Computing Distances between Reach Flowpipes

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ABSTRACT

We investigate quantifying the difference between two hybrid dynamical systems under noise and initial-state uncertainty. While the set of traces for these systems is infinite, it is possible to symbolically approximate trace sets using *reachpipes* that compute upper and lower bounds on the evolution of the reachable sets with time. We estimate distances between corresponding sets of trajectories of two systems in terms of distances between the reachpipes.

In case of two individual traces, the Skorokhod distance has been proposed as a robust and efficient notion of distance which captures both value and timing distortions. In this paper, we extend the computation of the Skorokhod distance to reachpipes, and provide algorithms to compute upper and lower bounds on the distance between two sets of traces. Our algorithms use new geometric insights that are used to compute the worst-case and best-case distances between two polyhedral sets evolving with time.

1. INTRODUCTION

The quantitative conformance problem between two dynamical systems asks how close the traces of the two systems are under a given metric on hybrid traces [1, 2, 9]. If the systems are deterministic and start from unique initial conditions, each has exactly one trace, and the quantitative conformance problem computes the distance between these two traces. In this case, we have shown in previous work that the *Skorokhod metric* between traces provides a robust and efficiently computable distance that captures the intuitive notion of closeness of two systems [18, 9]. However, if there is uncertainty in the initial states and noise in the inputs, each system defines not just a single trace but a set of traces. In this work, we investigate algorithms to compute distances between sets of trajectories of two dynamical systems under initial state and input uncertainties.

Given two sets F_1, F_2 of trajectories of two dynamical systems, the natural generalization of the Skorokhod distance between traces is to ask what is the farthest a trajectory in one set can be from a trajectory in the other, *i.e.*, to compute

$$\mathcal{D}_{\mathsf{var}}(F_1, F_2) = \sup_{f_1, f_2} \mathcal{D}_{\mathsf{tr}}(f_1, f_2)$$

where $\mathcal{D}_{\mathsf{tr}}$ is the given Skorokhod metric on traces[1].

Unfortunately, due to the continuous nature of systems, trace sets F_1 and F_2 are not available in closed form for most kinds of systems. Instead, given a trace set F, one approximates it using a *reachpipe*, a function $R : [0, T] \to 2^{\mathbb{R}^d}$, such that $R(t) = \cup_{f \in F}\{f(t)\}$, *i.e.*, $R(t)$ is the set of all trace values that can be observed at time t. A reachpipe R can be viewed as an approximation $\mathsf{Fp}(R)$ to the original set of traces, the approximation $\mathsf{Fp}(R)$ includes every trace f such that $f(t) \in R(t)$, not just those allowed by the dynamics. In practice, even the reachpipe may not have an exact representation, and instead, one computes over- or under-approximations to the reachpipe by computing a sequence of *reach set* samples at discrete timepoints $t_0, t_1, \ldots$. Indeed, there are several techniques to compute such approximations of reach sets [7, 17, 11, 13, 15, 10, 20, 8, 6], differing in the quality of the approximation, the efficiency of computation, or the representation of the reach set approximations.

We consider the problem of estimating trajectory set distances when we only have the sampled sequences of over- and under-approximations of reach sets. As a first step, we define a lower and an upper bound on the distance between F_1 and F_2 based on the reach set approximations.

Second, we show how to compute these bounds. To compute the distance, we re-formulate reachpipes as set-valued traces, *i.e.*, as traces over the time interval $[0, T]$ where the trace value at time t is the set $R(t) \subseteq \mathbb{R}^d$. This alternative viewpoint allows us to define trace distances $\mathcal{D}^\dagger$ between reachpipes by viewing them as set-valued traces. We derive relationships between the distances $\mathcal{D}^\dagger$ under this alternative viewpoint, and distances bounding the trace set distance (obtained using approximations to the reachpipes).

Finally, we derive algorithms to compute the $\mathcal{D}^\dagger$ distances between reachpipes in case the underlying metric on traces is given by the Skorokhod distance and the reach set sequences are given as polytopes in $\mathbb{R}^d$. The Skorokhod distance on traces takes into account both timing distortions and value differences; our algorithms lift the metric to reach sets viewed as time-varying polytopes. The algorithms allows for timing distortions, and generalize the Skorokhod distance algorithm over polygonal lines to polytopes which

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[1]In comparing sets, we use the term "distance" for similarity/dissimilarity functions $\mathcal{D}_{\mathsf{var}}$ satisfying the triangle inequality; these functions are not necessarily metrics, as $\mathcal{D}_{\mathsf{var}}(F, F)$ need not be zero.

vary with time. The main technical constructions in our algorithms are two novel geometric routines in a core part of the Skorokhod distance algorithm which allow us to move to the domain of time-varying polytopes for the set distances under consideration.

Putting everything together, we obtain polynomial time algorithms which compute bounds on traceset distances where the tracesets are observed only as reachset sample-polytopes at discrete timepoints.

Outline of the Paper. In Section 2, we recall the Skorokhod trace metric, and the related Fréchet metric. In Section 3, we formally present tracepipes and reachpipes, distances between trace sets, and bounds on these set distances. In Section 4 we explore the alternative viewpoint of reachpipes being set valued traces, and relate distances under this viewpoint and distances between reachpipes viewed as trace sets. In Section 5, we solve for the distance decision problems between reachpipes viewed as time-varying polytopes of $\mathbb{R}^d$. In Section 6 we put everything together and present various algorithms to compute bounds on Skorokhod traceset distances.

2. PRELIMINARIES: TRACE METRICS

A (finite) *trace* $f : [T_i, T_e] \to \mathbb{R}^d$ is a continuous mapping from a finite closed interval $[T_i, T_e]$ of $\mathbb{R}_+$, with $0 \leq T_i < T_e$, to $\mathbb{R}^d$.

2.1 The Skorokhod Trace Metric

We define a metric on the space of traces corresponding to a given metric on $\mathbb{R}^d$. A *retiming* $\mathsf{r} : I \mapsto I'$, for closed intervals I, I' of $\mathbb{R}_+$, is an order-preserving (i.e., monotone) continuous bijective function from I to I'; thus if $t < t'$ then $\mathsf{r}(t) < \mathsf{r}(t')$. Let $\mathsf{R}_{I \mapsto I'}$ be the class of retiming functions from I to I' and let id be the identity retiming. Given a trace $f : I_f \to \mathbb{R}^d$, and a retiming $\mathsf{r} : I \mapsto I_f$; the function $f \circ \mathsf{r}$ is another trace from I to $\mathbb{R}^d$.

Definition 1 (Skorokhod Metric)**.** Given a retiming $\mathsf{r} : I \mapsto I'$, define

$$|| \, \mathsf{r} - \mathsf{id} \, ||_{\sup} := \sup_{t \in I} | \, \mathsf{r}(t) - t|.$$

Given two traces $f : I_f \mapsto \mathbb{R}^d$ and $f' : I_{f'} \mapsto \mathbb{R}^d$, a norm L on $\mathbb{R}^d$, and a retiming $\mathsf{r} : I_f \mapsto I_{f'}$, define

$$\left\| f \; - \; f' \circ \mathsf{r} \right\|_{\sup} := \sup_{t \in I_f} \left\| f(t) - f'\left(\mathsf{r}(t)\right) \right\|_L.$$

The *Skorokhod metric*[2] between the traces f and f' is defined to be:

$$\mathcal{D}_{\mathcal{S}}(f, f') := \inf_{r \in \mathsf{R}_{I_f \mapsto I_{f'}}} \max \left(\left\| \mathsf{r} - \mathsf{id} \right\|_{\sup}, \, \left\| f \; - \; f' \circ \mathsf{r} \right\|_{\sup} \right). \quad \square$$

Intuitively, the Skorokhod metric incorporates two components: the first component quantifies the *timing discrepancy* of the timing distortion required to "match" the two traces, and the second quantifies the *value mismatch* (in the vector space $(\mathbb{R}^d, \|\cdot\|_L)$) of the values under the timing distortion. In the retimed trace $f \circ \mathsf{r}$, we see exactly the same values as in f, in exactly the same order, but the times at which the values are seen can be different.

[2]The two components of the Skorokhod metric (the retiming, and the value difference components) can be weighed with different weights – this simply corresponds to a change of scale.

2.2 The Fréchet Trace Metric

We showed in [18] that the Skorokhod metric is related to another metric, the Fréchet metric, over traces. We recall the definition and the relationship.

Definition 2 (Fréchet metric)**.** Let $\mathsf{C}_1 : I_1 \to \mathbb{R}^d$ and $\mathsf{C}_2 : I_2 \to \mathbb{R}^d$ be traces. The Fréchet metric between the two traces $\mathsf{C}_1, \mathsf{C}_2$ (given a norm L on $\mathbb{R}^d$) is defined to be

$$\mathcal{D}_{\mathcal{F}}(\mathsf{C}_1, \mathsf{C}_2) := \inf_{\substack{\alpha_1 : [0,1] \to I_1 \\ \alpha_2 : [0,1] \to I_2}} \max_{0 \leq \theta \leq 1} \left\| \mathsf{C}_1\left(\alpha_1(\theta)\right) - \mathsf{C}_2\left(\alpha_2(\theta)\right) \right\|_L$$

where α_1, α_2 range over continuous and strictly increasing bijective functions onto I_1 and I_2, respectively. $\square$

Intuitively, the *reparameterizations* α_1, α_2 control the "speed" of traversal along the two traces by two entities. The positions of the two entities in the two traces at "time" θ is given by $\alpha_1(\theta)$ and $\alpha_2(\theta)$ respectively; with the value of the traces at those positions being $\mathsf{C}_1\left(\alpha_1(\theta)\right)$, and $\mathsf{C}_2\left(\alpha_2(\theta)\right)$. The two entities always have a speed strictly greater than 0.

Given a trace $f : [T_i, T_e] \to \mathbb{R}^d$, we define the *time-explicit trace* $\mathsf{C}_f : [T_i, T_e] \to \mathbb{R}^d \times \mathbb{R}$ where we add the time value as an extra dimension, that is, $\mathsf{C}_f(t) = (f(t), t)$ for all $t \in [T_i, T_e]$. Given a value $\langle \boldsymbol{p}, t \rangle \in \mathbb{R}^d \times \mathbb{R}$, and a a norm L over $\mathbb{R}^d$, define the norm

$$\| \langle \boldsymbol{p}, t \rangle \|_{L^{\max}} = \max\left(\|\boldsymbol{p}\|_L, |t| \right). \tag{1}$$

Proposition 1 (From Skorokhod to Fréchet [18])**.** *Let $f : [T_i^f, T_e^f] \to \mathbb{R}^d$ and $g : [T_i^g, T_e^g] \to \mathbb{R}^d$ be two continuous traces. Consider the corresponding time-explicit traces $C_f : [T_i^f, T_e^f] \to \mathbb{R}^{d+1}$ and $C_g : [T_i^g, T_e^g] \to \mathbb{R}^{d+1}$. Consider the Skorokhod distance $\mathcal{D}_{\mathcal{S}}(f, g)$ with respect to a given norm L over $\mathbb{R}^d$. We have*

$$\mathcal{D}_{\mathcal{S}}(f, g) = \mathcal{D}_{\mathcal{F}}(\mathsf{C}_f, \mathsf{C}_g),$$

where the Fréchet distance $\mathcal{D}_{\mathcal{F}}(\mathsf{C}_f, \mathsf{C}_g)$ is with respect to the norm $L^{\max}$ over $\mathbb{R}^{d+1}$. $\square$

3. PIPES & PIPE-VARIATION DISTANCES

3.1 Tracepipes, Reachpipes and Set Distances

A *tracepipe* F is a nonempty collection of traces over some closed interval $[T_i, T_e]$. A *reachpipe* $R : [T_i, T_e] \to 2^{\mathbb{R}^d} \setminus \emptyset$ maps a finite closed interval $[T_i, T_e]$ of $\mathbb{R}_+$, denoted $\mathsf{tdom}(R)$, to non-empty subsets of $\mathbb{R}^d$. To a reachpipe R, we associate a tracepipe $\mathsf{Fp}(R)$ consisting of all continuous traces f over $\mathsf{tdom}(R)$ such that $f(t) \in R(t)$ for all $t \in \mathsf{tdom}(R)$. Dually, corresponding to each tracepipe F, we associate the reachpipe $\mathsf{Rp}(F)$, over the same time-domain, defined by $\mathsf{Rp}(F)\,(t) = \cup_{f \in F}\{f(t)\}$. Note that $F \subseteq \mathsf{Fp}\left(\mathsf{Rp}\left(F\right)\right)$, but equality need not hold: $\mathsf{Fp}\left(\mathsf{Rp}\left(F\right)\right)$ may contain more traces than F.

A reachpipe $R' : [T_i, T_e] \to 2^{\mathbb{R}^d}$ is an *over-approximation* (respectively, *under-approximation*) of a reachpipe $R : [T_i, T_e] \to 2^{\mathbb{R}^d}$ if for each $t \in [T_i, T_e]$, we have $R(t) \subseteq R'(t)$ (respectively, $R'(t) \subseteq R(t)$).

Example 1. Consider a linear dynamical system in $\mathbb{R}$ described by $\dot{x} = ax$, for $a > 0$ with initial state $x_0 \in [0, 0.1]$ over the time interval $[0, 10]$. For a fixed value of x_0, we get a trace $x_0 e^{at}$. Let $F = \{f_{x_0} \mid x_0 \in [0, 0.1]$ and $f_{x_0}(t) = x_0 e^{at}$ for $t \in [0, 10]\}$ be a tracepipe. The reachpipe $\mathsf{Rp}(F)$ corresponding to the tracepipe F is given by $\mathsf{Rp}(F)(t) = [0, 0.1e^{at}]$ for $t \in [0, 10]$. Observe that $\mathsf{Fp}\left(\mathsf{Rp}(F)\right)$ contains the more traces than F, for instance, the constant trace $f(t) = 0.1$. □

Let $\mathcal{D}_{\mathsf{tr}}$ be a given metric on traces. We define the *variation distance* $\mathcal{D}_{\mathsf{var}}(F_1, F_2)$ between two tracepipes F_1 and F_2 corresponding to the trace metric $\mathcal{D}_{\mathsf{tr}}$ as

$$\mathcal{D}_{\mathsf{var}}(F_1, F_2) := \sup_{f_1 \in F_1, f_2 \in F_2} \mathcal{D}_{\mathsf{tr}}(f_1, f_2) \tag{2}$$

The value $\mathcal{D}_{\mathsf{var}}(F_1, F_2)$ gives us the maximum possible inter-trace distance if one trace is from F_1 and the other from F_2. Notice that for all tracepipes F_1, F_2, F_3, we have that

1. $\mathcal{D}_{\mathsf{var}}(F_1, F_2) \geq 0$;
2. $\mathcal{D}_{\mathsf{var}}(F_1, F_2) = \mathcal{D}_{\mathsf{var}}(F_2, F_1)$; and
3. $\mathcal{D}_{\mathsf{var}}(F_1, F_3) \leq \mathcal{D}_{\mathsf{var}}(F_1, F_2) + \mathcal{D}_{\mathsf{var}}(F_2, F_3)$.

We may however have $\mathcal{D}_{\mathsf{var}}(F, F) > 0$, thus, $\mathcal{D}_{\mathsf{var}}$ need not be a metric over tracepipes. The value $\mathcal{D}_{\mathsf{var}}(F, F)$ gives us the maximum distance amongst traces in F according to the original trace metric $\mathcal{D}_{\mathsf{tr}}$.

Tracepipes cannot be constructed for most dynamical systems. However, *reachpipe* sets can be over/under-approximated at desired timepoints using analytic techniques. In the next subsection, we present a framework for bounding the tracepipe distance $\mathcal{D}_{\mathsf{var}}(F_1, F_2)$ using over/under-approximated reachpipes.

3.2 Approximating the Variation Distance

Let F_1 and F_2 be tracepipes. Since $F \subseteq \mathsf{Fp}(\mathsf{Rp}(F))$ for any tracepipe F, and Rp, Fp, and the variation distance $\mathcal{D}_{\mathsf{var}}$ are all monotonic, we have that

$$\mathcal{D}_{\mathsf{var}}(F_1, F_2) \leq \mathcal{D}_{\mathsf{var}}\big(\,\mathsf{Fp}\left(\lceil \mathsf{Rp}\left(F_1\right)\rceil\right), \mathsf{Fp}\left(\lceil \mathsf{Rp}\left(F_2\right)\rceil\right)\big) \tag{3}$$

for any over-approximations $\lceil \mathsf{Rp}(F_1)\rceil$ and $\lceil \mathsf{Rp}(F_2)\rceil$ of the reachpipes $\mathsf{Rp}(F_1)$ and $\mathsf{Rp}(F_2)$. Thus, in order to get an upper bound on $\mathcal{D}_{\mathsf{var}}(F_1, F_2)$ we can use over-approximations of the corresponding reachpipes.

Define the *minimum set distance*:

$$\mathcal{D}_{\min}(F_1, F_2) := \inf_{f_1 \in F_1, f_2 \in F_2} \mathcal{D}(f_1, f_2) \tag{4}$$

For this distance, it is clear that

$$\mathcal{D}_{\min}\big(\,\mathsf{Fp}\left(\mathsf{Rp}(F_1)\right), \mathsf{Fp}\left(\mathsf{Rp}(F_2)\right)\big) \leq \mathcal{D}_{\mathsf{var}}(F_1, F_2)$$

Combining this with Equation (3), we get the following Proposition for bounding the variation distance.

Proposition 2 (Tracepipe Variation Distance Bounds)**.** *Let F_1 and F_2 be tracepipes, and let $\lceil \mathsf{Rp}(F_1)\rceil$ and $\lceil \mathsf{Rp}(F_2)\rceil$ be over-approximations of the reachpipes $\mathsf{Rp}(F_1)$ and $\mathsf{Rp}(F_2)$. We have*

$$\mathcal{D}_{\min}\Big(\,\mathsf{Fp}\left(\lceil \mathsf{Rp}(F_1)\rceil\right), \mathsf{Fp}\left(\lceil \mathsf{Rp}(F_2)\rceil\right)\Big) \;\leq\; \mathcal{D}_{\mathsf{var}}(F_1, F_2)$$

$$\mathcal{D}_{\mathsf{var}}(F_1, F_2) \;\leq\; \mathcal{D}_{\mathsf{var}}\Big(\,\mathsf{Fp}\left(\lceil \mathsf{Rp}(F_1)\rceil\right), \mathsf{Fp}\left(\lceil \mathsf{Rp}(F_2)\rceil\right)\Big) \quad \square$$

Remark: Hausdorff Metric. A natural candidate for under-approximating the variation distance is the *Hausdorff* set metric, defined as:

$$\mathcal{D}_H(F_1, F_2) = \max\left\{ \sup_{f_1 \in F_1} \inf_{f_2 \in F_2} \mathcal{D}(f_1, f_2)\, ,\ \sup_{f_2 \in F_2} \inf_{f_1 \in F_1} \mathcal{D}(f_1, f_2) \right\} \tag{5}$$

Intuitively, if $\sup_{f_1 \in F_1} \inf_{f_2 \in F_2} \mathcal{D}(f_1, f_2)$ is less than δ, then given any trace $f_1 \in F_1$, there exists a trace $f_2 \in F_2$ such that $\mathcal{D}(f_1, f_2) < \delta$. Note that $\sup_{f_1 \in F_1} \inf_{f_2 \in F_2} \mathcal{D}(f_1, f_2) \leq \mathcal{D}_{\mathsf{var}}(F_1, F_2)$ and also $\sup_{f_2 \in F_2} \inf_{f_1 \in F_1} \mathcal{D}(f_1, f_2) \leq \mathcal{D}_{\mathsf{var}}(F_1, F_2)$, thus, we have

$$\mathcal{D}_H(F_1, F_2) \leq \mathcal{D}_{\mathsf{var}}(F_1, F_2) \tag{6}$$

Thus, on first glance, the Hausdorff metric appears to be a good candidate for under-approximating the variation distance. As mentioned earlier, obtaining tracepipe sets is usually not possible; we have to work with over or under-approximations obtained by way of reachpipes. Unfortunately, there is no obvious relationship between $\mathcal{D}_H(A, B)$ and $\mathcal{D}_H(A', B')$ for $A \subseteq A'$ and $B \subseteq B'$. This can be seen pictorially in Figure 1. The sets A, B, A', B' are subsets of the interval $[0, 10]$. In the first case, we have $\mathcal{D}_H(A, B) > \mathcal{D}_H(A', B')$ and in the second, $\mathcal{D}_H(A, B) < \mathcal{D}_H(A', B')$.

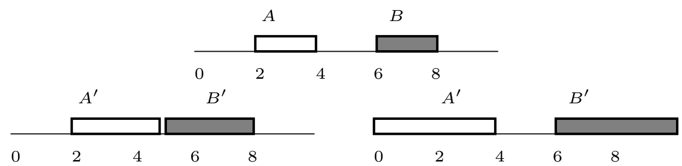

Figure 1: Sets A, B, and two cases of $A \subseteq A'$, $B \subseteq B'$

Thus, we cannot use the reachpipe over-approximations $\lceil \mathsf{Rp}(F_1)\rceil$ and $\lceil \mathsf{Rp}(F_2)\rceil$ to get a lower (or upper) bound on $\mathcal{D}_H(F_1, F_2)$. This problem occurs even even in the case of exact reachpipes $\mathsf{Rp}(F_1), \mathsf{Rp}(F_2)$ as we may have $F_1 \subsetneq \mathsf{Fp}\left(\mathsf{Rp}(F_1)\right)$ and $F_2 \subsetneq \mathsf{Fp}\left(\mathsf{Rp}(F_2)\right)$

For the special case where $F_1 = \{f_1\}$ is a singleton set, we have

$$\mathcal{D}_H(\{f_1\}, F_2) = \mathcal{D}_{\mathsf{var}}(\{f_1\}, F_2) \tag{7}$$

Thus, in case of a singleton $F_1 = \{f_1\}$, the value $\mathcal{D}_H\big(\,\mathsf{Fp}\left(\mathsf{Rp}(F_1)\right), \mathsf{Fp}\left(\mathsf{Rp}(F_2)\right)\big)$ is equal to the RHS of Equation (3), and hence only gives an upper bound on $\mathcal{D}_{\mathsf{var}}(F_1, F_2)$.

We note that even if we under-approximate the reach sets to obtain $\mathsf{Fp}\left(\lfloor \mathsf{Rp}(F_1)\rfloor\right)$, and $\mathsf{Fp}\left(\lfloor \mathsf{Rp}(F_2)\rfloor\right)$, we still do not have a lower bound for the Hausdorff distance as we cannot tell in which direction the distance changes on taking subsets (Figure 1). In addition, we may have $F \subsetneq \mathsf{Fp}\left(\lfloor \mathsf{Rp}(F)\rfloor\right)$ as for a traceset F, as $\mathsf{Fp}\left(\mathsf{Rp}(F)\right)$ over-approximate F, and competes with the fact that $\lfloor \mathsf{Rp}(F)\rfloor$ under-approximates $\mathsf{Rp}(F)$.

3.3 Constructing Reachpipes

For most dynamical systems, one cannot get a closed-form representation for the set of all traces. However, reachpipe sets can be over/under-approximated at desired timepoints using analytic techniques [7, 17, 11, 13, 15, 10, 20, 8, 6]. The procedure for bounding the tracepipe variation distance in this paper operates on reachpipes (the bounding quantities are as in Proposition 2). As a result it is necessary to choose an appropriate representation of reachpipes so that the distance computation procedure remains tractable.

Reachpipe Completion. Typically, reachset computation tools give us reach sets at sampled time-points, *i.e.*, the tools give us reachpipe samples $R(t_0), \ldots, R(t_m)$ at discrete time-points $t_0, \ldots, t_m$. We need to "complete" the reachpipes for intermediate time values. We do this completion by generalizing linear interpolation using scaling and Minkowski sums. Specifically, we define an over-approximated completion of R in between t_k, t_{k+1} as follows for $t_k \leq t \leq t_{k+1}$:

$$\lceil R\rceil(t) = \left\{ \boldsymbol{p} + \frac{t - t_k}{t_{k+1} - t_k} \cdot (\boldsymbol{q} - \boldsymbol{p}) \;\middle|\; \boldsymbol{p} \in R(t_k) \text{ and } \boldsymbol{q} \in R(t_{k+1}) \right\}.$$

For a set $A \subseteq \mathbb{R}^d$, given $\lambda \in \mathbb{R}$, let $\lambda \cdot A$ denote $\{\lambda \cdot \boldsymbol{p} \mid \boldsymbol{p} \in A\}$. The Minkowski sum of two sets A, B is defined as

$A+B=\{\boldsymbol{p}+\boldsymbol{q} \mid \boldsymbol{p}\in A \text{ and } \boldsymbol{q}\in B\}$. We also denote $-1\cdot A$ by $-A$. Under this notation, we have

$$\lceil R\rceil(t) = R(t_k) + \frac{t-t_k}{t_{k+1}-t_k}\cdot\left(R(t_{k+1})-R(t_k)\right). \quad (8)$$

Alternately, one can observe individual traces of the system at discrete times and complete the trace by linear interpolation at intermediate points. That is, suppose we observe a trace f at discrete points t_k and t_{k+1}: $f(t_k)=\boldsymbol{p}$ and $f(t_{k+1})=\boldsymbol{p}'$ and complete the trace as $f(t)=\boldsymbol{p}+\frac{t-t_k}{t_{k+1}-t_k}(\boldsymbol{p}'-\boldsymbol{p})$ for all points $t_k\leq t\leq t_{k+1}$. We explain why Equation (8) is an over-approximation for linearly interpolated completions of observed trace samples. Recall that

$$R(t)=\{\boldsymbol{p}\mid \text{ there exists some trace } f \text{ such that } f(t)=\boldsymbol{p}\}.$$

Under linear interpolation completion of traces, this set is

$$R(t)=\left\{\boldsymbol{p}+\frac{t-t_k}{t_{k+1}-t_k}\cdot(\boldsymbol{q}-\boldsymbol{p}) \;\middle|\; \begin{array}{l}\text{there exists a trace } f\\ \text{such that } f(t_k)=\boldsymbol{p} \text{ and}\\ f(t_{k+1})=\boldsymbol{q}\end{array}\right\} \quad (9)$$

In general $R(t)$ as defined in Equation (9) can be a strict subset of $\lceil R\rceil(t)$ as defined in Equation (8). For an example,

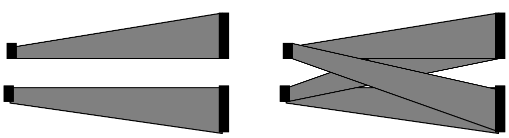

Figure 2: Reachpipe Completion (i) $R(t)$**; (ii)** $\lceil R\rceil(t)$

see Figure 2, where $R(t_k)\subseteq\mathbb{R}$ and $R(t_{k+1})\subseteq\mathbb{R}$ are the disjoint black line segments at the ends, and the shaded portions are the completions for $t\in(t_k,t_{k+1})$. The left side shows $R(t)$. The traces evolve from the top (resp. bottom) left black bars to the top (resp. bottom) right black bars. The figure on the right shows that $\lceil R\rceil$ over-approximates by assuming traces from the top left black bar to the bottom right black bar (and similarly from the bottom left bar). The strict inclusion can hold even if $R(t_k)$ and $R(t_{k+1})$ are convex sets.

Reachpipe Sample Sets. We now look at choosing appropriate forms of reachpipe sample sets $R(t_k)$. In hybrid systems literature the common forms of reach sets are (i) ellipsoids [17], (ii) support functions [15], (iii) zonotopes [11, 12], (iv) polyhedra and polytopes [10, 16, 7, 20, 20, 8], (v) polynomial approximations [19, 6].

In this work we use convex polytopes as reachpipe sample sets. A *polyhedron* is specified as: $A\cdot\boldsymbol{x}\leq\boldsymbol{b}$, where A is a $n\times d$ real-valued matrix, $\boldsymbol{x}=[x_1,\ldots,x_d]^\top$ is a column vector of d variables, $\boldsymbol{b}=[b_1,\ldots,b_d]^\top$ is a column vector with $b_k\in\mathbb{R}$ for every k, and "$\cdot$" denotes the standard matrix product. The polyhedron $A\cdot\boldsymbol{x}\leq\boldsymbol{b}$ consists of all points $(p_1,\ldots,p_d)\in\mathbb{R}^d$ such that for all $1\leq i\leq n$, we have $\sum_{k=1}^d A_{i,k}\cdot p_k\leq b_k$. A polyhedron is thus the intersection of n halfspaces, namely, the halfspaces $\sum_{k=1}^d A_{i,k}\cdot x_k\leq b_k$ for $1\leq i\leq n$. We use $\boldsymbol{a}_i\cdot\boldsymbol{x}\leq b_i$ as a shorthand to denote the i-th halfspace, where $\boldsymbol{a}_i$ is the i-th row vector of A. A *polytope* is a bounded polyhedron. Polytopes can also be specified as convex hulls of a finite set of points [14] (unfortunately, polynomial time algorithms are not known to obtain one representation from the other [4]). We use the halfspace representation as it has been shown to be amenable to computing over-approximations of reach sets of hybrid systems using the template polyhedra approach [16, 7, 20, 20, 8], in which the reachsets at sampled timepoints are over-approximated by polytopes by varying the constants in $\boldsymbol{b}$ (the matrix A stays unchanged). Zonotopes are special forms of polytopes, the algorithms developed in this work are also applicable for these special polytopes.

We note the property that if $R(t_k)$ and $R(t_{k+1})$ are polytopes (resp. zonotopes) in Equation (8), the completions $\lceil R\rceil(t)$ for every t are also polytopes (resp. zonotopes). This follows from the facts that for P_1 and P_2 polytopes (resp. zonotopes), (i) $\lambda\cdot P_1$ and $\lambda\cdot P_2$ are polytopes (resp. zonotopes) for λ a constant; and (ii) the Minkowski sum P_1+P_2 is also a polytope (resp. zonotope) [14].

Polygonal Polytope-Reachpipe (PPR). A *polygonal polytope-reachpipe* (PPR) is a reachpipe specified by reachpipe time-samples $R(0),\ldots R(m)$, such that for $k\in\{0,1,\ldots m-1\}$ (a) each $R(k)$ is a polytope in $\mathbb{R}^{d+1}$; and (b) $R(t)$ for $k<t<k+1$ is taken to be the linear interpolation as specified in Equation (8). Note that we take the reachpipe samples to occur at integer parameter values, this is WLOG as the actual time value can be added as an extra dimension as discussed in Subsection 2.2 with a slight modification: for a polygonal trace f consisting of affine segments starting at times $t_0,t_1,\ldots$, we let the corresponding (polygonal) time-explicit trace C be such that $C(k)=(f(t_k),t_k)$ for $k\in\{0,1,\ldots m\}$ (for non-integer $\rho\in[0,m]$, the trace C is specified by linear interpolation of the integer endpoints). Next, we study the variation distance between time-explicit PPRs with respect to the Fréchet trace metric in order to bound the Skorokhod distance between the corresponding tracepipes.

4. FRÉCHET DISTANCES BETWEEN POLYTOPE-REACHPIPES

We now investigate computing the pipe variation distance bounds given in Proposition 2 in the case of the Skorokhod trace metric. As a first step, we show it suffices to consider the Fréchet metric as the trace metric in the pipe variation distance.

Consider the setting of Subsection 3.3, which presented linear interpolation completion of sampled trace values. The traces so obtained by completion are continuous. We can define corresponding time-explicit traces $C_f:[T_i^f,T_e^f]\to\mathbb{R}^d\times\mathbb{R}$ for the traces $f:[T_i^f,T_e^f]\to\mathbb{R}^d$ obtained by completing the time sampled traces by linear interpolation. This makes Proposition 1 applicable. Corresponding to a tracepipe F over $\mathbb{R}^d$, we can define a time-explicit tracepipe F^* over $\mathbb{R}^d\times\mathbb{R}$ with traces $f\in F$ corresponding to time-explicit traces C_f in F^*. We then have (referring to trace metrics $\mathcal{S}$ or $\mathcal{F}$ explicitly in the variation distance through the notation $\mathcal{D}_{\mathcal{S}\mathsf{var}}$ or $\mathcal{D}_{\mathcal{F}\mathsf{var}}$):

$$\begin{aligned}\mathcal{D}_{\mathcal{S}\mathsf{var}}(F_1,F_2) &= \sup_{f_1\in F_1, f_2\in F_2}\mathcal{D}_{\mathcal{S}}(f_1,f_2)\\ &= \sup_{\mathsf{C}_{f_1}\in F_1^*,\mathsf{C}_{f_2}\in F_2^*}\mathcal{D}_{\mathcal{F}}(\mathsf{C}_{f_1},\mathsf{C}_{f_2})\\ &= \mathcal{D}_{\mathcal{F}\mathsf{var}}(F_1^*,F_2^*)\end{aligned}$$

Thus we focus on computing the pipe variation distances with respect to the Fréchet trace metric.

In Section 3, we considered distances between *sets* of traces, and investigated bounding the variation distance between sets of traces (*i.e.*, between tracepipes) using over-approximate tracesets obtained through reachpipes. In the next two subsections, we define a notion of Fréchet distance *directly* on reachpipes, by viewing a reachpipe as a trace from $[0, T]$ to polytopes of $\mathbb{R}^{d+1}$.

Let R_1, R_2 be PPRs from $[0, m_1]$ and $[0, m_2]$ to polytopes over $\mathbb{R}^{d+1}$. Our objective is to bound the tracepipe variation distance with respect to the Fréchet trace metric. From Proposition 2, we need to compute (a) $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2))$ and (b) $\mathcal{D}_{\mathcal{F}_{\min}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2))$.

4.1 Variation Distance on PPRs

In this subsection, we consider $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2))$. Recall that this value is defined as:

$$\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)) = \sup_{f_1 \in \mathsf{Fp}(R_1), f_2 \in \mathsf{Fp}(R_2)} \mathcal{D}_{\mathcal{F}}(f_1, f_2) \quad (10)$$

We define a new variation distance on reachpipes as follows.

Definition 3. Let R_1, R_2 be PPRs from $[0, m_1]$ and $[0, m_2]$ to polytopes over $\mathbb{R}^{d+1}$, and let L be a given norm on $\mathbb{R}^{d+1}$. The reachpipe variation distance $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2)$ is defined as:

$$\inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\le\theta\le1} \max_{\substack{\boldsymbol{p}_1\in R_1(\alpha_1(\theta))\\ \boldsymbol{p}_2\in R_2(\alpha_2(\theta))}} \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_L \quad (11)$$

where α_1, α_2 range over continuous and strictly increasing bijective functions onto $[0, m_1]$ and $[0, m_2]$ respectively. □

Note that $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}$ is defined over *reachpipes* R, as compared to $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}$ which is defined over tracepipes F or $\mathsf{Fp}(R)$. Also note that for any reparameterizations α_1, α_2, the sets $R_1(\alpha_1(\theta))$ and $R_2(\alpha_2(\theta))$ are closed and bounded. Thus, $\max_{\boldsymbol{p}_1\in R_1(\alpha_1(\theta)),\ \boldsymbol{p}_2\in R_2(\alpha_2(\theta))} \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_L$ is well defined. The function $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}$, like the function $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}$, is not a metric (notably, we can have $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R, R) > 0$).

Informally, we go along the PPRs R_1 and R_2 according to our chosen reparameterizations α_1, α_2, and compare the *polytopes* $R_1(\alpha_1(\theta))$ and $R_2(\alpha_2(\theta))$ for each value of $0 \le \theta \le 1$. If we view a PPR R as a mapping from $[0, m]$ to the set of polytopes of $\mathbb{R}^{d+1}$, then Definition 3 seems similar to the definition of the Fréchet distance over traces (Definition 2), where we use the following function to compare polytopes P_1, P_2:

$$\Phi_{\max}(P_1, P_2) = \max_{\boldsymbol{p}_1\in P_1, \boldsymbol{p}_2\in P_2} \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_L \quad (12)$$

Using $\Phi_{\max}$, Equation (11) can be written as:

$$\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2) = \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\le\theta\le1} \Phi_{\max}\Big(R_1(\alpha_1(\theta)),\ R_2(\alpha_2(\theta))\Big) \quad (13)$$

The following theorem shows that $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2)$ over-approximates the tracepipe distance $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2))$.

Theorem 1. *Let R_1, R_2 be* PPR*s from $[0, m_1]$ and $[0, m_2]$ to polytopes over $\mathbb{R}^{d+1}$, and let L be a given norm on $\mathbb{R}^{d+1}$. We have*

$$\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2) \ \ge\ \mathcal{D}_{\mathcal{F}_{\mathsf{var}}}\big(\mathsf{Fp}(R_1),\ \mathsf{Fp}(R_2)\big)$$

where the tracepipe distance $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}\big(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)\big)$ is as defined in Equation (10), and the reachpipe distance $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2)$ is as defined in Definition 3.

Proof. Consider any $f_1 \in \mathsf{Fp}(R_1)$, and any $f_2 \in \mathsf{Fp}(R_2)$. We have

$$\mathcal{D}_{\mathcal{F}}(f_1, f_2) = \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\le\theta\le1} \|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\|_L$$

Observe that $f_j(\alpha_j(\theta)) \in R_j(\alpha_j(\theta))$ for $j \in \{1, 2\}$. Thus, for every $\alpha_1, \alpha_2, \theta$,

$$\|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\|_L \le \Phi_{\max}(R_1(\alpha_1(\theta)),\ R_2(\alpha_2(\theta)))$$

Thus, we have

$$\mathcal{D}_{\mathcal{F}}(f_1, f_2) \le \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\le\theta\le1} \Phi_{\max}\Big(R_1(\alpha_1(\theta)),\ R_2(\alpha_2(\theta))\Big)$$

That is, for every $f_1 \in \mathsf{Fp}(R_1)$ and $f_2 \in \mathsf{Fp}(R_2)$, we have $\mathcal{D}_{\mathcal{F}}(f_1, f_2) \le \mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2)$. This implies that $\sup_{f_1\in\mathsf{Fp}(R_1),\ f_2\in\mathsf{Fp}(R_2)} \mathcal{D}_{\mathcal{F}}(f_1, f_2) \le \mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}(R_1, R_2)$. □

The above theorem can be applied with $R_1 = \lceil \mathsf{Rp}(F_1) \rceil$ and $R_2 = \lceil \mathsf{Rp}(F_2) \rceil$ in order to obtain the upper bound in Proposition 2 using the reachpipe variation distance $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}$ between $\lceil \mathsf{Rp}(F_1) \rceil$ and $\lceil \mathsf{Rp}(F_2) \rceil$. We next consider the lower bound.

4.2 Minimum Distance on PPRs

We now consider $\mathcal{D}_{\mathcal{F}_{\min}}\left(F^{R_1}, F^{R_2}\right)$ for PPRs R_1, R_2 from $[0, m_1]$ and $[0, m_2]$ to polytopes over $\mathbb{R}^{d+1}$ respectively. This distance is defined as:

$$\mathcal{D}_{\mathcal{F}_{\min}}\big(\mathsf{Fp}(R_1),\ \mathsf{Fp}(R_2)\big) = \inf_{f_1\in\mathsf{Fp}(R_1),\ f_2\in\mathsf{Fp}(R_2)} \mathcal{D}_{\mathcal{F}}(f_1, f_2) \quad (14)$$

Analogous to the $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}$ function of Definition 3, we define a minimum set distance $\mathcal{D}_{\mathcal{F}_{\min}}$ over reachpipes. We use the following function to compare polytopes (given a norm L over $\mathbb{R}^{d+1}$):

$$\Phi_{\min}(P_1, P_2) = \min_{\boldsymbol{p}_1\in P_1,\ \boldsymbol{p}_2\in P_2} \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|_L \quad (15)$$

Using this function, we define $\mathcal{D}_{\mathcal{F}_{\min}}$ as follows.

Definition 4. Let R_1, R_2 be PPRs from $[0, m_1]$ and $[0, m_2]$ to polytopes over $\mathbb{R}^{d+1}$, and let $\Phi_{\min}$ be the polytope comparison function as described previously. The reachpipe minimum set distance $\mathcal{D}_{\mathcal{F}_{\min}}^{\dagger}(R_1, R_2)$ is defined as:

$$\mathcal{D}_{\mathcal{F}_{\min}}^{\dagger}(R_1, R_2) = \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\le\theta\le1} \Phi_{\min}\Big(R_1(\alpha_1(\theta)),\ R_2(\alpha_2(\theta))\Big) \quad (16)$$

where α_1, α_2 range over continuous and strictly increasing bijective functions onto $[0, m_1]$ and $[0, m_2]$ respectively. □

The following theorem shows that $\mathcal{D}_{\mathcal{F}_{\min}}^{\dagger}(R_1, R_2)$ is equal to the tracepipe distance $\mathcal{D}_{\mathcal{F}_{\min}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2))$. The proof of the theorem can be found in the Appendix.

Theorem 2. *Let R_1, R_2 be* PPR*s from $[0, m_1]$ and $[0, m_2]$ to polytopes over $\mathbb{R}^{d+1}$, and let L be a given norm on $\mathbb{R}^{d+1}$. We have*

$$\mathcal{D}_{\mathcal{F}_{\min}}^{\dagger}(R_1, R_2) \ =\ \mathcal{D}_{\mathcal{F}_{\min}}\big(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)\big)$$

where the tracepipe distance $\mathcal{D}_{\mathcal{F}_{\min}}\big(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)\big)$ is as defined in Equation (14), and the reachpipe distance $\mathcal{D}_{\mathcal{F}_{\min}}^{\dagger}(R_1, R_2)$ is as defined in Definition 4. □

Theorems 1 and 2 allow us to bound to the tracepipe variation distance $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}$ using the reachpipe distances $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}^{\dagger}$ and $\mathcal{D}_{\mathcal{F}_{\min}}^{\dagger}$ that were defined in the current section. In the next section we present algorithms for computing these two reachpipe distances over PPRs.

5. FRÉCHET DISTANCES BETWEEN POLYTOPE-TRACES

Theorems 1 and 2 show that the distance functions $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}$ and $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}$ over PPRs can be used to bound the tracepipe distances $\mathcal{D}_{\mathcal{F}_{\mathsf{var}}}$ and $\mathcal{D}_{\mathcal{F}_{\min}}$. We now present procedures for computing $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}$ and $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}$ as follows. In Subsection 5.1 we extend the geometric free space concept used in [3, 18] to compute the Fréchet distance between two traces to the case of PPRs, and show how the PPR distance decision problem can be reduced to a two-dimensional reachability problem. In Subsection 5.2 we present algorithms for the reachability problems corresponding to $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}$ and $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}$.

5.1 The Free Space for Polytope-Traces

Let $\mathsf{PTopes}(\mathbb{R}^{d+1})$ denote the set of all polytopes in $\mathbb{R}^{d+1}$. A PPR R defined over the time interval $[0, m]$ can be viewed as a polytope-trace, defined as a function from $[0, m]$ to $\mathsf{PTopes}(\mathbb{R}^{d+1})$. Recall that a PPR R is specified by reachpipe time-samples $R(0), \dots R(m)$, such that for $k \in \{0, 1, \dots m-1\}$ the portion of R in between $(k, k+1)$ is assumed to be completed according to linear interpolation using $R(k)$ and $R(k+1)$. We denote this portion of R between $R(k)$ and $R(k+1)$ as $R^{[k]}$, *i.e.*, the portion of R defined over $k \leq t \leq k+1$.

Alt and Godau introduced *free spaces* [3] to compute the Fréchet distance between piecewise affine and continuous curves in $\mathbb{R}^d$. We show free spaces can also be used to compute the functions $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}$ and $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}$. First, we show how to extend free spaces to the domain of PPRs.

Definition 5 (Free Space)**.** Given PPRs $R_1 : [0, m_1] \to \mathsf{PTopes}(\mathbb{R}^{d+1})$ and $R_2 : [0, m_2] \to \mathsf{PTopes}(\mathbb{R}^{d+1})$, a real number $\delta \geq 0$, and a polytope comparison function $\Phi : \mathsf{PTopes}(\mathbb{R}^{d+1}) \times \mathsf{PTopes}(\mathbb{R}^{d+1}) \to \mathbb{R}_+$, the *$\delta$-Free Space* of R_1, R_2 with respect to Φ is defined as the set $\mathsf{Free}^{\Phi}_{\delta}(R_1, R_2) =$

$$\left\{(\rho_1, \rho_2) \in [0, m_1] \times [0, m_2] \;\middle|\; \Phi\Big(R_1(\rho_1), R_2(\rho_2)\Big) \leq \delta\right\} \quad \square$$

The free space for PPRs serves a similar role as in the case of the free space for traces. The tuples (ρ_1, ρ_1) belonging to $\mathsf{Free}^{\Phi}_{\delta}(R_1, R_2)$ denote the positions in the two reparameterizations such that the Φ value for those position pairs is at most δ. Thus $\mathsf{Free}^{\Phi}_{\delta}(R_1, R_2)$ collects the pairs (ρ_1, ρ_2) which could be used in valid reparameterizations of Definition 3 or 4. A pictorial representation of the free space is referred to as the *free space diagram*. The space $[0, m_1] \times [0, m_2]$ can be viewed as consisting of $m_1 m_2$ *cells*, with cell i, j being $[i, i+1] \times [j, j+1]$ for $0 \leq i < m_1$, and $0 \leq j < m_1$. Observe that $\mathsf{Free}^{\Phi}_{\delta}(R_1, R_2)$ intersected with cell i, j is just the free space corresponding to the PPR segments $R_1^{[i]}, R_2^{[j]}$; *i.e.*, the intersection of the cell i, j with $\mathsf{Free}^{\Phi}_{\delta}(R_1, R_2)$ is equal to $\mathsf{Free}^{\Phi}_{\delta}(R_1^{[i]}, R_2^{[j]})$.

Proposition 3 (Free Space & Reparameterizations)**.** *Given two* PPR*s R_1, R_2 from $[0, m_1]$ and $[0, m_2]$ to $\mathsf{PTopes}(\mathbb{R}^{d+1})$, we have $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}(R_1, R_2) \leq \delta$ (resp., $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}(R_1, R_2) \leq \delta$) iff there is a non-decreasing (in both dimensions) curve $\alpha : [0, 1] \to [0, m_1] \times [0, m_2]$ in $\mathsf{Free}^{\Phi_{\max}}_{\delta}(R_1, R_2)$ (resp. $\mathsf{Free}^{\Phi_{\min}}_{\delta}(R_1, R_2)$) from $(0, 0)$ to (m_1, m_2).* $\square$

The curve α can be thought of as a pair of parameterized curves (α_1, α_2), with $\alpha_1 : [0, 1] \to [0, m_1]$ and $\alpha_2 : [0, 1] \to [0, m_2]$. The functions α_1, α_2 can be viewed as the reparameterization functions in Definitions 3 and 4. The general shape of the free space for two PPRs is depicted in Figure 3. The unshaded portion is the free space. The figure also includes a continuous curve which is non-decreasing in both coordinates, from $(0, 0)$ to (m_1, m_2).

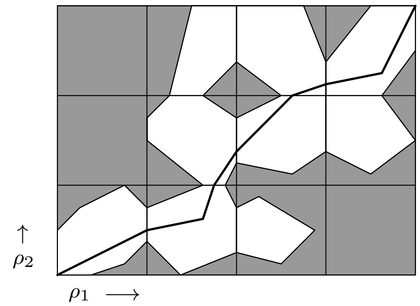

Figure 3: The Free Space $\mathsf{Free}^{\Phi}_{\delta}(R_1, R_2)$.

Note that the curve α (and hence also each of α_1, α_2) in Proposition 3 is non-decreasing; whereas the reparameterizations in Definitions 3 and 4 are strictly increasing. This is to account for the fact that optimal reparameterizations in Definitions 3 and 4 might not exist, as we have an "inf". It can be shown that $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}$ and $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}$ values do not change over PPRs if we allow non-decreasing reparameterizations since PPRs change smoothly due to the linear interpolation scheme. This issue also arises in the case of traces, and is discussed (for the case of traces) in more detail in [18]. We omit the technicalities, and henceforth assume that non-decreasing reparameterizations are allowed in Definitions 3 and 4.

5.2 The Polytope-Trace $\mathcal{D}^{\dagger}$ Decision Problems

In this section, we solve for the decision problems $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}(R_1, R_2) \leq \delta$ and $\mathcal{D}_{\mathcal{F}^{\dagger}_{\min}}(R_1, R_2) \leq \delta$, given a $\delta \geq 0$ and PPRs R_1, R_2. We use the free space reduction of Proposition 3 for these decision problems. The first step in this procedure is to compute the free space. Towards this step, we first show that the free spaces for the polytope comparison functions $\Phi_{\min}$ and $\Phi_{\max}$ are convex in individual cells of the free space diagram. This is done in Subsection 5.2.1. Using this convexity property, we show in Subsection 5.2.2 that in order to obtain the free space of a cell, it suffices to obtain the free space at the cell boundaries. We obtain algorithms to compute the free space cell boundaries in Subsection 5.2.3 (for $\Phi_{\min}$), and in 5.2.4 (for $\Phi_{\max}$). The procedure of Subsection 5.2.4 has a high time complexity, we present a polynomial time algorithm which works in case the PPRs satisfy certain conditions in Subsection 5.2.5. The results of the section are summarized in Propositions 5, 6 and 8.

5.2.1 Convexity of Free Space

The following lemma proves that the free space in the first cell (over $[0, 1] \times [0, 1]$) is convex for both the set comparison functions $\Phi_{\min}$ and $\Phi_{\max}$. Other cells are translations and have a similar proof.

Lemma 1 (Convexity of Free Space of Individual Cells)**.** *Let P^0_a, P^1_a, and P^0_b, P^1_b be polytopes in $\mathbb{R}^{d+1}$. Let $R_a : [0, 1] \to \mathsf{PTopes}(\mathbb{R}^{d+1})$ and $R_b : [0, 1] \to \mathsf{PTopes}(\mathbb{R}^{d+1})$ be (single-segment)* PPR*s constructed from the polytopes P^0_a, P^1_a and P^0_b, P^1_b respectively, via linear interpolation (as described in Equation* (8)*), taking $P^0_a = R_a(0)$ and $P^1_a = R_a(1)$ and $P^0_b = R_b(0)$ $P^1_b = R_b(1)$, respectively.*

The free space of R_a, R_b given a $\delta \geq 0$ for both

$\Phi_{\min}$ and $\Phi_{\max}$ is convex. That is, $\mathsf{Free}_\delta^{\Phi_{\min}}(R_a, R_b)$ *and* $\mathsf{Free}_\delta^{\Phi_{\max}}(R_a, R_b)$ *are both convex sets.*

Proof. Let Φ be $\Phi_{\min}$ or $\Phi_{\max}$. Suppose two points (in $[0,1] \times [0,1]$) belong to $\mathsf{Free}_\delta^{\Phi}(R_a, R_b)$. Let these points be $\rho = (\rho_a, \rho_b)$ and $\rho' = (\rho'_a, \rho'_b)$. We show that for any $0 \le \lambda \le 1$, the point $\rho^* = \lambda \cdot \rho + (1-\lambda) \cdot \rho'$ also belongs to $\mathsf{Free}_\delta^{\Phi}(R_a, R_b)$. The point ρ^* is the tuple

$$(\rho^*_a, \rho^*_b) = \big(\lambda \cdot \rho_a + (1-\lambda) \cdot \rho'_a \ , \ \lambda \cdot \rho_b + (1-\lambda) \cdot \rho'_b\big). \quad (17)$$

To show $(\rho^*_a, \rho^*_b) \in \mathsf{Free}_\delta^{\Phi}(R_a, R_b)$, we need to show that

$$\Phi\big(R_a(\rho^*_a), R_b(\rho^*_b)\big) \le \delta \quad (18)$$

We show this individually for $\Phi_{\min}$ and $\Phi_{\max}$.

(1) $\boldsymbol{\Phi_{\min}}$.
By the definition of $\Phi_{\min}$ (Equation (15)), and the facts that (ρ_a, ρ_b) and (ρ'_a, ρ'_b) are in $\mathsf{Free}_\delta^{\Phi_{\min}}(R_a, R_b)$, we have that:

- There exist points $\boldsymbol{p}_a \in R_a(\rho_a)$ and $\boldsymbol{p}_b \in R_b(\rho_b)$ such that $\|\boldsymbol{p}_a - \boldsymbol{p}_b\| \le \delta$.
- There exist points $\boldsymbol{p}'_a \in R_a(\rho'_a)$ and $\boldsymbol{p}'_b \in R_b(\rho'_b)$ such that $\|\boldsymbol{p}'_a - \boldsymbol{p}'_b\| \le \delta$.

Consider the points $\boldsymbol{p}^*_a = \lambda \cdot \boldsymbol{p}_a + (1-\lambda) \cdot \boldsymbol{p}'_a$; and $\boldsymbol{p}^*_b = \lambda \cdot \boldsymbol{p}_b + (1-\lambda) \cdot \boldsymbol{p}'_b$ (where λ is the same value as that used in Equation (17)). We have

$$\begin{aligned}
\|\boldsymbol{p}^*_a - \boldsymbol{p}^*_b\| &= \Big\|\Big(\lambda \cdot \boldsymbol{p}_a + (1-\lambda) \cdot \boldsymbol{p}'_a\Big) - \Big(\lambda \cdot \boldsymbol{p}_b + (1-\lambda) \cdot \boldsymbol{p}'_b\Big)\Big\| \\
&= \big\|\lambda \cdot (\boldsymbol{p}_a - \boldsymbol{p}_b) + (1-\lambda) \cdot \big(\boldsymbol{p}'_a - \boldsymbol{p}'_b\big)\big\| \\
&\le \lambda \cdot \|\boldsymbol{p}_a - \boldsymbol{p}_b\| + (1-\lambda) \cdot \big\|\boldsymbol{p}'_a - \boldsymbol{p}'_b\big\| \\
&\qquad\qquad \text{(by basic norm properties)} \\
&\le \lambda \cdot \delta + (1-\lambda) \cdot \delta \\
&= \delta
\end{aligned}$$

We now show $\boldsymbol{p}^*_a \in R_a(\rho^*_a)$, and $\boldsymbol{p}^*_b \in R_b(\rho^*_b)$ Observe that the polytope $R_a(\rho^*_a)$ which is defined to be the polytope

$$\begin{aligned}
&R_a(0) + \rho^*_a \cdot (R_a(0) - R_a(1)) \\
&= R_a(0) + \Big(\lambda \cdot \rho_a + (1-\lambda) \cdot \rho'_a\Big) \cdot (R_a(0) - R_a(1)) \\
&= \lambda \cdot (R_a(0) + \rho_a \cdot (R_a(0) - R_a(1))) \ + \\
&\qquad\qquad (1-\lambda) \cdot \big(R_a(0) + \rho'_a \cdot (R_a(0) - R_a(1))\big) \\
&= \lambda \cdot R_a(\rho_a) + (1-\lambda) \cdot R_a(\rho'_a)
\end{aligned} \quad (19)$$

Thus, $R_a(\rho^*_a)$ equals the polytope $\lambda \cdot R_a(\rho_a) + (1-\lambda) \cdot R_a(\rho'_a)$. Since $\boldsymbol{p}^*_a = \lambda \cdot \boldsymbol{p}_a + (1-\lambda) \cdot \boldsymbol{p}'_a$ for $\boldsymbol{p}_a \in R_a(\rho_a)$ and $\boldsymbol{p}'_a \in R_a(\rho'_a)$, this means that $\boldsymbol{p}^*_a \in R_a(\rho^*_a)$. Similarly, $\boldsymbol{p}^*_b \in R_b(\rho^*_b)$. Since we have demonstrated that $\|\boldsymbol{p}^*_a - \boldsymbol{p}^*_b\| \le \delta$, this means that $\Phi_{\min}(R_a(\rho^*_a), R_b(\rho^*_b)) \le \delta$. This shows that Equation (18) holds for $\Phi_{\min}$.

(2) $\boldsymbol{\Phi_{\max}}$.
Now we show that Equation (18) holds for $\Phi_{\max}$. By the definition of $\Phi_{\max}$ (Equation (12)), and the facts that (ρ_a, ρ_b) and (ρ'_a, ρ'_b) are in $\mathsf{Free}_\delta^{\Phi_{\min}}(R_a, R_b)$, we have that:

- For all points $\boldsymbol{p}_a \in R_a(\rho_a)$ and $\boldsymbol{p}_b \in R_b(\rho_b)$ we have that $\|\boldsymbol{p}_a - \boldsymbol{p}_b\| \le \delta$.
- For all points $\boldsymbol{p}'_a \in R_a(\rho'_a)$ and $\boldsymbol{p}'_b \in R_b(\rho'_b)$ we have that $\|\boldsymbol{p}'_a - \boldsymbol{p}'_b\| \le \delta$.

Consider any point $\boldsymbol{p}^*_a$ which belongs to $R_a(\rho^*_a)$ and any point $\boldsymbol{p}^*_b$ which belongs to $R_b(\rho^*_b)$. By Equation (19), we have $R_a(\rho^*_a) = \lambda \cdot R_a(\rho_a) + (1-\lambda) \cdot R_a(\rho'_a)$; and similarly for $R_b(\rho^*_b)$ Thus, by definition,

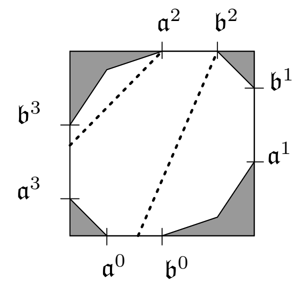

Figure 4: Cell Crossing with Non-Decreasing Curves.

- $\boldsymbol{p}^*_a = \lambda \cdot \boldsymbol{p}_a + (1-\lambda) \cdot \boldsymbol{p}'_a$ for some $\boldsymbol{p}_a \in R_a(\rho_a)$ and $\boldsymbol{p}'_a \in R_a(\rho'_a)$; and
- $\boldsymbol{p}^*_b = \lambda \cdot \boldsymbol{p}_b + (1-\lambda) \cdot \boldsymbol{p}'_b$ for some $\boldsymbol{p}_b \in R_a(\rho_b)$ and $\boldsymbol{p}'_b \in R_b(\rho'_b)$

It can be shown (as in the $\Phi_{\min}$ case) using the above two facts that $\|\boldsymbol{p}^*_a - \boldsymbol{p}^*_b\| \le \delta$. That is, we have that for any point $\boldsymbol{p}^*_a \in R_a(\rho^*_a)$, and any point $\boldsymbol{p}^*_b \in R_b(\rho^*_b)$, the value $\|\boldsymbol{p}^*_a - \boldsymbol{p}^*_b\|$ does not exceed δ. This means that

$$\sup_{\boldsymbol{p}^*_a \in R_a(\rho^*_a),\, \boldsymbol{p}^*_b \in R_b(\rho^*_b)} \|\boldsymbol{p}^*_a - \boldsymbol{p}^*_b\| \le \delta$$

Thus, $\Phi_{\max}(R_a(\rho^*_a), R_b(\rho^*_b)) \le \delta$. This shows that Equation (18) holds also for $\Phi_{\max}$ (in addition to $\Phi_{\min}$). ☐

5.2.2 Computing the Free Space

The convexity demonstrated by Lemma 1 simplifies the problem of computing a non-decreasing curve in the free space. As a result of the convexity of the free space for a cell, it suffices to only compute the free space boundaries at the cell boundaries. We refer to Figure 4. The dotted lines are example non-decreasing curves that cross the cell. As can be seen, to check if we can go from the left free space boundary to the top free space boundary of the cell, we only need the top free space boundary (and the precondition that the left free space boundary is non-empty). A similar situation arises for checking traversal from the bottom to top or bottom to right boundaries via non-decreasing curves. Convexity makes the internal shape of the free space inside a cell irrelevant. Invoking convexity again, we actually only need to compute the points $\mathfrak{a}^k, \mathfrak{b}^k$ for $k \in \{0, 3\}$. We present the computation procedure next.

We compute the bottom free space boundaries of cells (the other boundaries have similar algorithmic solutions). We need to compute the points $\mathfrak{a}^0, \mathfrak{b}^0$ in Figure 4. We do this for the first cell (over $[0,1] \times [0,1]$), other cells are translations and are similar. The point $\mathfrak{a}^0 = \langle \lambda^{\min}, 0 \rangle$, and the point $\mathfrak{b}^0 = \langle \lambda^{\max}, 0 \rangle$ for some $\lambda^{\min}$ and $\lambda^{\max}$ in $[0,1]$. It hence suffices to compute $\lambda^{\min}$ and $\lambda^{\max}$. We solve for $\lambda^{\min}$ (the solution for $\lambda^{\max}$ is similar) . This value $\lambda^{\min}$ is the solution of the following optimization problem (where $R_1(0), R_1(1), R_2(0)$ are given polytope samples of PPRs R_1 and R_2) :

$$\begin{aligned}
&\text{minimize} && \lambda \\
&\text{subject to} && \Phi\left(R_1(\lambda), R_2(0)\right) \le \delta \\
& && 0 \le \lambda \le 1
\end{aligned}$$

Expanding $R_1(\lambda)$, we get:

$$\begin{aligned}
&\text{minimize} && \lambda \\
&\text{subject to} && \Phi\big(\lambda \cdot R_1(0) + (1-\lambda) \cdot R_1(1),\ R_2(0)\big) \le \delta \\
& && 0 \le \lambda \le 1
\end{aligned} \quad (20)$$

The solution to the above problem depends on the function Φ. We solve each case $\Phi_{\min}$ and $\Phi_{\max}$ individually.

5.2.3 Free Space Cell Boundaries for $\Phi_{\min}$

In this subsection, we compute the bottom free space boundary of the first cell (over $[0,1]\times[0,1]$). The optimization problem (20) for $\Phi=\Phi_{\min}$ has the same solution as:

$$\begin{aligned}&\text{minimize} \quad \lambda\\ &\text{such that} \quad \begin{array}{l}\exists \text{ point } \boldsymbol{p}\in\lambda\cdot R_1(0)+(1-\lambda)\cdot R_1(1),\\ \exists \text{ point } \boldsymbol{q}\in R_2(0)\end{array}\\ &\qquad\qquad\qquad \text{s.t. } \|\boldsymbol{p}-\boldsymbol{q}\|\le\delta\\ &\qquad 0\le\lambda\le 1\end{aligned}$$

Let $R_1(0)$ be the polytope $A_1^0\cdot\boldsymbol{x}\le\boldsymbol{b}_1^0$, $R_1(1)$ be the polytope $A_1^1\cdot\boldsymbol{x}\le\boldsymbol{b}_1^1$, and $R_2(0)$ be the polytope $A_2\cdot\boldsymbol{x}\le\boldsymbol{b}_2$; where the As are $n\times(d+1)$ matrices of given constants, and $\boldsymbol{b}$s are column vectors of size $d+1$ containing given constants; and $\boldsymbol{x}$s are column vectors of variables. The previous optimization problem can be stated using these polytopes as:

$$\begin{aligned}\text{minimize} \quad & \lambda\\ \text{subject to} \quad & \left\|\lambda\cdot\boldsymbol{x}^0+(1-\lambda)\cdot\boldsymbol{x}^1-\boldsymbol{y}\right\|\le\delta\\ & A_1^0\cdot\boldsymbol{x}^0\le\boldsymbol{b}_1^0\\ & A_1^1\cdot\boldsymbol{x}^1\le\boldsymbol{b}_1^1\\ & A_2\cdot\boldsymbol{y}\le\boldsymbol{b}_2\\ & 0\le\lambda\le 1\end{aligned}\tag{21}$$

The optimization above is over the variables $\lambda,\boldsymbol{x}^0,\boldsymbol{x}^1,\boldsymbol{y}$. The values for $A_1^0,A_1^1,A_2,\boldsymbol{b}_1^0,\boldsymbol{b}_1^1,\boldsymbol{b}_2,\delta$ are given. We would like to reduce the problem to Linear Programming (LP), however we note that, as stated, the problem is an instance of quadratic programming due to the multiplication of the parameter λ with parameter column vectors $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$. We show that these multiplicative constraints can be removed. Towards this, we need the following lemma.

Lemma 2. *Suppose $A\cdot\boldsymbol{x}\le\boldsymbol{b}$ is a non-empty polytope in $\mathbb{R}^{d+1}$ and $\boldsymbol{b}\ne\boldsymbol{0}$. Then $A\cdot\boldsymbol{x}\le\boldsymbol{0}$ either has no solution, or contains the only point $\boldsymbol{x}=\boldsymbol{0}$.* □

Using the above lemma, the following result can be shown (the proof is in the Appendix).

Lemma 3. *Let $A_1^0\cdot\boldsymbol{x}^0\le\boldsymbol{b}_1^0$, and $A_1^1\cdot\boldsymbol{x}^1\le\boldsymbol{b}_1^1$, and $A_2\cdot\boldsymbol{y}\le\boldsymbol{b}_2$ be non-empty polytopes in $\mathbb{R}^{d+1}$. The following optimization problem has the same solution as Problem (21).*

$$\begin{aligned}\textit{minimize} \quad & \lambda\\ \textit{subject to} \quad & \left\|\boldsymbol{z}^0+\boldsymbol{z}^1-\boldsymbol{y}\right\|\le\delta\\ & A_1^0\cdot\boldsymbol{z}^0\le\lambda\cdot\boldsymbol{b}_1^0\\ & A_1^1\cdot\boldsymbol{z}^1\le(1-\lambda)\cdot\boldsymbol{b}_1^1\\ & A_2\cdot\boldsymbol{y}\le\boldsymbol{b}_2\\ & 0\le\lambda\le 1\end{aligned}\tag{22}$$

□

We thus can take $\lambda_{\min}$ to be the solution of the optimization problem (22). Consider the norms $L_1^{\max}$ (recall the derived norms given in Equation (1)); or $L_\infty^{\max}$ (which is just the same as the L_∞ norm). Let us use any of these norms as the norm in $\left\|\boldsymbol{z}^0+\boldsymbol{z}^1-\boldsymbol{y}\right\|$. The optimization problem (22) as stated is not a LP instance. However, we showed in [18] how constraint problems involving the $L_1^{\max}$, or L_∞ norms can be framed as LP by doubling the number of variables. A similar approach works here, thus, Problem (22) can be solved using linear programming. We solved for the minimal λ. We can employ the same techniques for finding the maximal λ. This gives us the following result.

Proposition 4 (Free Space Cell Boundaries for $\Phi_{\min}$)**.** *Given two PPRs R_1,R_2, the set $\mathsf{Free}_\delta^{\Phi_{\min}}(R_1,R_2)$ at cell-(i,k) boundaries can be computed in time $O\left(\mathsf{LP}\left(S_1^i+S_1^{i+1}+S_2^k+S_2^{k+1}\right)\right)$, where S_j^l denotes the halfspace representation size of polytope $R_j(l)$, and $\mathsf{LP}(\cdot)$ is the (polynomial time) upper bound for solving linear programming instances.* □

After computing the free space cell boundaries, we can employ a dynamic programming algorithm to check if there is a non-decreasing curve travelling through the free space from the point $(0,0)$ to (m_1,m_2).

Proposition 5 ($\mathcal{D}_{\mathcal{F}^\dagger_{\min}}$ Decision Problem)**.** *Given PPRs R_1,R_2 represented as m_1, m_2 polytopes respectively , and a $\delta\ge 0$, we can decide the question $\mathcal{D}_{\mathcal{F}^\dagger_{\min}}(R_1,R_2)\le\delta$ in time $O\left(m_1\cdot m_2\cdot\mathsf{LP}(S_{\max})\right)$ for both $L_1^{\max}$ and L_∞ norms on $\mathbb{R}^{d+1}$, where $S_{\max}$ is the maximum of the halfspace representation sizes of the given polytopes, and $\mathsf{LP}(\cdot)$ is the (polynomial time) upper bound for solving linear programming.* □

5.2.4 Free Space Cell Boundaries for $\Phi_{\max}$

In this subsection, we compute the bottom free space boundary of the first cell (over $[0,1]\times[0,1]$). The optimization problem (20) for $\Phi=\Phi_{\max}$ has the same solution as:

$$\begin{aligned}&\text{minimize} \quad \lambda\\ &\text{such that} \quad \begin{array}{l}\forall \text{ points } \boldsymbol{p}\in\lambda\cdot R_1(0)+(1-\lambda)\cdot R_1(1),\\ \forall \text{ points } \boldsymbol{q}\in R_2(0)\end{array}\\ &\qquad\qquad\qquad \text{we have } \|\boldsymbol{p}-\boldsymbol{q}\|\le\delta\\ &\qquad 0\le\lambda\le 1\end{aligned}$$

Unfortunately, this cannot be converted into an LP instance as in the $\Phi_{\min}$ case because of the "for all" quantifier in the constraints. The above optimization problem can be expressed in the theory of reals which is decidable [5]. This gives us a procedure to compute the free space cell boundaries for $\Phi_{\max}$. Once we have the free space boundaries, we can use a dynamic programming algorithm (as in the $\Phi_{\min}$ case) to obtain the following result.

Proposition 6 ($\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}$ Decision Problem)**.** *Given PPRs R_1,R_2 represented as m_1, m_2 polytopes respectively , and a $\delta\ge 0$, it is decidable to check $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}(R_1,R_2)\le\delta$ for both $L_1^{\max}$ and L_∞ norms on $\mathbb{R}^{d+1}$.* □

The check in Proposition 6 uses the theory of reals and has a high complexity. We show in the next subsection that under certain assumptions on the PPRs, we can obtain a polynomial time procedure.

5.2.5 $\Phi_{\max}$ Free Space: Polynomial Time Special Case

In this subsection, we obtain a polynomial time algorithm for computing the free space for $\boldsymbol{\Phi}_{\max}$, under mild conditions on the PPRs.

For a *fixed* λ, we can check if

$$\Phi_{\max}\Big(\lambda\cdot R_1(0)+(1-\lambda)\cdot R_1(1),R_2(0)\Big)\le\delta.$$

This is done as follows. Consider the optimization problem

$$\begin{aligned}\text{maximize} \quad & \Delta\\ \text{such that} \quad & \left\|\lambda\cdot\boldsymbol{x}^0+(1-\lambda)\cdot\boldsymbol{x}^1-\boldsymbol{y}\right\|\ge\Delta\\ & A_1^0\cdot\boldsymbol{x}^0\le\boldsymbol{b}_1^0\\ & A_1^1\cdot\boldsymbol{x}^1\le\boldsymbol{b}_1^1\\ & A_2\cdot\boldsymbol{y}\le\boldsymbol{b}_2\\ & 0\le\Delta\end{aligned}\tag{23}$$

The following cases arise.

- If the optimal Δ is strictly bigger than δ, then
$$\Phi_{\max}\big(\lambda \cdot R_1(0) + (1-\lambda)\cdot R_1(1),\ R_2(0)\big) > \delta$$
because in this case the constraints in (23) imply that there exist points $\boldsymbol{x}^0 \in R_1(0)$ and $\boldsymbol{x}^1 \in R_1(1)$ and $\boldsymbol{y} \in R_2(0)$ such that $\left\|\lambda \cdot \boldsymbol{x}^0 + (1-\lambda)\cdot \boldsymbol{x}^1 - \boldsymbol{y}\right\| \geq \Delta > \delta$. Hence $\langle \lambda, 0\rangle$ does not belong to the free space.
- If $\Delta \leq \delta$, it implies that $\Phi_{\max}\big(\lambda \cdot R_1(0) + (1-\lambda)\cdot R_1(1),\ R_2(0)\big) \leq \delta$. Hence $\langle \lambda, 0\rangle$ belongs to the free space.

Finally, note that the feasible region of (23) is never empty since for $\Delta = 0$ the variables $\boldsymbol{x}^0, \boldsymbol{x}^1, \boldsymbol{y}$ can range over values in $R_1(0), R_1(1), R_2(0)$ respectively; hence one of the above cases will hold. Problem (23) can be framed as an LP instance by adding additional variables using the same methods as in the case for $\Phi_{\min}$ for $L_1^{\max}$ or L_∞ norms.

If we can find *one* λ value such that $\Phi_{\max}\big(\lambda \cdot R_1(0) + (1-\lambda)\cdot R_1(1),\ R_2(0)\big) \leq \delta$, then we can do binary search over the interval $[0, \lambda]$ to get $\lambda^{\min}$ (and similarly for $\lambda^{\max}$). We next present a heuristic to do this in polynomial time. Fix an integer K, partition $[0, 1]$ into K equal intervals, and check for $\lambda = 0, \frac{1}{K}, \frac{2}{K}, \ldots, 1$ whether $\langle \lambda, 0\rangle$ belongs to the free space.

Once the first $\lambda \in \{0, \frac{1}{K}, \frac{2}{K}, \ldots, 1\}$ is found such that $\langle \lambda, 0\rangle$ belongs to the free space, we perform a binary search around it over the interval $(\lambda - 1/K,\ \lambda]$ to obtain $\lambda^{\min}$ to a desired degree of accuracy (which we take to be less than 2^{-cK} for a constant c for convenience), and similarly for $\lambda^{\max}$. If the binary search fails to obtain a lower or upper boundary, we set the corresponding lower or upper boundary to λ. In total, we solve $O(K)$ instances of problem (23). Suppose that the actual free space interval at the bottom boundary of the cell is $[\lambda^{\min}, \lambda^{\max}] \times \{0\}$. If $\lambda^{\max} - \lambda^{\min} < 1/K$, we *may* find an empty subinterval. If $\lambda^{\max} - \lambda^{\min} \geq 1/K$, we are *guaranteed* to find the interval (to any desired degree of accuracy).

Observe that if the bottom boundary of cell i, j is $[\lambda^{\min}, \lambda^{\max}] \times \{j\}$, then it means that the set of *all* optimal reparameterizations α_1, α_2 in Equation (13) in addition satisfy $\big(\alpha_2(\theta) = j\big) \rightarrow \big(\alpha_1(\theta) \in [\lambda^{\min}, \lambda^{\max}]\big)$. In other words, the polytope at time $\alpha_2(\theta)$ in the PPR R_2 can only be mapped to R_1 polytopes in between times $[\lambda^{\min}, \lambda^{\max}]$. The smaller the interval $[\lambda^{\min}, \lambda^{\max}]$, the more restricted the allowable timing distortions which witness $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}(R_1, R_2) \leq \delta$, and thus, the smaller the degree of freedom of time-distorting of the time-point j in R_2; which in turn means the less robust the possible reparameterizations..

Proposition 7 ($\Phi_{\max}$ Free Space in Polyomial time)**.** *Given two* PPR*s R_1 and R_2, the set $\mathsf{Free}_\delta^{\Phi_{\max}}(R_1, R_2)$ at the boundaries of cell i, k can be computed to a precision of $O(K)$ bits in time $O\left(K \cdot \mathsf{LP}\left(S_1^i + S_1^{i+1} + S_2^k + S_2^{k+1}\right)\right)$, provided the free space intervals at the cell boundaries, if non-empty, are of length at least $\frac{1}{K}$, where S_j^l denotes the halfspace representation size of polytope $R_j(l)$, and $\mathsf{LP}(\cdot)$ is the (polynomial time) upper bound for solving linear programming.* □

This gives us the following decision procedure using a dynamic programming algorithm, and improves Proposition 6 time complexity if the PPRs satisfy certain conditions.

Proposition 8 ($\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}$ Decision Problem in Polynomial Time)**.** *Given* PPR*s R_1, R_2 represented by m_1, m_2 polytopes respectively, $\delta \geq 0$, and integer $K > 0$, we can decide the question $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}(R_1, R_2) \leq \delta$ under the two conditions:*

1. *$\forall\, i \in \{0..m_1\}$, and $\forall\, j \in \{0..m_2 - 1\}$, either (a) there exists a sub-interval $[\lambda^{\min}, \lambda^{\max}] \subseteq [j, j+1]$, with $\lambda^{\max} - \lambda^{\min} \geq 1/K$, such that $\Phi_{\max}\left(R_1(i), R_2(t)\right) \leq \delta$ for all $t \in [\lambda^{\min}, \lambda^{\max}]$, or (b) for all $t \in [j, j+1]$, we have $\Phi_{\max}\left(R_1(i), R_2(t)\right) > \delta$; and*
2. *$\forall\, j \in \{0..m_2\}$, and $\forall\, i \in \{0..m_1 - 1\}$, either (a) there exists a sub-interval $[\lambda^{\min}, \lambda^{\max}] \subseteq [i, i+1]$, with $\lambda^{\max} - \lambda^{\min} \geq 1/K$, such that $\Phi_{\max}\left(R_1(t), R_2(j)\right) \leq \delta$ for all $t \in [\lambda^{\min}, \lambda^{\max}]$, or (b) for all $t \in [i, i+1]$, we have $\Phi_{\max}\left(R_1(t), R_2(j)\right) > \delta$*

in time $O\left(m_1 \cdot m_2 \cdot K \cdot \mathsf{LP}(S_{\max})\right)$ for both $L_1^{\max}$, L_∞ norms where $S_{\max}$ is the maximum of the halfspace representation sizes of the given polytopes, and $\mathsf{LP}()$ is the (polynomial time) upper bound for solving linear programming. □

An analysis of the dynamic programming reachability algorithm shows that the two conditions in Proposition 8 are only required for an i, j pair collection for which a cell-i, j from the collection occurs in *every* path from $0, 0$ to m_1, m_2 in the free space diagram of the two PPRs. As a result, for a sufficiently large K, we expect the algorithm of this subsection to work in all except for certain pathological cases.

Proposition 8 gives us a conservative procedure in case the validity of the two stated conditions is not known: if for a chosen $K > 0$, the procedure returns that the distance is less than or equal to δ, then indeed $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}(R_1, R_2) \leq \delta$. Also note that as δ increases, the corresponding free space and the free space boundaries become larger, and when δ is increases enough, the PPR conditions are satisfied. Since we intend to use the $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}$ distances of PPRs as over-approximations of tracepipes, the conservative nature of Proposition 8 does not break the over-approximation scheme.

6. VARIATION DISTANCE BOUNDS

We now put everything together, using the results of the preceding sections to obtain bounds on the variation distance $\mathcal{D}_{\mathcal{S}_{\mathsf{var}}}(F_1, F_2)$ for PPRs F_1 and F_2. From Propositions 2, 1, and Theorems 1, 2, and using binary search on the decision algorithms of Propositions 5 and 6 we get the following theorem.

Theorem 3. *Suppose tracepipes F_1 and F_2 correspond to sampled over-approximate reach set polytopes $\lceil\mathsf{Rp}(F_1)\rceil(t_1^1), \ldots, \lceil\mathsf{Rp}(F_1)\rceil(t_1^{m_1})$ at time-points $t_1^1, \ldots, t_1^{m_1}$, and $\lceil\mathsf{Rp}(F_2)\rceil(t_2^1), \ldots, \lceil\mathsf{Rp}(F_2)\rceil(t_2^{m_2})$ at time-points $t_2^1, \ldots, t_2^{m_2}$ respectively. Let $\lceil\mathsf{Rp}(F_1)\rceil$ and $\lceil\mathsf{Rp}(F_1)\rceil$ be corresponding reachpipe completions constructed by linear interpolation. We can compute $\beta_{\min}, \beta_{\max}$ with*
$$\beta_{\min} \leq \mathcal{D}_{\mathcal{S}_{\mathsf{var}}}(F_1, F_2) \leq \beta_{\max}$$
for the Skorokhod trace metric over L_1, L_∞ norms on $\mathbb{R}^d$ such that

- *$\beta_{\min} = \mathcal{D}_{\mathcal{S}_{\min}}\big(\mathsf{Fp}\left(\lceil\mathsf{Rp}(F_1)\rceil\right), \mathsf{Fp}\left(\lceil\mathsf{Rp}(F_2)\rceil\right)\big)$ and*
- *$\beta_{\max}$ is an upper-bound of the variation distance $\mathcal{D}_{\mathcal{S}_{\mathsf{var}}}\big(\mathsf{Fp}\left(\lceil\mathsf{Rp}(F_1)\rceil\right), \mathsf{Fp}\left(\lceil\mathsf{Rp}(F_2)\rceil\right)\big)$; and is equal to the the Skorokhod distance $\mathcal{D}_{\mathcal{S}^\dagger_{\mathsf{var}}}(\lceil\mathsf{Rp}(F_1)\rceil, \lceil\mathsf{Rp}(F_2)\rceil)$ between the reachpipes $\lceil\mathsf{Rp}(F_1)\rceil$ and $\lceil\mathsf{Rp}(F_2)\rceil$ (where $\mathcal{D}_{\mathcal{S}^\dagger_{\mathsf{var}}}$ is defined analogously to $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}$).* □

In order to do binary searches on the decision procedures used in Theorem 3, we need an upper bound U on $\beta_{\max}$. This upper bound can be obtained as follows (in polynomial time). We pick one pair of reparameterizations and use these to get an upper bound U on $\mathcal{D}_{\mathcal{F}^\dagger_{\mathsf{var}}}(R_1, R_2)$ (and thus on $\mathcal{D}_{\mathcal{S}^\dagger_{\mathsf{var}}}(R_1, R_2)$) for $R_1 = \lceil\mathsf{Rp}(F_1)\rceil$, and $R_2 = \lceil\mathsf{Rp}(F_2)\rceil$.

Assume $m_2 \geq m_1$. Fix $\alpha_1 : [0,1] \to [0, m_1]$ to be any non-decreasing reparameterization such that $\alpha_1(\theta) = m_1$ for $\theta \geq 0.5$; and let $\alpha_2 : [0,1] \to [0, m_1]$ be a non-decreasing reparameterization such that $\alpha_2(\theta) = \alpha_1(\theta)$ for $\theta \leq 0.5$, and α_2 over $[0.5, 1]$ being non-decreasing to $[m_1, m_2]$. An upper bound of $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}(R_1, R_2)$ is

$$\max_{0 \leq \theta \leq 1} \Phi_{\max}\big(R_1(\alpha_1(\theta))\, ,\, R_2(\alpha_2(\theta))\big) \tag{24}$$

The stated reparameterizations are such that $R_1(i)$ is compared to $R_2(i)$ for $0 \leq i \leq m_1$ in $\Phi_{\max}$, and $R_2(i)$ for $i > m_1$ is compared to $R_1(m_1)$. It can be shown that the value of Expression (24) is the maximum of $\max_{i \in \{0,1,\ldots,m_1\}} \Phi_{\max}(R_1(i), R_2(i))$ and $\max_{j \in \{m_1,\ldots,m_2\}} \Phi_{\max}(R_1(m_1), R_2(j))$. These two maximums can be computed in polynomial time by computing $\Phi_{\max}(R_1(i), R_2(j))$ for required i, j pairs using linear programming (Lemmas 4, and 5 in the Appendix). Once the upper bound U is obtained, we can compute $\beta_{\min}$ in $O((\lg(U) + B) \cdot m_1 \cdot m_2 \cdot \mathsf{LP}(S_{\max}))$ time, where B is the number of desired bits of the fractional part in $\beta_{\min}$, and $S_{\max}$ is the maximum of the halfspace representation sizes of the given polytopes, and $\mathsf{LP}(\cdot)$ is the (polynomial time) upper bound for solving linear programming.

Polynomial Time Case for $\beta_{\max}$. Theorem 3 uses the theory of reals to obtain $\beta_{\max}$. In case an upper bound U on $\beta_{\max}$ is given and the PPRs and $\delta < U$ are such that the conditions of Proposition 8 are satisfied, we can employ the polynomial time algorithm of the proposition in the decision question queries for obtaining $\beta_{\max}$. This procedure runs in $O((\lg(U) + B) \cdot m_1 \cdot m_2 \cdot K \cdot \mathsf{LP}(S_{\max}))$ time, where K is an integer governing the robustness of retiming functions (in the sense discussed above Proposition 7). Note that if the PPRs do not satisfy the the conditions of Proposition 8, then this procedure will still give an upper bound on $\mathcal{D}_{\mathsf{var}}\big(\mathsf{Fp}\left(\lceil \mathsf{Rp}(F_1) \rceil\right), \mathsf{Fp}\left(\lceil \mathsf{Rp}(F_2) \rceil\right)\big)$, but it may be larger than the Skorokhod distance $\mathcal{D}_{\mathcal{S}^{\dagger}_{\mathsf{var}}}(\lceil \mathsf{Rp}(F_1) \rceil, \lceil \mathsf{Rp}(F_2) \rceil)$ between the reachpipes $\lceil \mathsf{Rp}(F_1) \rceil$ and $\lceil \mathsf{Rp}(F_2) \rceil$.

Using Sliding Windows. The Skorokhod metric allows matching an F_1 trace segment in between times t_1^0, t_1^1 to F_2 trace segments in between times $t_2^{m_2-1}, t_2^{m_2}$, *i.e.*, the retimings put no limit on the timing distortions. In practice, we have bounds on timing distortions. As a result, we can restrict the retimings to be in a window W: we require that trace segment j of one trace only be matched to portions of other traces consisting of segments $j - W$ though $j + W$. Under this restriction, the algorithm of Theorem 3 can be improved to run in time $O(((\lg(U) + B) \cdot m \cdot W \cdot K \cdot \mathsf{LP}(S_{\max}))$, where $m = \max(m_1, m_2)$. Usually W, B and K can be taken to be constants, thus we get a practical running time of $O(m \cdot \lg(U) \cdot \mathsf{LP}(S_{\max}))$, which is linear in the number of given polytope reachsets, and linear in the LP solving time involving the largest given polytope representation.

7. CONCLUSIONS

We have considered the problem of determining the distance between two tracepipes. Such problems arise in the analysis of dynamical systems under the presence of uncertainties and noise. Our starting point was the polynomial-time algorithm to compute the Skorokhod metric between individual traces [18]. Our algorithm takes as input discrete sequences of polyhedral approximations to the reach set, such as those provided by symbolic tools such as SpaceEx [13, 10]. Our main result shows polynomial time algorithms to approximate the distance from above and from below.

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9. APPENDIX

Proof of Theorem 2. We prove inequalities in both directions.

(1) $\mathcal{D}_{\mathcal{F}_{\min}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)) \geq \mathcal{D}_{\mathcal{F}^\dagger_{\min}}(R_1, R_2)$.
Consider any $f_1 \in \mathsf{Fp}(R_1)$, and any $f_2 \in \mathsf{Fp}(R_2)$. We have

$$\mathcal{D}_{\mathcal{F}}(f_1, f_2) = \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\leq\theta\leq 1} \|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\|$$

As in the proof of Theorem 1, we have that for every $\alpha_1, \alpha_2, \theta$,

$$\|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\| \geq \Phi_{\min}(R_1(\alpha_1(\theta)), R_2(\alpha_2(\theta)))$$

Thus, for every $f_1 \in \mathsf{Fp}(R_1)$, and $f_2 \in \mathsf{Fp}(R_2)$, we have

$$\mathcal{D}_{\mathcal{F}}(f_1, f_2) \geq \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\leq\theta\leq 1} \Phi_{\min}\Big(R_1(\alpha_1(\theta)), R_2(\alpha_2(\theta))\Big)$$

i.e., $\mathcal{D}_{\mathcal{F}}(f_1, f_2) \geq \mathcal{D}_{\mathcal{F}^\dagger_{\min}}(R_1, R_2)$. This implies that $\inf_{f_1\in\mathsf{Fp}(R_1), f_2\in\mathsf{Fp}(R_2)} \mathcal{D}_{\mathcal{F}}(f_1, f_2) \geq \mathcal{D}_{\mathcal{F}^\dagger_{\min}}(R_1, R_2)$. This completes the proof of the first direction.

(2) $\mathcal{D}_{\mathcal{F}_{\min}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)) \leq \mathcal{D}_{\mathcal{F}^\dagger_{\min}}(R_1, R_2)$.
Recall that $\mathcal{D}_{\mathcal{F}_{\min}}(\mathsf{Fp}(R_1), \mathsf{Fp}(R_2)) =$

$$\inf_{f_1\in\mathsf{Fp}(R_1), f_2\in\mathsf{Fp}(R_2)} \inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\leq\theta\leq 1} \|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\|.$$

This equals (switching the inf order):

$$\inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \inf_{f_1\in\mathsf{Fp}(R_1), f_2\in\mathsf{Fp}(R_2)} \max_{0\leq\theta\leq 1} \|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\|.$$

We need to show that the above expression is $\leq$ than:

$$\inf_{\substack{\alpha_1:[0,1]\to[0,m_1]\\ \alpha_2:[0,1]\to[0,m_2]}} \max_{0\leq\theta\leq 1} \Phi_{\min}\Big(R_1(\alpha_1(\theta)), R_2(\alpha_2(\theta))\Big)$$

To show this direction of the inequality, it suffices to show that for every pair of valid reparameterizations α_1, α_2, we have:

$$\inf_{f_1\in\mathsf{Fp}(R_1), f_2\in\mathsf{Fp}(R_2)} \max_{0\leq\theta\leq 1} \|f_1(\alpha_1(\theta)) - f_2(\alpha_2(\theta))\| \leq \max_{0\leq\theta\leq 1} \Phi_{\min}\Big(R_1(\alpha_1(\theta)), R_2(\alpha_2(\theta))\Big) \quad (25)$$

The formal proof of the above inequality is technical. We sketch the main ideas. Fix α_1, α_2 reparameterizations. Define the function $\mathsf{minpairs}$ from $[0,1]$ to subsets of $\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$ as $\mathsf{minpairs}(\theta) =$

$$\left\{ \langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle \;\middle|\; \begin{array}{l} \boldsymbol{p}_1 \in R_1(\alpha_1(\theta)), \text{ and } \boldsymbol{p}_2 \in R_2(\alpha_2(\theta)), \\ \text{and } \|\boldsymbol{p}_1 - \boldsymbol{p}_2\| \leq \Phi_{\min}\Big(R_1(\alpha_1(\theta)), R_2(\alpha_2(\theta))\Big) \end{array} \right\}$$

That is, $\mathsf{minpairs}(\theta)$ contains point pairs $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 \rangle$ with $\boldsymbol{p}_1 \in R_1(\alpha_1(\theta))$, and $\boldsymbol{p}_2 \in R_2(\alpha_2(\theta))$ such that $\boldsymbol{p}_1, \boldsymbol{p}_2$ are the closest points in the corresponding polytopes $R_1(\alpha_1(\theta))$ and $R_2(\alpha_2(\theta))$ (there may be several such pairs for the two polytopes). It can be shown that for each θ, we can pick a single point tuple from $\mathsf{minpairs}(\theta)$, namely $\langle \boldsymbol{p}_1^\theta, \boldsymbol{p}_2^\theta \rangle$ such that the functions $\mathsf{C}_1(\alpha_1(\theta)) = \boldsymbol{p}_1^\theta$ and $\mathsf{C}_2(\alpha_2(\theta)) = \boldsymbol{p}_2^\theta$ are continuous functions from $[0, m_1]$ and $[0, m_2]$ to $\mathbb{R}^{d+1}$, ie. they are continuous traces. This can be done due to the fact that R_1 and R_2 are PPRs and thus the polygons $R_1(\alpha_1(\theta))$ and $R_2(\alpha_2(\theta))$ change smoothly with respect to θ.

Observe that the curves C_1 and C_2 are such that

$$\max_{0\leq\theta\leq 1} \|\mathsf{C}_1(\alpha_1(\theta)) - \mathsf{C}_2(\alpha_2(\theta))\| \leq \max_{0\leq\theta\leq 1} \Phi_{\min}\Big(R_1(\alpha_1(\theta)), R_2(\alpha_2(\theta))\Big)$$

This prove Inequality 25. This concludes the second part of the theorem proof. □

Proof of Lemma 3. The basic idea is that we introduce variables $\boldsymbol{z}^0 = \lambda \cdot \boldsymbol{x}^0$ and $\boldsymbol{z}^1 = (1-\lambda) \cdot \boldsymbol{x}^0$, and we multiply both sides of $A_1^0 \cdot \boldsymbol{x}^0 \leq \boldsymbol{b}_1^0$ by λ, and of $A_1^1 \cdot \boldsymbol{x}^1 \leq \boldsymbol{b}_1^1$ by $1-\lambda$. For the two optimization problems to be the same, it suffices to show that for any $0 \leq \lambda \leq 1$, and for any $\boldsymbol{y}$ satisfying $A_2 \cdot \boldsymbol{y} \leq \boldsymbol{b}_2$,

$$\begin{array}{ll} \text{there exist } \boldsymbol{x}^0, \boldsymbol{x}^1 \text{ such that:} & \left\|\lambda \cdot \boldsymbol{x}^0 + (1-\lambda) \cdot \boldsymbol{x}^1 - \boldsymbol{y}\right\| \leq \delta \\ \text{with} & A_1^0 \cdot \boldsymbol{x}^0 \leq \boldsymbol{b}_1^0 \\ & A_1^1 \cdot \boldsymbol{x}^1 \leq \boldsymbol{b}_1^1 \end{array} \quad (26)$$

iff

$$\begin{array}{ll} \text{there exist } \boldsymbol{z}^0, \boldsymbol{z}^1 \text{ such that:} & \left\|\boldsymbol{z}^0 + \boldsymbol{z}^1 - \boldsymbol{y}\right\| \leq \delta \\ \text{with} & A_1^0 \cdot \boldsymbol{z}^0 \leq \lambda \cdot \boldsymbol{b}_1^0 \\ & A_1^1 \cdot \boldsymbol{z}^1 \leq (1-\lambda) \cdot \boldsymbol{b}_1^1 \end{array} \quad (27)$$

Fix a λ, and a $\boldsymbol{y}$ vector. We show the above equivalence.
"Only if". Suppose there exist $\boldsymbol{x}^0, \boldsymbol{x}^1$ satisfying constraints 26. Let $\boldsymbol{z}^0 = \lambda \cdot \boldsymbol{x}^0$ and $\boldsymbol{z}^1 = (1-\lambda) \cdot \boldsymbol{x}^0$. Observe that $\boldsymbol{z}^0, \boldsymbol{z}^1$ satisfy the conditions of the second system, and also $\left\|\boldsymbol{z}^0 + \boldsymbol{z}^1 - \boldsymbol{y}\right\| \leq \delta$ as $\left\|\lambda \cdot \boldsymbol{x}^0 + (1-\lambda) \cdot \boldsymbol{x}^1 - \boldsymbol{y}\right\| \leq \delta$. This concludes the proof of the "Only if" direction.
"If". Suppose there exist $\boldsymbol{z}^0, \boldsymbol{z}^1$ satisfying constraints 27. If $\lambda \neq 0$ and $\lambda \neq 1$, then take $\boldsymbol{x}^0 = \frac{1}{\lambda} \cdot \boldsymbol{z}^0$, and $\boldsymbol{x}^1 = \frac{1}{1-\lambda} \cdot \boldsymbol{z}^1$. It can be checked that $\boldsymbol{x}^0, \boldsymbol{x}^1$ satisfy constraints 26.

Now suppose $\lambda = 0$. The point $\boldsymbol{z}^0, \boldsymbol{z}^1$ thus also satisfy:

$$\begin{array}{ll} & \left\|\boldsymbol{z}^0 + \boldsymbol{z}^1 - \boldsymbol{y}\right\| \leq \delta \\ \text{with} & A_1^0 \cdot \boldsymbol{z}^0 \leq \boldsymbol{0} \\ & A_1^1 \cdot \boldsymbol{z}^1 \leq \boldsymbol{b}_1^1 \end{array}$$

If $\boldsymbol{b}^0 = \boldsymbol{0}$, then $\boldsymbol{x}^0 = \boldsymbol{z}^0$ and $\boldsymbol{x}^1 = \boldsymbol{z}^1$ satisfy constraints 26.

Suppose $\boldsymbol{b}^0 \neq \boldsymbol{0}$. From Lemma 2, since $A_1^0 \cdot \boldsymbol{z}^0 \leq \boldsymbol{0}$, we must have that $\boldsymbol{z}^0 = \boldsymbol{0}$. Thus, we have $\left\|\boldsymbol{z}^1 - \boldsymbol{y}\right\| \leq \delta$ with $A_1^1 \cdot \boldsymbol{z}^1 \leq \boldsymbol{b}_1^1$. Now we let $\boldsymbol{x}^0$ be any point in the polytope $A_1^0 \cdot \boldsymbol{x}^0 \leq \boldsymbol{b}_1^0$, and $\boldsymbol{x}^1 = \boldsymbol{z}^1$. It can be seen that these $\boldsymbol{x}^0, \boldsymbol{x}^1$ satisfy

$$\begin{array}{ll} & \left\|\boldsymbol{x}^1 - \boldsymbol{y}\right\| \leq \delta \\ \text{with} & A_1^0 \cdot \boldsymbol{x}^0 \leq \boldsymbol{b}^0 \\ & A_1^1 \cdot \boldsymbol{x}^1 \leq \boldsymbol{b}_1^1 \end{array}$$

The case of $\lambda = 1$ is similar. This concludes the proof of he "If" part, and thus also the proof of the lemma. □

Lemma 4. *Let R_1, R_2 be PPRs represented by m_1, m_2 polytopes respectively with $m_2 \geq m_1$. Fix $\alpha_1 : [0,1] \rightarrow [0, m_1]$ to be any non-decreasing reparameterization such that $\alpha_1(\theta) = m_1$ for $\theta \geq 0.5$; and let $\alpha_2 : [0,1] \rightarrow [0, m_1]$ be a non-decreasing reparameterization such that $\alpha_2(\theta) = \alpha_1(\theta)$ for $\theta \leq 0.5$, and α_2 over $[0.5, 1]$ being non-decreasing to $[m_1, m_2]$. The value of $\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}(R_1, R_2)$ is at most the maximum of $\max_{i \in \{0,1,\ldots,m_1\}} \Phi_{\max}(R_1(i), R_2(i))$ and $\max_{j \in \{m_1,\ldots,m_2\}} \Phi_{\max}(R_1(m_1), R_2(j))$.*

Proof. Since α_1, α_2 are valid non-decreasing reparameterizations, we have

$$\mathcal{D}_{\mathcal{F}^{\dagger}_{\mathsf{var}}}(R_1, R_2) \;\leq\; \max_{0 \leq \theta \leq 1} \Phi_{\max}\big(R_1(\alpha_1(\theta))\,,\, R_2(\alpha_2(\theta))\big)$$

It is clear that $\max_{0 \leq \theta \leq 1} \Phi_{\max}\big(R_1(\alpha_1(\theta))\,,\, R_2(\alpha_2(\theta))\big)$ cannot be smaller than the maximum of $\max_{i \in \{0,1,\ldots,m_1\}} \Phi_{\max}(R_1(i), R_2(i))$ and $\max_{j \in \{m_1,\ldots,m_2\}} \Phi_{\max}(R_1(m_1), R_2(j))$. We prove that the two quantities are equal. To prove this, it suffices to show that if (a) $\Phi_{\max}(R_1(i), R_2(i)) \leq \delta$, and (b) $\Phi_{\max}(R_1(i+1), R_2(i+1)) \leq \delta$, then for all $0 \leq \lambda \leq 1$, we have

$$\Phi_{\max}\left(\begin{array}{l} \lambda \cdot R_1(i) + (1-\lambda) \cdot R_1(i+1), \\ \lambda \cdot R_2(i) + (1-\lambda) \cdot R_2(i+1) \end{array}\right) \leq \delta.$$

We prove the above as follows. Assume (a) $\Phi_{\max}(R_1(i), R_2(i)) \leq \delta$, and (b) $\Phi_{\max}(R_1(i+1), R_2(i+1)) \leq \delta$. Let $\boldsymbol{p}_1^i \in R_1(i)$, and $\boldsymbol{p}_1^{i+1} \in R_1(i+1)$, and $\boldsymbol{p}_2^i \in R_2(i)$, and $\boldsymbol{p}_2^{i+1} \in R_2(i+1)$. We have

$$\begin{aligned} &\left\| \lambda \boldsymbol{p}_1^i + (1-\lambda)\boldsymbol{p}_1^{i+1} - \left(\lambda \boldsymbol{p}_2^i + (1-\lambda)\boldsymbol{p}_2^{i+1}\right) \right\| \\ &\quad \leq \left\| \lambda \left(\boldsymbol{p}_1^i - \boldsymbol{p}_2^i\right) \right\| + \left\| (1-\lambda) \left(\boldsymbol{p}_1^{i+1} - \boldsymbol{p}_2^{i+1}\right) \right\| \\ &\qquad\qquad \leq \lambda\delta + (1-\lambda)\delta = \delta \end{aligned}$$

This concludes the proof. □

Lemma 5. *Let Q_1 and Q_2 be polytopes in $\mathsf{PTopes}(\mathbb{R}^{d+1})$. The value $\Phi_{\max}(Q_1, Q_2)$ can be computed in time $O(\mathsf{LP}(|Q_1| + |Q_2|))$ where $|Q_1|$ and $|Q_2|$ denote the halfspace representation sizes of the respective polytopes, and $\mathsf{LP}()$ is the (polynomial time) upper bound for solving linear programming.*

Proof. Let Q_1 have the halfspace representation $A_1^0 \cdot \boldsymbol{x}^1 \leq \boldsymbol{b}_1$, and let Q_2 be $A_2 \cdot \boldsymbol{x}^2 \leq \boldsymbol{b}_2$ for $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ column vectors of $d{+}1$ variables taking values in $\mathbb{R}$. The value of $\Phi_{\max}(Q_1, Q_2)$ is the solution to the following constraint problem:

$$\begin{aligned} \text{maximize} \quad & \Delta \\ \text{such that} \quad & \left\| \boldsymbol{x}^1 - \boldsymbol{x}^2 \right\| \geq \Delta \\ & A_1^0 \cdot \boldsymbol{x}^1 \leq \boldsymbol{b}_1 \\ & A_2 \cdot \boldsymbol{x}^2 \leq \boldsymbol{b}_2 \\ & 0 \leq \Delta \end{aligned} \tag{28}$$

The optimization problem (28) can be solved using linear programming. Suppose the solution of the optimization problem (28) is δ. It means that (i) there are no points $\boldsymbol{p}^1 \in Q_1$, and $\boldsymbol{p}^2 \in Q_2$ such that $\left\| \boldsymbol{p}^1 - \boldsymbol{p}^2 \right\| > \delta$, and (ii) there exist points $\boldsymbol{p}^1 \in Q_1$, and $\boldsymbol{p}^2 \in Q_2$ such that $\left\| \boldsymbol{p}^1 - \boldsymbol{p}^2 \right\| \leq \delta$. These two facts imply $\Phi_{\max}(Q_1, Q_2) = \delta$. □