perfectly known at the receiver but only partially available at the transmitter through a finite-rate feedback link of B bits per channel update between the transmitter and receiver. More specifically, a quantization codebook $\mathcal{C} = \{\widehat{\mathbf{v}}_1, \cdots, \widehat{\mathbf{v}}_N\}$, which is composed of unit-norm transmit beamforming vectors, is assumed known to both the receiver and the transmitter. Based on the channel realization $\mathbf{h}$, the receiver selects the best code point $\widehat{\mathbf{v}}$ from the codebook and sends the corresponding index back to the transmitter. At the transmitter, the unit-norm vector $\widehat{\mathbf{v}}$ is employed as the beamforming vector, i.e.,

$$y = \langle \mathbf{h}, \widehat{\mathbf{v}} \rangle \cdot s + n = \|\mathbf{h}\| \cdot \langle \mathbf{v}, \widehat{\mathbf{v}} \rangle \cdot s + n, \quad E[|s|^2] = \rho \quad (16)$$

where $\mathbf{v}$ is the channel directional vector given by $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$. The corresponding ergodic capacity or the maximum system mutual information rate of the quantized MISO beamforming system is given by

$$C_{\text{Q}} = E[\log_2(1 + \rho \cdot \|\mathbf{h}\|^2 \cdot |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2)]. \qquad (17)$$

With perfect channel state information available at the transmitter, which corresponds to the case of infinite rate feedback $B = \infty$, it is optimal to choose $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$ as the transmit beamforming vector, and the corresponding system ergodic capacity[5] is given by

$$C_{\text{P}} = E\left[\log_2(1 + \rho \cdot \|\mathbf{h}\|^2)\right]. \qquad (18)$$

Therefore, the performance of a CSI-feedback-based MISO system can be characterized by the capacity loss C_{Loss} due to

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[3]This replacement can be extended to other variables and definitions. In the rest of this paper, we will directly use $\widetilde{a}$ to represent a quantity that is obtained by replacing m_{opt} with $\widetilde{m}_{\text{opt}}$ when a is a function of m_{opt}, i.e., $a = a(m_{\text{opt}})$

[4]For the sake of fair comparisons, we normalize the channel covariance matrix such that the mean of the eigenvalues equals to one (equal to the i.i.d. channel case $\boldsymbol{\Sigma}_{\text{h}} = I_t$).

[5]It is shown in [31] that insignificant capacity gain can be achieved by utilizing a temporal water-filling power allocation. Hence, capacities C_{Q} and C_{P} given by (17) and (18) are obtained by assuming an equal power allocation over the time domain.

the finite-rate quantization of the transmit beamforming vectors, which is defined as the expectation of the instantaneous mutual information rate loss $C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}})$, i.e.,

$$
C_{\mathrm{Loss}} = C_{\mathrm{P}} - C_{\mathrm{Q}} = E[C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}})],
$$
$$
C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}}) = -\log_2\left(1 - \frac{\rho \cdot \|\mathbf{h}\|^2}{1 + \rho \cdot \|\mathbf{h}\|^2} \cdot (1 - |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2)\right).
$$
$$(19)$$

This performance metric was also used in [14] and [24].

B. Reformulation of the Quantized Feedback-Based MISO Beamforming System

MISO beamforming systems with CSI feedback can be reformulated as a general vector quantization problem by adopting a direct mapping between CSI and source variables, given by $(\mathbf{v}, \alpha) \rightarrow (\mathbf{y}, \mathbf{z})$. To be specific, the source variable to be quantized is denoted as $\overline{\mathbf{v}} = [\mathbf{v}_{\mathrm{R}}^{\mathsf{T}}, \mathbf{v}_{\mathrm{I}}^{\mathsf{T}}]^{\mathsf{T}}$ of $2t$ real dimensions with $\mathbf{v}_{\mathrm{R}}$ and $\mathbf{v}_{\mathrm{I}}$ representing the real and imaginary parts of the complex channel directional vector $\mathbf{v}$. The encoder side information is denoted as $\alpha = \|\mathbf{h}\|^2$, which is of dimension $k_\alpha = 1$ and represents the power of the vector channel. By definition, source variable $\overline{\mathbf{v}}$ is a unit norm vector, i.e., $\|\overline{\mathbf{v}}\|^2 = 1$. Moreover, consider a subset $[\mathbf{v}_1] \triangleq \{\mathbf{v} \mid \mathbf{v} = e^\phi \mathbf{v}_1, \; \forall \; \phi \in [0, 2\pi]\}$, it is evident that each element has the same inner product with (any) $\widehat{\mathbf{v}}$, only up to a phase rotation ambiguity, i.e., $\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle = e^\phi \cdot \langle \mathbf{v}_1, \widehat{\mathbf{v}} \rangle$. This means that all of the vectors in $[\mathbf{v}_1]$ will be quantized into the same $\widehat{\mathbf{v}}$ and yield the same instantaneous capacity (or capacity loss). Therefore, we only have to consider one of the vectors in this subset when analyzing the average system distortion and picking whichever one makes no difference. In order to simplify the mathematical derivation, we choose the one that satisfies $\angle \langle \mathbf{v}, \widehat{\mathbf{v}} \rangle = 0$ for any vectors in the Voronoi cell belonging to code point $\widehat{\mathbf{v}}$. Written in a concise format, the constraint function is given by

$$
\mathbf{g}(\mathbf{v}) = \begin{bmatrix} \mathbf{v}_{\mathrm{R}}^{\mathsf{T}} \mathbf{v}_{\mathrm{R}} + \mathbf{v}_{\mathrm{I}}^{\mathsf{T}} \mathbf{v}_{\mathrm{I}} - 1 \\ \mathbf{v}_{\mathrm{R}}^{\mathsf{T}} \widehat{\mathbf{v}}_{\mathrm{I}} - \mathbf{v}_{\mathrm{I}}^{\mathsf{T}} \widehat{\mathbf{v}}_{\mathrm{R}} \end{bmatrix} = 0 \qquad (20)
$$

where the first element represents the norm constraint, and the second one represents the phase constraint. Function $\mathbf{g}(\mathbf{v})$ has size $k_{\mathrm{c}} = 2$, which leads to the actual degrees of freedom of the quantization variable $\mathbf{v}$ to be $k_{\mathrm{q}}' = 2t - 2$. The instantaneous capacity loss due to the effects of finite-rate CSI quantization is taken to be the system distortion function $D_{\mathrm{Q}}(\mathbf{v}, \widehat{\mathbf{v}}; \alpha)$ given by the following form [from (19)]:

$$
D_{\mathrm{Q}}(\mathbf{v}, \widehat{\mathbf{v}}; \alpha) = C_{\mathrm{L}}(\mathbf{h}, \widehat{\mathbf{v}})
$$
$$
\triangleq -\log_2\left(1 - \frac{\rho\alpha}{1 + \rho\alpha} \cdot (1 - |\langle \mathbf{v}, \widehat{\mathbf{v}} \rangle|^2)\right) \qquad (21)
$$

where α is the instantaneous channel power given by $\alpha = \|\mathbf{h}\|^2$.

C. High-Resolution Distortion Analysis of CSI-Quantized MISO System

Due to space limitations and to avoid overlap with our previous work, the derivations have been condensed by skipping some manipulations used in obtaining the final

expressions. Please refer to [24] and [30] for more details. For correlated MISO fading channels $\mathbf{h} \sim \mathcal{N}_c(\mathbf{0}, \boldsymbol{\Sigma}_{\mathrm{h}})$ with channel correlation matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$ having distinct eigenvalues,[6] i.e., $\lambda_{\mathrm{h},1} > \cdots > \lambda_{\mathrm{h},t} > 0$, the marginal probability density functions of α and $\mathbf{v}$ can be shown to have the following form [30], [32]:

$$
p_{\mathbf{v}}(\mathbf{x}) = \gamma_t^{-1} \cdot |\boldsymbol{\Sigma}_{\mathrm{h}}|^{-1} \cdot \left(\mathbf{x}^{\mathsf{H}} \boldsymbol{\Sigma}_{\mathrm{h}}^{-1} \mathbf{x}\right)^{-t}, \qquad (22)
$$

$$
p_\alpha(x) = \sum_{i=1}^t \prod_{j \neq i} \left(1 - \frac{\lambda_{\mathrm{h},j}}{\lambda_{\mathrm{h},i}}\right)^{-1} \cdot \frac{1}{\lambda_{\mathrm{h},i}} \exp\left(-\frac{x}{\lambda_{\mathrm{h},i}}\right). \qquad (23)
$$

The corresponding conditional distributions are given by

$$
p_{\mathbf{v}|\alpha}(\mathbf{x}) = \left(\sum_{i=1}^t \prod_{j \neq i} \left(1 - \frac{\lambda_{\mathrm{h},j}}{\lambda_{\mathrm{h},i}}\right)^{-1} \cdot \frac{1}{\lambda_{\mathrm{h},i}} \exp\left(-\frac{\alpha}{\lambda_{\mathrm{h},i}}\right)\right)^{-1}
$$
$$
\times \frac{\alpha^{t-1} \cdot \exp\left(-\alpha \cdot \mathbf{x}^{\mathsf{H}} \boldsymbol{\Sigma}_{\mathrm{h}}^{-1} \mathbf{x}\right)}{\pi^{t-1} \cdot |\boldsymbol{\Sigma}_{\mathrm{h}}|} \qquad (24)
$$

$$
p_{\alpha|\mathbf{v}}(x) = \frac{x^{t-1} \cdot \left(\mathbf{v}^{\mathsf{H}} \boldsymbol{\Sigma}_{\mathrm{h}}^{-1} \mathbf{v}\right)^t \cdot \exp\left(-x \cdot \mathbf{v}^{\mathsf{H}} \boldsymbol{\Sigma}_{\mathrm{h}}^{-1} \mathbf{v}\right)}{(t-1)!} \qquad (25)
$$

The constrained sensitivity matrix, defined in (14), of the finite-rate quantized MISO beamforming system can be shown to be given by

$$
\mathbf{W}_{\mathrm{c},\alpha}(\widehat{\mathbf{v}}) = \frac{\rho\alpha}{\ln 2 \cdot (1 + \rho\alpha)} \cdot I_{2t-2}. \qquad (26)
$$

By substituting (26) into the hyper-ellipsoidal approximation given by (6), the optimal inertial profile is tightly lower bounded (or approximated) by the following form:

$$
\widetilde{m}_{\mathrm{c,opt}}(\widehat{\mathbf{v}}; \alpha) = \frac{(t-1) \cdot \gamma_t^{-\frac{1}{t-1}} \cdot \rho\alpha}{\ln 2 \cdot t \cdot (1 + \rho\alpha)}
$$
$$
\gamma_t = \frac{\pi^{t-1}}{(t-1)!}. \qquad (27)
$$

Having obtained the inertial profile (27), one can then derive the following two distortion lower bounds.

- *Distortion lower bound $D_{\mathrm{c\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}})$*

By substituting the conditional PDF $p_{\alpha|\mathbf{v}}(x)$ given by (25), the marginal PDF $p_{\mathbf{v}}(\mathbf{x})$ given by (22) and the inertial profile $\widetilde{m}_{\mathrm{c,opt}}(\mathbf{v}; \alpha)$ given by (27) into the distortion lower bound (9), the system asymptotic distortion lower bound $\widetilde{D}_{\mathrm{c\text{-}Low},1}$ can be expressed in the following form:

$$
\widetilde{D}_{\mathrm{c\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}}) = \frac{(t-1)\,\gamma_t^{-\frac{t}{t-1}} \cdot \rho \cdot \beta_1(\rho, t, \boldsymbol{\Sigma}_{\mathrm{h}})}{\ln 2 \cdot |\boldsymbol{\Sigma}_{\mathrm{h}}|} \cdot 2^{-\frac{B}{t-1}} \qquad (28)
$$

where $\beta_1(\rho, t, \boldsymbol{\Sigma}_{\mathrm{h}})$ is a constant coefficient that only depends on the number of antennas t, channel correlation matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$ and

---

[6]In this paper, we provide distortion analysis for correlated MISO channels whose channel covariance matrices $\boldsymbol{\Sigma}_{\mathrm{h}}$ have distinct positive eigenvalues. It is straightforward to extend the result to any covariance matrix that is positive definite. If the channel covariance matrix is singular, the channel quantization should actually be carried out in a space with reduced dimension.

system SNR ρ. It is given by

$$
\begin{aligned}
&\beta_1(\rho, t, \mathbf{\Sigma}_{\mathrm{h}}) \\
&= \left(\int_{\mathbf{v}:\mathbf{g}(\mathbf{v})=0} \left(\left(\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\mathbf{v}\right)^{-(t+1)} \right. \right. \\
&\quad \left. \left. \cdot {}_2F_0\left(t+1,1;;-\frac{\rho}{\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\mathbf{v}}\right) \right)^{\frac{t-1}{t}} d\mathbf{v} \right)^{\frac{t}{t-1}}
\end{aligned} \tag{29}
$$

with ${}_2F_0(;;)$ representing the generalized hypergeometric function. The optimal point density $\lambda^*(\mathbf{v})$ that achieves the minimal distortion is given by

$$
\begin{aligned}
\lambda^*(\mathbf{v}) &= \beta_1(\rho, t, \mathbf{\Sigma}_{\mathrm{h}})^{-\frac{t-1}{t}} \cdot \left(\left(\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\mathbf{v}\right)^{-(t+1)} \right. \\
&\quad \left. \cdot {}_2F_0\left(t+1,1;;-\frac{\rho}{\mathbf{v}^{\mathsf{H}}\mathbf{\Sigma}_{\mathrm{h}}^{-1}\mathbf{v}}\right) \right)^{\frac{t-1}{t}}.
\end{aligned} \tag{30}
$$

Some discussion on the evaluation of the coefficient β_1 is provided in the following subsection.

• *Distortion lower bound $D_{\text{c-Low,2}}(\mathbf{\Sigma}_{\mathrm{h}})$*

Similarly, by substituting the conditional PDF $p_{\mathbf{v}|\alpha}(\mathbf{x})$ given by (24), the marginal PDF $p_\alpha(x)$ given by (23) and the inertial profile $\widetilde{m}_{\text{c,opt}}(\mathbf{v};\alpha)$ given by (27) into the distortion lower bound (12), the asymptotic distortion lower bound $\widetilde{D}_{\text{c-Low,2}}$ can be expressed in the following form:

$$
\begin{aligned}
&\widetilde{D}_{\text{c-Low,2}}(\mathbf{\Sigma}_{\mathrm{h}}) \\
&= \frac{((t-1)! \cdot |\mathbf{\Sigma}_{\mathrm{h}}|)^{\frac{1}{t-1}} \cdot t^{t-1} \cdot \beta_2(\rho, t, \mathbf{\Sigma}_{\mathrm{h}})}{\ln 2 \cdot (t-1)^{t-1}} \cdot 2^{-\frac{B}{t-1}}
\end{aligned} \tag{31}
$$

where $\beta_2(\rho, t, \mathbf{\Sigma}_{\mathrm{h}})$ is a constant coefficient depends on system SNR ρ, number of antennas t and channel covariance matrix $\mathbf{\Sigma}_{\mathrm{h}}$. It is given by

$$
\beta_2(\rho, t, \mathbf{\Sigma}_{\mathrm{h}}) = \int_0^\infty \frac{\rho}{1+\rho x} \cdot p_\alpha\left(\frac{x(t-1)}{t}\right)^{\frac{t}{t-1}} dx. \tag{32}
$$

The evaluation of the coefficient β_2 is briefly discussed here.

• *Interesting Observations of the Distortion Bounds*

Based on the expressions for the average distortion lower bounds $\widetilde{D}_{\text{c-Low,1}}(\mathbf{\Sigma}_{\mathrm{h}})$ and $\widetilde{D}_{\text{c-Low,2}}(\mathbf{\Sigma}_{\mathrm{h}})$, the following observations can be made.

1) The asymptotic distortion lower bounds provided by (28) and (31) are described in a general format and are suitable for arbitrary channel correlations with covariance matrix $\mathbf{\Sigma}_{\mathrm{h}}$. The average distortion of i.i.d. MISO channels is a special case where the covariance matrix $\mathbf{\Sigma}_{\mathrm{h}}$ equals to the identity matrix. By substituting $\mathbf{\Sigma}_{\mathrm{h}} = I_t$ into (29), the coefficient β_1 reduces to be the following form:

$$
\beta_1(\rho, t, I_t) = {}_2F_0(t+1, 1;; -\rho) \cdot \gamma_t^{\frac{t}{t-1}}. \tag{33}
$$

Moreover, by substituting β_1 given by (33) into (28), the average system distortion lower bound $\widetilde{D}_{\text{c-Low,1}}$ for i.i.d. MISO systems can be obtained as

$$
\widetilde{D}_{\text{c-Low,1}} = \left(\frac{t-1}{\ln 2} \cdot {}_2F_0(t+1, 1;; -\rho) \cdot \rho \right) \cdot 2^{-\frac{B}{t-1}}. \tag{34}
$$

Similar simplification can be carried out by substituting $\mathbf{\Sigma}_{\mathrm{h}} = I_t$ into (31) and (32). It can be shown that for i.i.d. fading channels $D_{\text{c-Low,2}}$ equals to $D_{\text{c-Low,1}}$, which is given by (34). This result is consistent with the capacity loss expression obtained in [13] and [23].

2) Since the sensitivity matrix $\mathbf{W}_{\text{c},\alpha}(\widehat{\mathbf{v}})$ given by (26) satisfies the factorable condition given by (13), the distortion lower bound $D_{\text{c-Low,1}}$ is, hence, achievable and equal to the asymptotic distortions of the optimal quantizer, i.e.

$$
D_{\text{c-Low,1}} = D_{\text{Opt}} \gtrapprox \widetilde{D}_{\text{c-Low,1}} = \widetilde{D}_{\text{Q-opt}}. \tag{35}
$$

3) Both the distortion lower bounds $\widetilde{D}_{\text{c-Low,1}}$ and $\widetilde{D}_{\text{c-Low,2}}$ of correlated MISO channels, as well as the distortion of i.i.d. MISO channels, can be expressed as a weighted exponential function given by

$$
D = c \cdot 2^{-\frac{B}{t-1}},
$$

where c is a constant coefficient that is independent of the quantization (feedback) rate B.

4) Due to the multidimensional integration required to evaluate the coefficient $\beta_1(\rho, t, \mathbf{\Sigma}_{\mathrm{h}})$ given by (29), the distortion lower bound $\widetilde{D}_{\text{c-Low,1}}$ lacks a closed-form expression and can only be evaluated through a Monte Carlo simulation or a $(2t-2)$-dimensional numerical integration. Compared to the distortion lower bound $\widetilde{D}_{\text{c-Low,1}}$, $\widetilde{D}_{\text{c-Low,2}}$ (or the coefficient β_2) can be evaluated through a one-dimensional integration.

5) The distortion bounds of correlated MISO channels are smaller than that of the i.i.d. MISO channels, and satisfy the following inequality:

$$
0 < \widetilde{D}_{\text{c-Low,2}}(\mathbf{\Sigma}_{\mathrm{h}}) \overset{a}{\le} \widetilde{D}_{\text{c-Low,1}}(\mathbf{\Sigma}_{\mathrm{h}}) \overset{b}{\le} \widetilde{D}_{\text{c-Low,1}}(I_t). \tag{36}
$$

with equality of (a) and (b) if and only if $\mathbf{\Sigma}_{\mathrm{h}} = I_t$. This means that i.i.d. channels are the worst channel to quantize in a sense of having the largest distortion (or capacity loss). [7] This result is proved in Proposition 1 (in Appendix B). Detailed comparisons of the above distortion bounds and the distortions of mismatched quantizers are provided in Sections III-D and V.

6) Note that the above observations are drawn from the proposed distortion lower bounds, which are derived based on the high-resolution assumption. However, as a well known result in conventional source coding area, the high-rate distortion bounds agree well with the real simulation results when the resolution is larger than 3 bits per dimensions $(B/k_{\mathrm{q}} \geq 3)$ [33]. In this paper, due to "log-like" nature of the distortion function (system capacity loss), the distortion bounds converge even faster (about 1.5 bits per dimension), which is verified by simulation results in the following subsection. Moreover, the proposed distortion lower bounds

---

[7]This does not necessarily mean that correlated MISO channels have larger system capacities than i.i.d. channels. Since the capacity of i.i.d. MISO channels are better than that of correlated MISO channels with ideal CSI at the transmitter, the overall capacity of the finite-rate feedback-based MISO system still favors i.i.d. fading channels in the capacity sense.
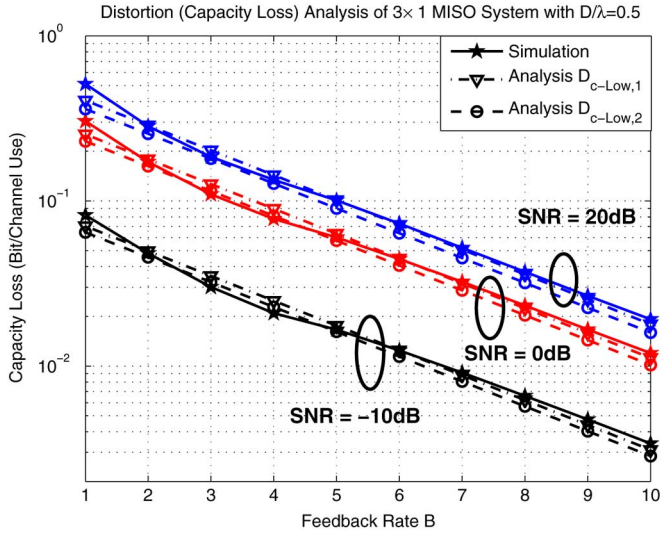
Fig. 1. Capacity loss versus CSI feedback rate B of a 3×1 correlated MISO transmit beamforming system with normalized antenna spacing $D/\lambda = 0.5$, and SNR $\rho = -10, 0$ and 20 dB.

Fig. 2. Capacity loss versus CSI feedback rate B of a 3×1 correlated MISO transmit beamforming system with normalized antenna spacing $D/\lambda = 0.2, 0.3, 0.5, 2.0$, and SNR $\rho = 20$ dB.

are tight, and hence are able to characterize the system performance well even for CSI quantizers with small to moderate quantization rates.

• *Numerical and Simulation Examples*

Some numerical experiments are now presented to provide a better feel for the utility of the bounds. Fig. 1 shows the capacity loss due to the finite-rate quantization of the CSI versus feedback rate B for a 3×1 MISO system over correlated Rayleigh fading channels under different system SNRs at $\rho = -10, 0$, and 20 dB, respectively. The spatially correlated channel is simulated by the correlation model in [34]: A linear antenna array with antenna spacing of half wavelength, i.e., $D/\lambda = 0.5$, uniform angular-spread in $[-30°, 30°]$ and angle of arrival $\phi = 0°$. The simulation results are obtained from a MISO system using optimal CSI quantizers whose codebooks are generated by the mean-squared weighted inner-product (MSwIP) criterion proposed in [11]. The distortion lower bounds $\widetilde{D}_{\text{c-Low},1}$ and $\widetilde{D}_{\text{c-Low},2}$ given by (28) and (31) are also included in the plot for comparisons. It can be observed from the plot that the proposed distortion (or the capacity loss) lower bounds are tight and predict very well the actual system capacity loss obtained from Monte Carlo simulations.

In order to see the effects of channel correlation on CSI quantization in a MISO system, we show in Fig. 2 the curves of capacity loss versus quantization rate (both simulation and analytical lower bound $\widetilde{D}_{\text{c-Low},1}$) of the same MISO system under different channel correlations obtained with adjacent antenna spacing $D/\lambda = 0.2, 0.3, 0.5, 2.0$ at SNR $\rho = 20$ dB. As a comparison to uncorrelated MISO channels, we also show in Fig. 3 the ratio of the distortion for correlated MISO channels over the distortion for i.i.d. fading channels with quantization rate $B = 10$ bits, SNR $\rho = 5$ dB, and under different channel correlations. It can be observed from the plot that the system distortion of correlated MISO channels is strictly less than that of the i.i.d. channels and the analytical result agree well with the actual simulation results.
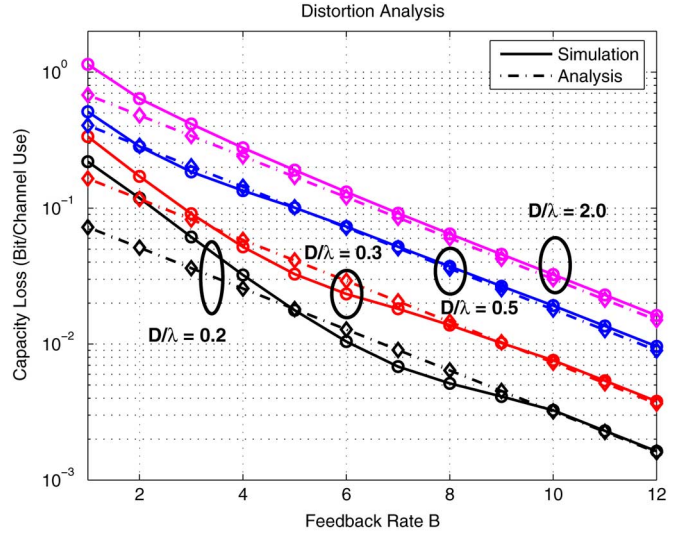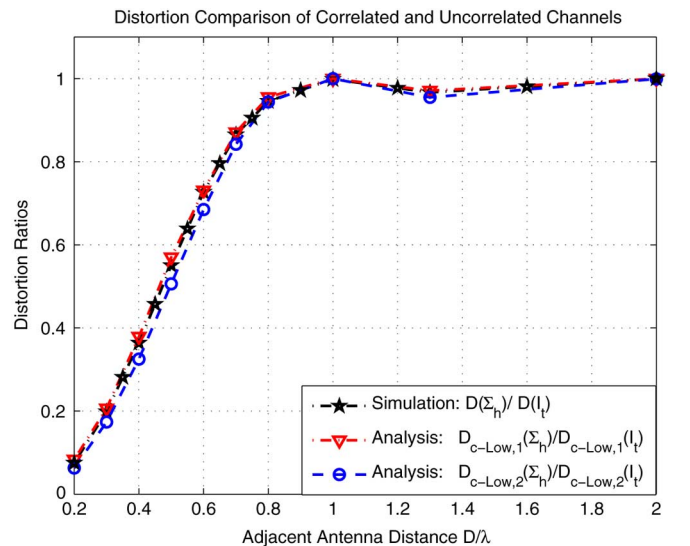

Fig. 3. Normalized capacity loss (w.r.t. the capacity loss of uncorrelated fading channels) versus transmit antenna spacing D/λ of a 3×1 MISO transmit beamforming system with SNR $\rho = 5$ dB under CSI feedback rate $B = 10$ bits.

D. Distortion Analysis in High-SNR and Low-SNR Regimes

• *High-SNR Distortion Analysis*

In high SNR regimes, the constrained sensitivity matrix $\mathbf{W}_{\text{c},\alpha}$ reduces to

$$\mathbf{W}_{\text{c},\alpha}^{\text{H-snr}}(\mathbf{v}, \alpha) = \lim_{\rho \to \infty} \frac{\rho \alpha}{\ln 2 \cdot (1 + \rho \alpha)} \cdot I = \frac{I}{\ln 2} \quad (37)$$

which is independent of $\mathbf{v}$, the side information α, as well as the SNR ρ. This means that 1) the encoder can discard the available side information α without any loss of system performance; 2) one single codebook is used for different system SNRs in high SNR regions. In this case, the inertial profile $\widetilde{m}_{\text{opt}}(\mathbf{v}, \alpha)$ and

the average inertial profile $\widetilde{m}_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{v})$ also reduce to be a constant independent of the location $\mathbf{v}$, as well as side information α

$$\widetilde{m}_{\mathrm{opt}}^{\mathrm{H\text{-}snr}} = \widetilde{m}_{\mathrm{opt}}^{\mathrm{w, H\text{-}snr}} = \frac{(t-1) \cdot \gamma_t^{-\frac{1}{t-1}}}{\ln 2 \cdot t}. \tag{38}$$

By substituting (38) into the distortion lower bound given by (9), the system capacity loss of i.i.d. MISO channels in high SNR regime is given by

$$\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr}}(\mathbf{\Sigma}_{\mathrm{h}} = I_t) = \frac{t-1}{t} \cdot 2^{-\frac{B}{t-1}} \tag{39}$$

which is consistent with the analysis obtained in [11] based on a statistical approach. For correlated MISO fading channels, the high-SNR asymptotic distortion lower bound $\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr}}$ can be shown to have the following closed-form expression:

$$\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr}}(\mathbf{\Sigma}_{\mathrm{h}}) = \frac{(t-1) \cdot \left(\prod_{i=1}^{t} \lambda_{\mathrm{h},i}\right)^{\frac{1}{t-1}}}{\ln 2 \cdot t}$$
$$\cdot \left((t-1) \sum_{i=1}^{t} \frac{(\ln \lambda_{\mathrm{h},i})/\lambda_{\mathrm{h},i}}{\prod_{k \neq i}(1 - \lambda_{\mathrm{h},k}/\lambda_{\mathrm{h},i})}\right)^{\frac{t}{t-1}} \cdot 2^{-\frac{B}{t-1}}. \tag{40}$$

The details are provided in Appendix A.

- *Low-SNR Distortion Analysis*

In low SNR regimes, i.e., $\rho \to 0$, the constrained sensitivity matrix $\mathbf{W}_{\mathrm{c},\alpha}$ reduces to be

$$\mathbf{W}_{\mathrm{c},\alpha}^{\mathrm{L\text{-}snr}} = \frac{2\rho\alpha}{\ln 2 \cdot (1 + \rho\alpha)} \cdot I \overset{\rho \to 0}{\approx} \frac{2\rho\alpha}{\ln 2} \cdot I. \tag{41}$$

Therefore, the inertial profile $\widetilde{m}_{\mathrm{opt}}(\mathbf{v}, \alpha)$ and the average inertial profile $\widetilde{m}_{\mathrm{opt}}^{\mathrm{w}}(\mathbf{v})$ are given by

$$\widetilde{m}_{\mathrm{opt}}^{\mathrm{L\text{-}snr}}(\mathbf{v}, \alpha) = \frac{(t-1) \cdot \gamma_t^{-\frac{1}{t-1}} \cdot \rho\alpha}{\ln 2 \cdot t}$$
$$\widetilde{m}_{\mathrm{opt}}^{\mathrm{w, L\text{-}snr}}(\mathbf{v}) = \frac{(t-1) \cdot \gamma_t^{-\frac{1}{t-1}} \cdot \rho}{\ln 2 \cdot \left(\mathbf{v}^{\mathrm{H}} \mathbf{\Sigma}_{\mathrm{h}}^{-1} \mathbf{v}\right)}. \tag{42}$$

Similarly, by substituting (42) into the distortion lower bound given by (9), the MISO system capacity loss in low SNR regimes over both i.i.d. and correlated fading channels can be represented as

$$\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr}}(\mathbf{\Sigma}_{\mathrm{h}} = I_t) = \frac{(t-1)\rho}{\ln 2} \cdot 2^{-\frac{B}{t-1}}, \tag{43}$$

$$\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr}}(\mathbf{\Sigma}_{\mathrm{h}}) = \frac{(t-1)\rho \cdot \gamma_t^{-\frac{t}{t-1}} \cdot \beta_3(t, \mathbf{\Sigma}_{\mathrm{h}})}{\ln 2 \cdot |\mathbf{\Sigma}_{\mathrm{h}}|} \cdot 2^{-\frac{B}{t-1}} \tag{44}$$

where $\beta_3(t, \mathbf{\Sigma}_{\mathrm{h}})$ is a constant coefficient given by

$$\beta_3(t, \mathbf{\Sigma}_{\mathrm{h}}) = \left(\int_{\mathbf{v}\,:\,\mathbf{g}(\mathbf{v})=0} (\mathbf{v}^{\mathrm{H}} \mathbf{\Sigma}_{\mathrm{h}}^{-1} \mathbf{v})^{-\frac{t^2-1}{t}} d\mathbf{v}\right)^{\frac{t}{t-1}}. \tag{45}$$

Moreover, when there are a large number of transmit antennas, the high-dimensional approximation of the distortion

lower bound $\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr}}$ can be represented by the following closed-form expression (obtained after some manipulation of (43)):

$$\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr, H\text{-}dim}} = \frac{\rho \cdot (t-1) \cdot \left(\prod_{i=1}^{t} \lambda_{\mathrm{h},i}\right)^{\frac{1}{t-1}}}{\ln 2} \cdot 2^{-\frac{B}{t-1}}. \tag{46}$$

- *Distortion Comparisons With i.i.d. MISO channels*

Through numerical evaluations, the second product term in the right-hand side (RHS) of the (40) is found to be close to 1 in most cases leading to the following approximate relationship:

$$\frac{\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr, H\text{-}dim}}(\mathbf{\Sigma}_{\mathrm{h}})}{\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr, H\text{-}dim}}(I_t)} = \frac{\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr, H\text{-}dim}}(\mathbf{\Sigma}_{\mathrm{h}})}{\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr, H\text{-}dim}}(I_t)} \approx \eta(\mathbf{\Sigma}_{\mathrm{h}}) \tag{47}$$

where the constant coefficient $\eta(\mathbf{\Sigma}_{\mathrm{h}})$ is given by

$$\eta(\mathbf{\Sigma}_{\mathrm{h}}) = \frac{\left(\prod_{i=1}^{t} \lambda_{\mathrm{h},i}\right)^{\frac{1}{t}}}{\sum_{i=1}^{t} \lambda_{\mathrm{h},i}/t} \leq 1 \tag{48}$$

and represents the relative capacity loss of quantizing a correlated MISO channel as compared to that of an i.i.d. MISO channel in high-SNR and low-SNR regimes with large number of antennas. This means that 1) the ratio of the geometric mean over the arithmetic mean of the eigenvalues of the channel covariance matrix is a key parameter that characterizes the system performance; 2) the capacity loss of a MISO system with finite-rate CSI feedback is approximately proportional to this ratio in both high-SNR and low-SNR regimes with a large number of transmit antennas.

As a numerical example, we show in Fig. 4 the normalized capacity loss (distortion ratio of correlated MISO channels over i.i.d. fading channels) versus antenna spacing D/λ in high-SNR regimes with $\rho = 20$ dB and quantization rate $B = 10$ bits. For comparison purpose, the ratio of the distortion lower bound, i.e., $\widetilde{D}_{\mathrm{c\text{-}Low},1}(\mathbf{\Sigma}_{\mathrm{h}})/\widetilde{D}_{\mathrm{c\text{-}Low},1}(I_t)$, as well as its high-SNR and high-dimensional approximation $\eta(\mathbf{\Sigma}_{\mathrm{h}})$ given by (48) are also included in the plot. Interestingly, it can be observed from Fig. 4 that the obtained high-dimensional approximation of the distortion ratio agree well with the simulation results even for cases with a small number of antennas $t = 3$.

IV. ASYMPTOTIC DISTORTION ANALYSIS OF MISMATCHED QUANTIZERS

In the previous section and in past work, the analytical results were derived under the assumption that both the encoder and the decoder have perfect knowledge of the source distribution, distortion function, and are using the most efficient quantization algorithm. This is clearly not always true as practical constraints often result in approximations and various types of suboptimal choices in the design of feedback-based wireless communication systems. These suboptimal choices often result in various types of mismatches. In this subsection, asymptotic analysis of mismatched quantizers is provided for the following three different categories: dimensionality mismatch, distortion function mismatch and source distribution mismatch.
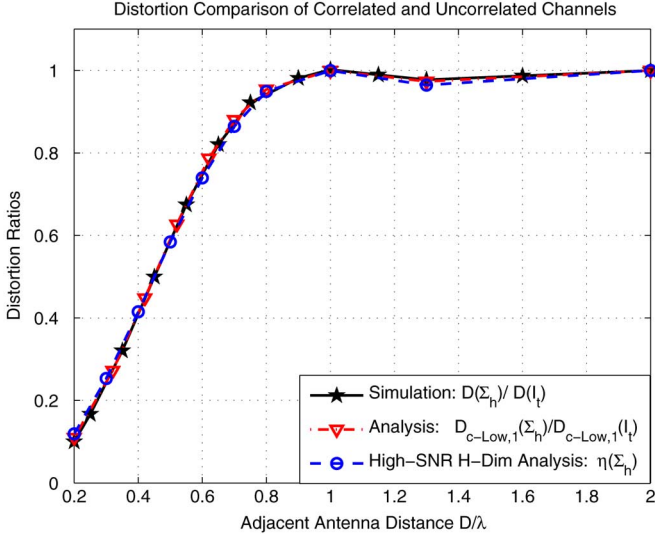
Distortion Comparison of Correlated and Uncorrelated Channels

Fig. 4. Normalized capacity loss (w.r.t. the capacity loss of uncorrelated fading channels) versus transmit antenna spacing D/λ of a 3×1 MISO transmit beamforming system in high-SNR regime with $\rho = 20$ dB.

A. Dimensionality Mismatch

The optimal quantizer is designed to quantize the source variable $\mathbf{y}$ (or objective) with the minimal free dimensions k_q or $k'_q = k_q - k_c$ in the case of constrained source. Dimensionality mismatch occurs when the suboptimal quantizer is designed to quantize a redundant source variable $\mathbf{y}_R$. As an example, for the MISO problem one may quantize directly the channel vector $\mathbf{h}$ instead of the directional vector $\mathbf{v} = \mathbf{h}/\|\mathbf{h}\|$. Hence, vector quantization is carried out in a space with dimension $k_{q,R}(k_{q,R} > k_q)$, and the distortion function D_Q is represented in its redundant form $D_{Q\text{-}R}(\mathbf{y}_R, \hat{\mathbf{y}}_R; \mathbf{z})$. In this case, by following the methodology provided in [23], [24], the final asymptotic analysis of the mismatched quantizer can be obtained which has a form similar to the lower bounds give by (9) and (12), i.e.,

$$D_{\text{mis-R-Low}} = c \cdot 2^{-\frac{2B}{k_{q,R}}} \qquad (49)$$

where c is a constant coefficient that depends on $k_{q,R}$, distortion function $D_{Q\text{-}R}$, and the source distribution $p(\mathbf{y}_R, \mathbf{z})$. An important and general observation from (49) is that by quantizing the redundant source variable $\mathbf{y}_R$, the system asymptotic distortion will have a smaller exponential slope $(-2/k_{q,R})$ when compared to that of quantizing the minimal free-dimensional vector $\mathbf{y}$ with exponential distortion slope $(-2/k_q)$.

B. Distortion Function Mismatch

In some cases, the quantizer (or the codebook) is designed or trained by using a distortion measure $D_{\text{mis-Q}}$ that is different from the actual system distortion function D_Q. An example of such a situation in practice is when the approximated distortion function $D_{\text{mis-Q}}$ leads to simple and efficient quantization schemes and codebook design algorithms [35]. More specifically for the MISO problem, one can envision designing a quantizer based on an SNR maximization criteria for simplicity and evaluating it using the capacity loss criteria. We provide in this subsection an asymptotic analysis of the general vector quantizer with mismatched distortion function.

The distortion of interest is denoted by D_Q and the distortion function used for designing the quantizer is denoted by $D_{\text{mis-Q}}$. Similar to the case of D_Q, a continuous second order derivative is assumed for the mismatched distortion function $D_{\text{mis-Q}}$. A parameter of interest in this context is the sensitivity matrix (or the Hessian matrix of $D_{\text{mis-Q}}$ w.r.t. vector $\mathbf{y}$) of the mismatched distortion function $D_{\text{mis-Q}}$, which is denoted by $\mathbf{W}_{\text{mis},\mathbf{z}}(\mathbf{y})$. Since $D_{\text{mis-Q}}$ is the basis of the quantizer, it determines the Voronoi region and the point density function. Codebook generated or trained by the mismatched sensitivity matrix leads to a mismatched Voronoi region $\mathbb{E}_{\text{mis},\mathbf{z}}(\mathbf{y})$, which can be approximated by a hyper ellipsoid $\mathcal{T}(\mathbf{0}, \mathbf{W}_{\text{mis},\mathbf{z}}(\hat{\mathbf{y}}_i), V(\mathbb{E}_{\text{mis},\mathbf{z}}(\mathbf{y}))$ with its definition given by (7), where $V(\mathbb{E}_{\text{mis},\mathbf{z}}(\mathbf{y}))$ is the volume of the mismatched Voronoi region. Unfortunately, the performance of the quantizer is evaluated using the true distortion function D_Q. Therefore, by substituting the approximated $\mathbb{E}_{\text{mis},\mathbf{z}}(\mathbf{y})$ into (4), the mismatched inertial profile utilizing the suboptimal codebook can be closely approximated by (50), shown at the bottom of the page. Following the multidimensional integration approach provided in [27] and [35], the mismatched inertial profile $\tilde{m}_{\text{mis-D}}(\hat{\mathbf{y}}_i; \mathbf{z})$ can be shown to be given by the following closed form expression:

$$\tilde{m}_{\text{mis-D}}(\mathbf{y}; \mathbf{z}) = \frac{1}{k_q + 2} \left(\frac{|\mathbf{W}_{\text{mis},\mathbf{z}}(\mathbf{y})|}{\kappa_{k_q}^2} \right)^{\frac{1}{k_q}}$$
$$\cdot \mathbf{tr}\left(\mathbf{W}_{\text{mis},\mathbf{z}}^{-1}(\mathbf{y}) \mathbf{W}_{\mathbf{z}}(\mathbf{y}) \right) \geq \tilde{m}_{\text{opt}}(\mathbf{y}; \mathbf{z}) \quad (51)$$

where κ_{k_q} is defined in (6). Consequently, the average mismatched inertial profile $\tilde{m}_{\text{mis-D}}^{\text{w}}(\mathbf{y})$ can be represented as

$$\tilde{m}_{\text{mis-D}}^{\text{w}}(\mathbf{y}) = \int_{\mathbb{Z}} \tilde{m}_{\text{mis-D}}(\mathbf{y}; \mathbf{z}) \cdot p(\mathbf{z}|\mathbf{y}) \, d\mathbf{z}. \quad (52)$$

In addition, the mismatched sensitivity matrix also leads to a mismatched point density function having the following form, from (11):

$$\lambda_{\text{mis-D}}(\mathbf{y}) = \left(\tilde{m}_{\text{opt-mis}}^{\text{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_q}{2+k_q}}$$
$$\cdot \left(\int_{\mathbb{Q}} \left(\tilde{m}_{\text{opt-mis}}^{\text{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \right)^{\frac{k_q}{2+k_q}} \, d\mathbf{y} \right)^{-1} \quad (53)$$

$$m_{\text{mis-D}}(\mathbf{y}; \mathbf{z}) \approx \tilde{m}_{\text{mis-D}}(\mathbf{y}; \mathbf{z}) = V(\mathbb{E}_{\text{mis},\mathbf{z}}(\mathbf{y}))^{-\frac{2+k_q}{k_q}}$$
$$\times \int_{\mathbf{y}' \in \mathcal{T}(\mathbf{0}, \mathbf{W}_{\text{mis},\mathbf{z}}(\mathbf{y}), V(\mathbb{E}_{\text{mis},\mathbf{z}}(\mathbf{y})))} (\mathbf{y}' - \mathbf{y})^\top \mathbf{W}_{\mathbf{z}}(\mathbf{y}) (\mathbf{y}' - \mathbf{y}) d\mathbf{y}'. \quad (50)$$

where $\widetilde{m}_{\text{opt-mis}}^{\text{w}}(\mathbf{y})$ is the optimal average inertia profile of a system with actual distortion function equal to $D_{\text{mis-Q}}$. Finally, by substituting the above mismatched average inertial profile (52) and mismatched point density (53) into the distortion lower bound $D_{\text{Low},1}(\widetilde{D}_{\text{Low},1})$ given by (9), the average distortion of a quantizer with mismatched distortion function can be obtained as

$$
\begin{aligned}
&\widetilde{D}_{\text{mis-D-Low},1} \\
&= 2^{-\frac{2B}{k_{\text{q}}}} \cdot \int_{\mathbb{Q}} \widetilde{m}_{\text{mis-D}}^{\text{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \cdot \widetilde{\lambda}_{\text{mis-D}}(\mathbf{y})^{-\frac{2}{k_{\text{q}}}} \, d\mathbf{y}. \quad (54)
\end{aligned}
$$

Utilizing a similar approach, the other mismatched distortion lower bound $\widetilde{D}_{\text{mis-D-Low},2}$ can also be obtained.

C. Source Distribution Mismatch

It is evident that the optimal quantizer (or the optimal codebook) is designed to match not only the distortion function D_{Q} but also the underlying source distribution $p(\mathbf{y}, \mathbf{z})$. In situations where the source distribution is hard to obtain or is subject to errors, the performance of the quantized system will degrade with the use of the suboptimal codebook generated using the mismatched source distribution, which is denoted as $p_{\text{mis}}(\mathbf{y}, \mathbf{z})$. As an example, for the MISO problem one may use a codebook designed assuming i.i.d. channels for correlated channels. The mismatched source distribution results in a mismatched average inertial profile, which is given by

$$
\widetilde{m}_{\text{mis-P}}^{\text{w}}(\mathbf{y}) = \int_{\mathbb{Z}} \widetilde{m}_{\text{opt}}(\mathbf{y}; \mathbf{z}) \cdot p_{\text{mis}}(\mathbf{z}|\mathbf{y}) d\mathbf{z}. \quad (55)
$$

The mismatched average inertial profile $\widetilde{m}_{\text{mis-P}}^{\text{w}}(\mathbf{y})$ together with $p_{\text{mis}}(\mathbf{y})$ further lead to a mismatched point density function $\lambda_{\text{mis-P}}(\mathbf{y})$

$$
\begin{aligned}
\lambda_{\text{mis-P}}(\mathbf{y}) &= (\widetilde{m}_{\text{mis-P}}^{\text{w}}(\mathbf{y}) \cdot p_{\text{mis}}(\mathbf{y}))^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \\
&\quad \cdot \left(\int_{\mathbb{Q}} (\widetilde{m}_{\text{mis-P}}^{\text{w}}(\mathbf{y}) \cdot p_{\text{mis}}(\mathbf{y}))^{\frac{k_{\text{q}}}{2+k_{\text{q}}}} \, d\mathbf{y} \right)^{-1} \quad (56)
\end{aligned}
$$

which is not optimized to match the actual source distribution as compared to the optimal point density function given by (11). Therefore, the asymptotic distortion of a suboptimal quantizer with mismatched source distribution is given by

$$
\begin{aligned}
&\widetilde{D}_{\text{mis-P-Low},1} \\
&= 2^{-\frac{2B}{k_{\text{q}}}} \cdot \int_{\mathbb{Q}} \widetilde{m}_{\text{opt}}^{\text{w}}(\mathbf{y}) \cdot p(\mathbf{y}) \cdot \widetilde{\lambda}_{\text{mis-P}}(\mathbf{y})^{-\frac{2}{k_{\text{q}}}} \, d\mathbf{y}. \quad (57)
\end{aligned}
$$

Again, the other asymptotic distortion bound $\widetilde{D}_{\text{mis-P-Low},2}$ can also be obtained.

In summary, the mismatched analysis provided in this section shows that the system performance degradation (or the distortion increment) due to the mismatch in the distortion function as well the source distribution only impacts the coefficient in front of the exponential term $2^{-2B/k_{\text{q}}}$. However, the dimensionality mismatch caused by quantizing a redundant source vector $\mathbf{y}_{\text{R}}$ has a more significant effect on the system performance. It re-

duces the slope of the exponential components in (9) and (12), and hence leads to a larger distortion $2^{-2B/k_{\text{q,R}}}(2^{-2B/k_{\text{q,R}}} \gg 2^{-2B/k_{\text{q}}})$ than that of an optimal quantizer especially in the high resolution regimes.

V. MISMATCHED ANALYSIS OF QUANTIZED MISO BEAMFORMING SYSTEMS

As an application of the mismatched analysis provided in Section IV, this section provides a capacity loss analysis of a finite-rate feedback-based MISO beamforming system when the CSI quantizer is mismatched and suboptimal. This is in contrast to the distortion analysis provided in Section III of MISO systems with optimal CSI quantizers wherein the codebook and the encoding algorithm were designed to perfectly match the distortion function as well as the source distribution. Imperfect codebook and suboptimal quantizer are quite prevalent in practice which makes this study interesting.

A. Dimensionality Mismatch and Quantization Criterion Mismatch

Here, we present the analysis of a suboptimal (mismatched) quantizer that directly quantizes the CSI using the mean-squared error (MSE) as the distortion measure. The results illustrate the importance of encoding the appropriate parameters as well as the distortion function of interest.

For an MMSE channel quantizer, the channel state information $\mathbf{h}$ is directly quantized and results in a conventional vector quantization problem with the source variable having $2t$ free (real) dimensions and with no encoder side information. The corresponding distortion function of the MMSE channel quantizer is given by

$$
D_{\text{mis-R}}(\mathbf{h}, \widehat{\mathbf{h}}) = \|\mathbf{h} - \widehat{\mathbf{h}}\|^2 \quad (58)
$$

whose sensitivity matrix $\mathbf{W}_{\text{mis-R}}$ is given by $\mathbf{W}_{\text{mis-R}} = I_{2t}$. At the transmitter, the unit norm beamforming vector $\widehat{\mathbf{v}}$ is obtained by normalizing the quantized channel vector $\widehat{\mathbf{h}}$, i.e., $\widehat{\mathbf{v}} = \widehat{\mathbf{h}}/\|\widehat{\mathbf{h}}\|$. Hence, the actual system distortion function (or the capacity loss) can be expressed in terms of vectors $\mathbf{h}$ and $\widehat{\mathbf{h}}$ as

$$
D_{\text{Q-R}}(\mathbf{h}, \widehat{\mathbf{h}}) = \log_2(1+\rho \cdot \|\mathbf{h}\|^2) - \log_2\left(1+\rho \cdot \frac{|\langle \mathbf{h}, \widehat{\mathbf{h}} \rangle|^2}{\|\widehat{\mathbf{h}}\|^2}\right). \quad (59)
$$

Its corresponding sensitivity matrix can be shown to have the following form (please refer to [30] for detailed derivations):

$$
\mathbf{W}(\widehat{\mathbf{h}}) = \frac{\rho}{\ln 2 \cdot (1+\rho \cdot \|\widehat{\mathbf{h}}\|^2)} \cdot (I - \mathbf{\Omega}) \quad (60)
$$

where matrix $\mathbf{\Omega} \in \mathbb{R}^{2k_{\text{q}} \times 2k_{\text{q}}}$ is given by

$$
\mathbf{\Omega} = \left[\begin{array}{c|c} \widehat{\mathbf{v}}_{\text{R}}\widehat{\mathbf{v}}_{\text{R}}^{\mathsf{T}} + \widehat{\mathbf{v}}_{\text{I}}\widehat{\mathbf{v}}_{\text{I}}^{\mathsf{T}} & \widehat{\mathbf{v}}_{\text{R}}\widehat{\mathbf{v}}_{\text{I}}^{\mathsf{T}} - \widehat{\mathbf{v}}_{\text{I}}\widehat{\mathbf{v}}_{\text{R}}^{\mathsf{T}} \\ \hline \widehat{\mathbf{v}}_{\text{I}}\widehat{\mathbf{v}}_{\text{R}}^{\mathsf{T}} - \widehat{\mathbf{v}}_{\text{R}}\widehat{\mathbf{v}}_{\text{I}}^{\mathsf{T}} & \widehat{\mathbf{v}}_{\text{R}}\widehat{\mathbf{v}}_{\text{R}}^{\mathsf{T}} + \widehat{\mathbf{v}}_{\text{I}}\widehat{\mathbf{v}}_{\text{I}}^{\mathsf{T}} \end{array} \right]. \quad (61)
$$

The MMSE channel quantizer being analyzed suffers from two types of mismatches: 1) The quantizer is designed to quantize a redundant channel state information vector $\mathbf{h}$ of dimensions $2t$ instead of $2t-2$ in the optimal quantizer, which leads to

a dimensionality mismatch; 2) The quantizer uses a mismatched distortion function $D_{\text{mis-R}}$ given by (58) as compared to D_{Q} given by (21). Since the MMSE codebook is designed to match the mismatched sensitivity matrix $\mathbf{W}_{\text{mis-R}}$, the Voronoi region of the MMSE quantizer is close to a hypersphere of dimension $2t$, which leads to a suboptimal point density given by [26]

$$\lambda_{\text{mis-R}}(\mathbf{h}) = p(\mathbf{h})^{\frac{t}{t+1}} \cdot \left(\int p(\mathbf{h})^{\frac{t}{t+1}} \, d\mathbf{h} \right)^{-1} \qquad (62)$$

where $p(\mathbf{y})$ is the PDF of the MISO channel impulse response $\mathbf{h}$. Furthermore, from (51), the suboptimal MMSE quantizer also leads to a mismatched normalized inertial profile given by

$$m_{\text{mis-R}}(\mathbf{h}) = \frac{(t-1)(t!)^{\frac{1}{t}} \rho}{\ln 2 \cdot (t+1) \cdot \pi \cdot (1 + \rho \|\mathbf{h}\|^2)} . \qquad (63)$$

By substituting (62) and (63) into the asymptotic distortion integration given by (3), it can be shown that the average system distortion of a mismatched MMSE channel quantizer has the following closed-form expression, shown in (64) at the bottom of the page, where $\beta_4(\rho, t, \mathbf{\Sigma}_{\text{h}})$ is a constant coefficient given by

$$\beta_4(\rho, t, \mathbf{\Sigma}_{\text{h}}) = -\frac{t}{t+1} \sum_{i=1}^{t} \left(\lambda_{\text{h},i} \prod_{k \neq i} \left(1 - \frac{\lambda_{\text{h},k}}{\lambda_{\text{h},i}} \right) \right)^{-1}$$
$$\times \exp\left(\frac{t}{\rho(t+1)\lambda_{\text{h},i}} \right) \cdot E_{\text{i}}\left(-\frac{t}{\rho(t+1)\lambda_{\text{h},i}} \right) \qquad (65)$$

with $E_{\text{i}}(\cdot)$ representing the exponential integral function. The detailed derivations of (64) and (65) are presented in Appendix A. It can be observed from (64) that the system distortion of the mismatched MMSE channel quantizer decays slower (with slope $-1/t$ in the exponent) than that of the optimal quantizer (with slope $-1/(t-1)$). This is a significant system performance degradation especially for systems with a small number of antennas, emphasizing the importance of choosing an appropriate CSI quantization scheme.

In order to get a better understanding of the degradation caused by the mismatched MMSE channel quantizer, we plot in Fig. 5 the capacity loss due to the finite-rate CSI quantization versus feedback rate B for a 3×1 MISO system over correlated fading channels with adjacent antenna spacing $D/\lambda = 0.5$ and different system SNRs of $\rho = -10$, and 20 dB, respectively. Codebooks are designed by using both the optimal mean-squared weighted inner-product (MSwIP) criterion proposed in [14] and the simple MMSE criterion mentioned in this section. The analytical evaluations of the system distortion lower bound $\widetilde{D}_{\text{c-Low},1}$ provided by (28) and the mismatched distortion $\widetilde{D}_{\text{mis-R-Low},1}$ provided by (64) are also included in
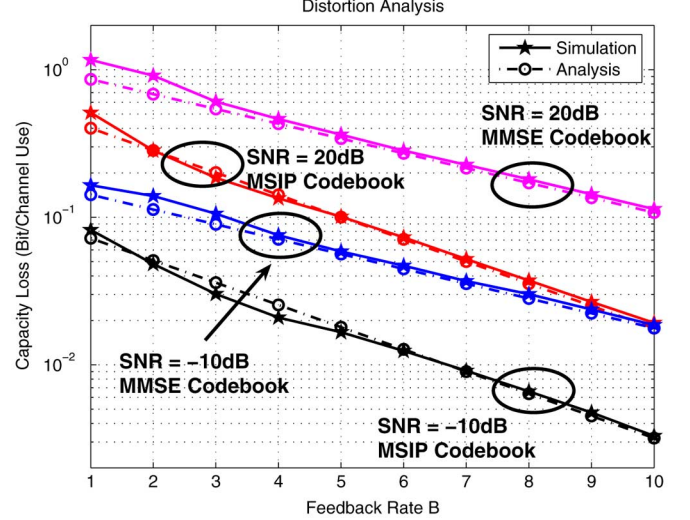

Fig. 5. Capacity loss of a 3×1 correlated MISO system with normalized antenna spacing $D/\lambda = 0.5$ versus CSI feedback rate B using different channel quantization codebooks (Optimal codebook versus MMSE quantizer).

the plot for comparisons. It can be observed from the plot that the system performance is significantly degraded by the mismatched quantizer, especially for systems with small number of antennas. Moreover, the proposed distortion analysis is tight and matches very well the actual system capacity loss obtained from Monte Carlo simulations.

B. Source Distribution Mismatch (Or Point Density Mismatch)

For the correlated MISO channels, the channel distribution depends on the covariance matrix $\mathbf{\Sigma}_{\text{h}}$, which needs to be estimated and is subject to estimation error. Moreover, it is also practically infeasible to redesign codebooks for every $\mathbf{\Sigma}_{\text{h}}$, store them and use them adaptively. Therefore, in practical situations, only very limited codebooks are available and so the mismatched channel covariance matrix $\mathbf{\Sigma}_{\text{h}}^{\text{m}}$ will cause performance degradation.

Based on the mismatched covariance matrix $\mathbf{\Sigma}_{\text{h}}^{\text{m}}$, a suboptimal codebook is generated with the mismatched point density given by (from (30))

$$\lambda_{\text{mis-P}}(\mathbf{v}) = \beta_1\left(\rho, t, \mathbf{\Sigma}_{\text{h}}^{\text{m}}\right)^{-\frac{t-1}{t}} \left(\left(\mathbf{v}^{\text{H}} \left(\mathbf{\Sigma}_{\text{h}}^{\text{m}}\right)^{-1} \mathbf{v}\right)^{-(t+1)} \right.$$
$$\left. \times \, {}_2F_0\left(t+1, 1; ; -\frac{\rho}{\mathbf{v}^{\text{H}} \left(\mathbf{\Sigma}_{\text{h}}^{\text{m}}\right)^{-1} \mathbf{v}} \right) \right)^{\frac{t-1}{t}} . \qquad (66)$$

Due to the fact that correct distortion function D_{Q} given by (21) is used in the codebook design, there is no mismatch in the inertial profile. By substituting the mismatched point density $\lambda_{\text{mis-P}}$ given by (66) into the distortion integral (57), the system

$$\widetilde{D}_{\text{mis-R-Low},1}(\mathbf{\Sigma}_{\text{h}}) = \frac{(t-1) \cdot (t! \cdot |\mathbf{\Sigma}_{\text{h}}|)^{\frac{1}{t}} \cdot \left(\frac{t+1}{t}\right)^{t} \cdot \beta_4(\rho, t, \mathbf{\Sigma}_{\text{h}})}{\ln 2 \cdot t} \cdot 2^{-\frac{B}{t}} \qquad (64)$$

distortion lower bound of the source-distribution-mismatched quantizer can be obtained as

$$\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1} = \left(\int_{\alpha} \int_{\mathbf{v}:\mathbf{g}(\mathbf{v})=0} \widetilde{m}_{\mathrm{c,\,opt}}(\mathbf{v},\,\alpha) \cdot p(\mathbf{v},\,\alpha) \right.$$
$$\left. \cdot \lambda_{\mathrm{mis\text{-}P}}(\mathbf{v})^{-\frac{1}{t-1}} d\mathbf{v} \, d\alpha \right) \cdot 2^{-\frac{B}{t-1}} \quad (67)$$

where $\widetilde{m}_{\mathrm{c,\,opt}}(\mathbf{x})$ is the optimal inertial profile given by (27). As a special case, if the codebook designed for i.i.d. MISO channels is used for correlated MISO systems[8], i.e., $\boldsymbol{\Sigma}_{\mathrm{h}}^{\mathrm{m}} = I_t$, the mismatched point density $\lambda_{\mathrm{mis}}(\mathbf{v})$ is uniform and the asymptotic distortion of the mismatched quantizer has the following closed-form expression:

$$\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}}) = \frac{(t-1) \cdot \beta_5(\rho, \boldsymbol{\Sigma}_{\mathrm{h}})}{\ln 2 \cdot t} \cdot 2^{-\frac{B}{t-1}} \quad (68)$$

where the constant coefficient $\beta_5(\rho, \boldsymbol{\Sigma}_{\mathrm{h}})$ is given by

$$\beta_5(\rho, \boldsymbol{\Sigma}_{\mathrm{h}}) = 1 + \sum_{i=1}^{t} \left(\rho \, \lambda_{\mathrm{h},i} \prod_{j \neq i} \left(1 - \frac{\lambda_{\mathrm{h},j}}{\lambda_{\mathrm{h},i}} \right) \right)^{-1}$$
$$\cdot \exp\left(\frac{1}{\rho \, \lambda_{\mathrm{h},i}} \right) \cdot E_{\mathrm{i}}\left(\frac{-1}{\rho \, \lambda_{\mathrm{h},i}} \right). \quad (69)$$

The detailed derivations of (68) and (69) are provided in Appendix A.

As a numerical example, we demonstrate in Fig. 6 the capacity loss due to the finite-rate CSI quantization versus feedback rate B for the same 3×1 MISO system over correlated fading channels with adjacent antenna spacing $D/\lambda = 0.5$ and different system SNRs at $\rho = -10$, and 20 dB, respectively. Both the optimal codebooks with correct channel covariance matrix as well as the mismatched i.i.d. codebooks are employed for simulation. The analytical evaluations of the distortion lower bound $\widetilde{D}_{\mathrm{c\text{-}Low},1}$ provide by (28) and the mismatched distortion $\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}$ provided by (68) are also included in the plot for comparison. It can be observed from the plot that the system performance is degraded by the mismatched i.i.d. codebook but with the same exponential decaying factor $2^{-B/(t-1)}$. Moreover, the proposed distortion analysis closely matches the system capacity loss obtained from simulations.

C. Comparisons With Other Channel Quantizers

In order to understand how the mismatched channel covariance matrix $(\boldsymbol{\Sigma}_{\mathrm{h}}^{\mathrm{m}} = I_t)$ affects the MISO system performance, a distortion comparison between optimal and mismatched quantizers under both correlated and i.i.d. fading channels is performed. By utilizing the concavity property of function $\beta_5(\rho, \boldsymbol{\Sigma}_{\mathrm{h}})$ given by (69) w.r.t. matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$, it is proved in Proposition 1 (in Appendix B) that the average distortion $\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}})$ of a mismatched quantizer using i.i.d. codebook in a correlated environments with channel covariance matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$ satisfies the following inequality:

$$\widetilde{D}_{\mathrm{c\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}}) \leq \widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}}) \leq \widetilde{D}_{\mathrm{c\text{-}Low},1}(I_t) \quad (70)$$

[8]This can be also viewed as the case where the channel covariance matrix is completely unavailable at both the transmitter and the receiver, and hence one single codebook is used for any channel correlation.
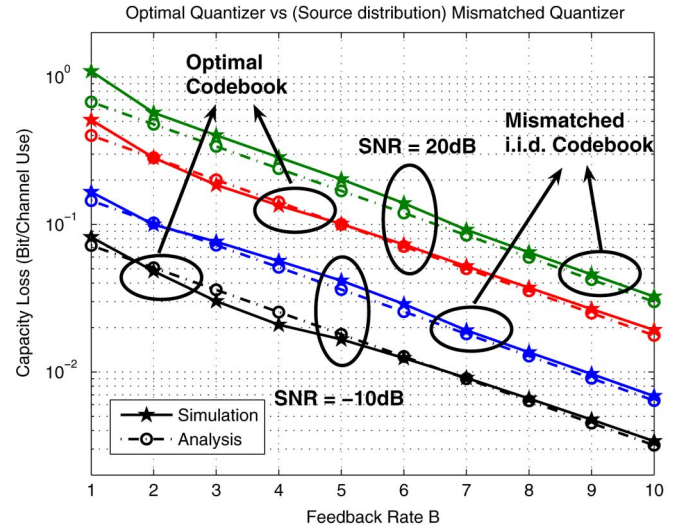

Fig. 6. Capacity loss of a 3×1 correlated MISO system with normalized antenna spacing $D/\lambda = 0.5$ versus CSI feedback rate B using different channel quantization codebooks (Optimal codebook versus Mismatched codebook for i.i.d. fading channels).

whereas $\widetilde{D}_{\mathrm{c\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}})$ represents the system distortion of an optimal quantizer with codebook designed to match the same correlated MISO fading channel with channel covariance matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$. Moreover, it is also proved in Proposition 1 (in Appendix B) that the mismatched system distortion $\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}(\boldsymbol{\Sigma}_{\mathrm{h}})$ converges to the distortion of i.i.d. MISO channels with optimal quantizers in high-SNR and low-SNR regimes, i.e.,

$$\frac{\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}^{\mathrm{H\text{-}snr}}(\boldsymbol{\Sigma}_{\mathrm{h}})}{\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr}}(I_t)} = \frac{\widetilde{D}_{\mathrm{mis\text{-}P\text{-}Low},1}^{\mathrm{L\text{-}snr}}(\boldsymbol{\Sigma}_{\mathrm{h}})}{\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{L\text{-}snr}}(I_t)} = 1. \quad (71)$$

The above results mean that: 1) the capacity loss of a correlated MISO channel by using the mismatched quantizer is larger than that of the optimal quantizer, but still less than that of an uncorrelated MISO channel even with optimal codebook; 2) the performance of the mismatched quantizer is strongly affected by the suboptimality caused by the mismatched codebook. In high-SNR and low-SNR regimes, mismatched CSI quantizers using i.i.d. codebooks will lead to the same "worst" system distortion $\widetilde{D}_{\mathrm{c\text{-}Low},1}^{\mathrm{H\text{-}snr}}(I_t)$ regardless of the actual fading channel correlations, or channel covariance matrix $\boldsymbol{\Sigma}_{\mathrm{h}}$. Once again, note that these two observations are drawn from the proposed high-resolution lower bounds. However, for the same reason described in Section III-C, they are tight and well predict the system performance even suitable for small to moderate B, demonstrated by the following simulation results.

We plot in Fig. 7 the normalized capacity loss, defined as the ratio of mismatched distortion of correlated channels to that of i.i.d. fading channels with optimal codebooks, of a correlated 3×1 MISO system versus antenna spacing D/λ with the mismatched i.i.d. codebooks, and with system SNR $\rho = -10, 20$ dB and quantization rate $B = 10$ bits. For comparison purpose, the normalized distortions using optimal codebooks of the same MISO system with the same correlated channel conditions are also included in the plot. The curves provided in Fig. 7 further confirm the two observations made prior.
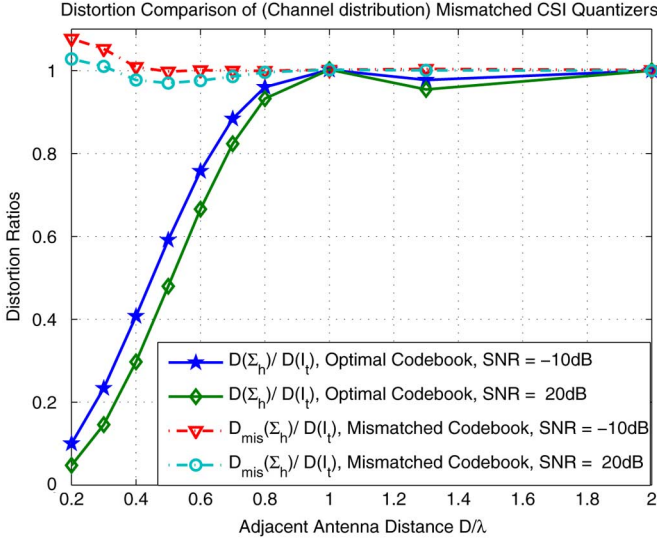
Fig. 7. Normalized capacity loss (w.r.t. the capacity loss of uncorrelated fading channels) comparison of a 3×1 MISO transmit beamforming with optimal and mismatched codebooks versus antenna spacing $d = D/\lambda$, in high and low SNR regimes ($\rho = -10$ and 20 dB).

VI. CONCLUSION

This paper employed high resolution quantization theory to study the effects of finite-rate quantization of the CSI on the performance of MISO systems over correlated fading channels. The contributions of this paper are twofold. First, as an extended application of the general distortion analysis, tight lower bounds on the capacity loss of correlated MISO systems due to the finite-rate channel quantization were provided. Interestingly, in high-SNR and low-SNR regimes, the capacity loss of correlated MISO channels was shown to be related to that of i.i.d. fading channels by a simple multiplicative factor which is given by the ratio of the geometric mean to the arithmetic mean of the eigenvalues of the channel covariance matrix. Second, the analysis framework is extended to the general asymptotic analysis of suboptimal quantizers resulting from mismatches in the distortion functions, source statistics, and quantization criteria. As an illustration, two types of mismatched MISO CSI quantizers were investigated: quantizers designed with MMSE criterion, and quantizers whose codebooks are designed with a mismatched channel covariance matrix. Bounds on the channel capacity loss of the mismatched codebooks were provided and compared to that of the optimal quantizers. Finally, numerical and simulation results were presented and they confirm the accuracy of the obtained theoretical distortion bounds.

APPENDIX A
DERIVATION OF DISTORTION BOUNDS $\widetilde{D}_{\text{c-Low},1}^{\text{H-snr}}$, $\widetilde{D}_{\text{mis-R-Low},1}$, AND $\widetilde{D}_{\text{mis-P-Low},1}$

Proof:

1) By substituting the average inertia profile $\widetilde{m}_{\text{opt}}^{\text{w, H-snr}}$ given by (38) as well as the marginal pdf $p_{\mathbf{v}}(\mathbf{x})$ given by (22) into the distortion bound given by (9), lower bound $\widetilde{D}_{\text{c-Low},1}^{\text{H-snr}}(\boldsymbol{\Sigma}_{\text{h}})$ has the following form (after some

manipulations):

$$
\widetilde{D}_{\text{c-Low},1}^{\text{H-snr}}(\boldsymbol{\Sigma}_{\text{h}})
$$
$$
= \left(\frac{(t-1) \cdot \left(\prod_{i=1}^{t} \lambda_{\text{h},i} \right)^{\frac{1}{t-1}}}{\ln 2 \cdot t} \cdot (\beta_6)^{\frac{t}{t-1}} \right) \cdot 2^{-\frac{B}{t-1}} \quad (72)
$$

where the coefficient β_6 is given by

$$
\beta_6 = E\left[\frac{\mathbf{h}^{\mathsf{H}} \boldsymbol{\Sigma}_{\text{h}}^{-1} \mathbf{h}}{\mathbf{h}^{\mathsf{H}} \mathbf{h}} \right] \quad (73)
$$

which is a ratio of Gaussian quadratic variables. The moments of ratios of random variables, including central quadratic forms in normal variables, were investigated in [36], and the results can be described by the following integral:

$$
E\left[\left(\frac{X}{Y} \right)^n \right] = \Gamma(n)^{-1} \int_0^{\infty} v^{n-1} M_{X,Y}^{(n)}(0, -v) \, dv \quad (74)
$$

where $M_{X,Y}(u, v)$ is the joint moment generating function (m.g.f.) of random variables X and Y, and $M_{X,Y}^{(n)}(0, -v)$ stands for $\partial^n M_{X,Y}(u, -v)/\partial v^n$ evaluated at $u = 0$. Therefore, by setting $X = \mathbf{h}^{\mathsf{H}} \boldsymbol{\Sigma}_{\text{h}}^{-1} \mathbf{h}$ and $Y = \mathbf{h}^{\mathsf{H}} \mathbf{h}$, the joint m.g.f. of variables X and Y can be shown to be given by

$$
M_{X,Y}(u, v) = \frac{1}{\det(I - (u \cdot I + v \cdot \boldsymbol{\Sigma}_{\text{h}}))}
$$
$$
= \left(\prod_{k=1}^{t} (1 - u - v \cdot \lambda_{\text{h},k}) \right)^{-1}. \quad (75)
$$

By substituting the joint m.g.f. given by (75) into the moments integration function (74), coefficient β_6 has the following closed-form expression:

$$
\beta_6 = (t-1) \sum_{i=1}^{t} \frac{(\ln \lambda_{\text{h},i})/\lambda_{\text{h},i}}{\prod_{k \neq i}(1 - \lambda_{\text{h},k}/\lambda_{\text{h},i})}. \quad (76)
$$

Finally, distortion lower bound (40) can be readily obtained by substituting (76) into (72).

2) In order to prove distortion bound $\widetilde{D}_{\text{mis-R-Low},1}(\boldsymbol{\Sigma}_{\text{h}})$ given by (64), it is sufficient to prove that the coefficient β_4, given by the following form:

$$
\beta_4 = E\left[\frac{\rho}{1 + \rho \cdot (t+1)/t \cdot \mathbf{h}^{\mathsf{H}} \mathbf{h}} \right] \quad (77)
$$

is identical to the expression given by (65). Again, by setting $X = \rho$ and $Y = 1 + \rho \cdot (t+1)/t \cdot \mathbf{h}^{\mathsf{H}} \mathbf{h}$, the joint m.g.f. of variables X and Y can be shown to be given by $M_{X,Y}(u, v)$

$$
= \frac{\exp(u \rho + v)}{\det(I - \rho \cdot (t+1)/t \cdot v \cdot \boldsymbol{\Sigma}_{\text{h}})}
$$
$$
= \exp(u \rho + v) \cdot \left(\prod_{k=1}^{t} \left(1 - \rho \cdot \frac{t+1}{t} \cdot v \cdot \lambda_{\text{h},k} \right) \right)^{-1}. \quad (78)
$$

By substituting the joint m.g.f. (78) into (74), closed-form expression of coefficient β_4 given by (65) can be obtained.

3) Similarly, in order to prove distortion lower bound $\widetilde{D}_{\text{mis-P-Low},1}(\boldsymbol{\Sigma}_{\text{h}})$ given by (68), it is sufficient to prove that the coefficient β_5, given by the following form:

$$\beta_5 = E\left[\frac{\rho \cdot \mathbf{h}^{\mathsf{H}}\mathbf{h}}{1 + \rho \cdot \mathbf{h}^{\mathsf{H}}\mathbf{h}}\right] \quad (79)$$

is identical to the expression given by (69). It is evident from (79) that β_5 is related to β_4 in the following manner:

$$\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}}) = 1 - \frac{1}{\rho}\beta_4\left(\frac{t\rho}{t+1}, \boldsymbol{\Sigma}_{\text{h}}\right). \quad (80)$$

Hence, the closed-form expression of β_5 given by (69) can be obtained. ∎

APPENDIX B
ORDERING OF THE DISTORTION BOUNDS

Proposition 1: For a MISO system with finite-rate CSI feedback, the following ordering of the system distortions is valid for any correlated fading channels with covariance matrix $\boldsymbol{\Sigma}_{\text{h}}$ satisfying $\mathbf{tr}(\boldsymbol{\Sigma}_{\text{h}}) = t$,

$$0 < \widetilde{D}_{\text{c-Low},2}(\boldsymbol{\Sigma}_{\text{h}}) \overset{a}{\leq} \widetilde{D}_{\text{c-Low},1}(\boldsymbol{\Sigma}_{\text{h}})$$
$$\overset{b}{\leq} \widetilde{D}_{\text{mis-P-Low},1}(\boldsymbol{\Sigma}_{\text{h}}) \overset{c}{\leq} \widetilde{D}_{\text{c-Low},1}(I_t). \quad (81)$$

Moreover, the mismatched system distortion $\widetilde{D}_{\text{mis-P-Low},1}(\boldsymbol{\Sigma}_{\text{h}})$ converges to the distortion of i.i.d. MISO channels with optimal quantizers in high-SNR and low-SNR regimes, i.e.,

$$\lim_{\rho \to 0, \infty} \frac{\widetilde{D}_{\text{mis-P-Low},1}(\boldsymbol{\Sigma}_{\text{h}})}{\widetilde{D}_{\text{c-Low},1}(I_t)} = 1. \quad (82)$$

Proof: First of all, (a) and (b) can be proved easily by their definition. For inequality (c), first note that (after some manipulations) the asymptotic distortion of the mismatched quantizer can be represented as

$$\widetilde{D}_{\text{mis-P-Low},1}(\boldsymbol{\Sigma}_{\text{h}}) = \frac{(t-1) \cdot \beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}})}{\ln 2 \cdot t} \cdot 2^{-\frac{B}{t-1}} \quad (83)$$

where the constant coefficient $\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}})$ is given by the following form:

$$\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}}) = E\left[\frac{\rho \cdot \|\mathbf{h}\|^2}{1 + \rho \cdot \|\mathbf{h}\|^2}\right] = E\left[\frac{\rho \cdot \mathbf{h}_0^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}\mathbf{h}_0}{1 + \rho \cdot \mathbf{h}_0^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}\mathbf{h}_0}\right] \quad (84)$$

where vectors $\mathbf{h}$ and $\mathbf{h}_0$ have the following distribution:

$$\mathbf{h} \sim \mathcal{N}_{\text{c}}(\mathbf{0}, \boldsymbol{\Sigma}_{\text{h}}), \quad \mathbf{h}_0 \sim \mathcal{N}_{\text{c}}(\mathbf{0}, I_t). \quad (85)$$

It is evident from (84) that β_5 is invariant under the following transformation:

$$\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}}) = \beta_5(\rho, \mathbf{U}\boldsymbol{\Sigma}_{\text{h}}\mathbf{U}^{\mathsf{H}}) \quad (86)$$

where $\mathbf{U}$ is any unitary matrix. Hence according to (86), if the unitary matrix $\mathbf{U}$ is set to be the eigenvectors of $\boldsymbol{\Sigma}_{\text{h}}$, we only need to focus our attention on the case where $\boldsymbol{\Sigma}_{\text{h}}$ is a diagonal

matrix. Furthermore, it is also true that β_5 is invariant to any permutations on the diagonal elements of $\boldsymbol{\Sigma}_{\text{h}}$,

$$\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}}) \overset{a}{=} \frac{1}{t!}\sum_{\mathbf{P}}\beta_5\left(\rho, \mathbf{P}^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}\mathbf{P}\right)$$
$$\overset{b}{\leq} \beta_5\left(\rho, \left(\frac{1}{t!}\sum_{\mathbf{P}}\mathbf{P}^{\mathsf{H}}\boldsymbol{\Sigma}_{\text{h}}\mathbf{P}\right)\right) = \beta_5(\rho, I_t) \quad (87)$$

where $\mathbf{P}$ is any permutation matrix, equality (a) follows the same reasoning as the invariant transformation (86), and (b) follows from the concavity property of function $f(x) = x/(1+x)$. At this point, by substituting the inequality (87) into the system distortion expression given by (83), inequality (c) of the distortion ordering given by (81) can be obtained.

According to the definition of β_5 given by (84), the following equations can be obtained:

$$\lim_{\rho \to 0}\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}}) = \rho t, \quad \lim_{\rho \to \infty}\beta_5(\rho, \boldsymbol{\Sigma}_{\text{h}}) = 1 \quad (88)$$

which further lead to the convergence of the system distortions given by (82). ∎

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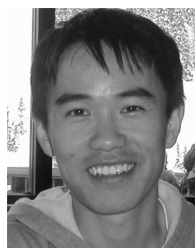
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