Lattice-Based Cryptography: Short Integer Solution (SIS) and Learning With Errors (LWE)

Chris Peikert
Georgia Institute of Technology

crypt@b-it 2013

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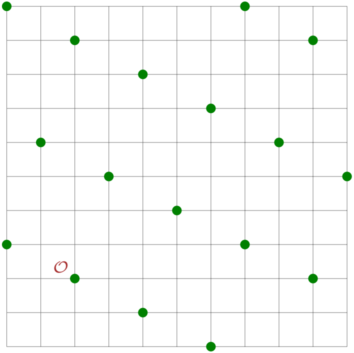

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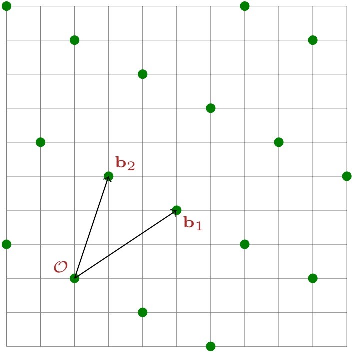

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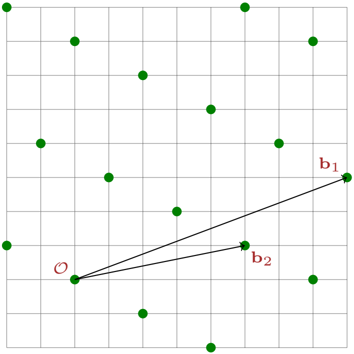

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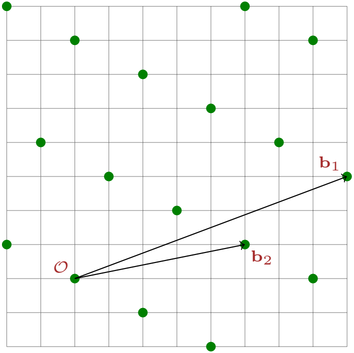

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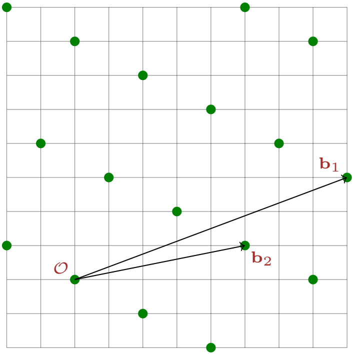

Hard Problems

- Find/detect short nonzero lattice vector(s): SVP, GapSVP, SIVP
- Decode under small amount of error: BDD

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- Goal: find nontrivial small $z_1, \ldots, z_m \in \mathbb{Z}$ such that:

$$z_1 \cdot \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + z_2 \cdot \begin{pmatrix} | \\ \mathbf{a}_2 \\ | \end{pmatrix} + \cdots + z_m \cdot \begin{pmatrix} | \\ \mathbf{a}_m \\ | \end{pmatrix} = \begin{pmatrix} | \\ 0 \\ | \end{pmatrix} \in \mathbb{Z}_q^n$$

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- Set $m > n \lg q$. Define $f_{\mathbf{A}} : \{0,1\}^m \to \mathbb{Z}_q^n$ as

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  ...yields solution $\mathbf{z} = \mathbf{x} - \mathbf{x}' \in \{0, \pm 1\}^m$, of norm $\|\mathbf{z}\| \leq \sqrt{m}$.

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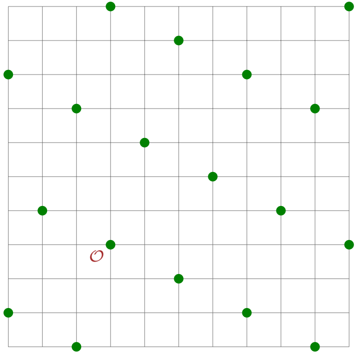

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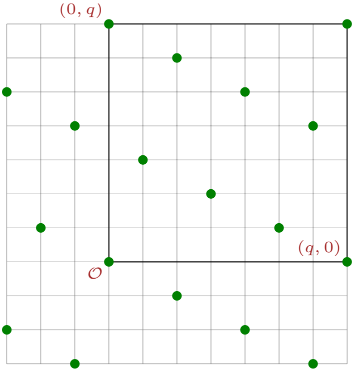

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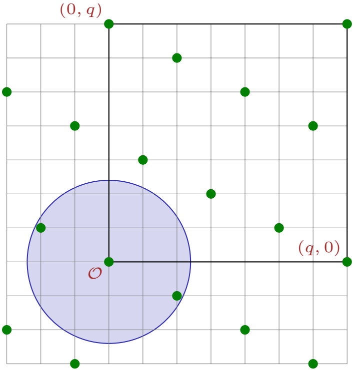

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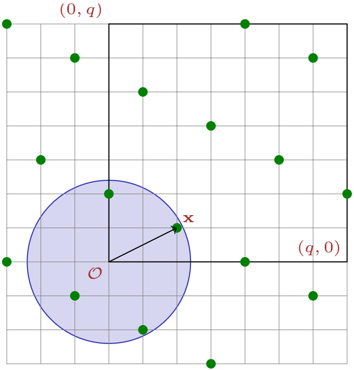

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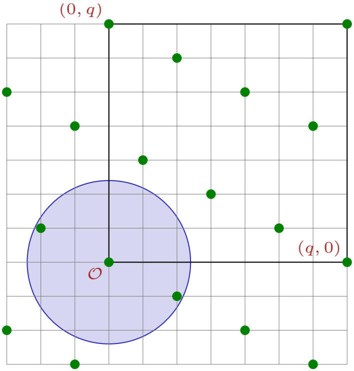

Worst-Case/Average-Case Connection [Ajtai'96,...]

Finding short ($\|\mathbf{z}\| \leq \beta \ll q$) nonzero $\mathbf{z} \in \mathcal{L}^{\perp}(\mathbf{A})$

for uniformly random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$

$$\Downarrow$$

solving $\text{GapSVP}_{\beta\sqrt{n}}$ and $\text{SIVP}_{\beta\sqrt{n}}$ on any n-dim lattice.

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Dirty Little Secret

- This trick — reducing a short vector modulo a lattice — is the only one-way function used in all of lattice crypto!

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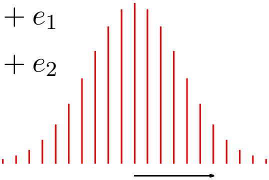

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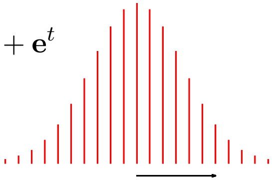

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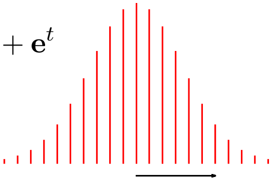

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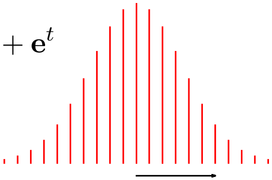

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- Foundation for a huge amount of crypto

[R'05,PW'08,GPV'08,PVW'08,CDMW'08,AGV'09,ACPS'09,CHKP'10,ABB'10a,ABB'10b,GKV'10,BV'11,BGV'12,. . .]

LWE as a Lattice Problem

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$$\mathcal{L}(\mathbf{A}) = \{\mathbf{z}^t \equiv \mathbf{s}^t\mathbf{A} \bmod q\}$$

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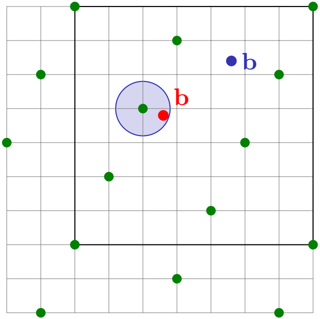
Distinguishing $\mathbf{b}$ vs. $\mathbf{b}$: decision-BDD.



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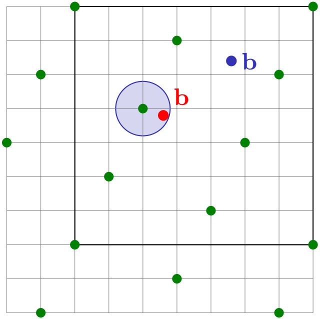
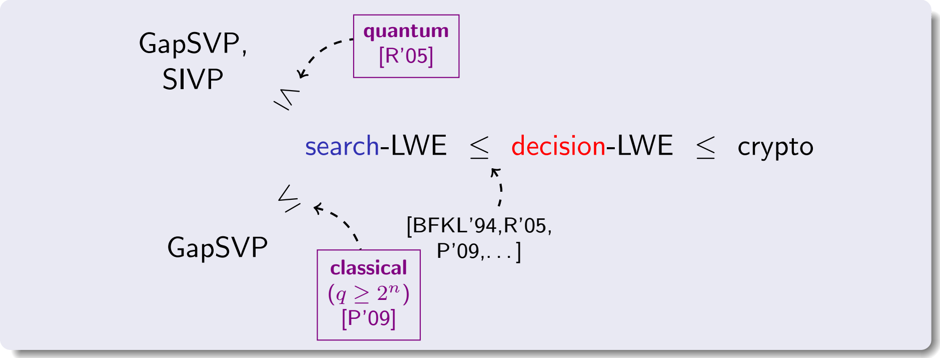
- Also enjoys worst-case hardness [R'05,P'09]
  . . . but results are more subtle.



Overview of LWE Hardness



- Dim-modulus tradeoff [BLPRS'13]: e.g., $n, q = 2^n$ for $n^2, q = \text{poly}(n)$.
- Why error $\alpha q > \sqrt{n}$?
  - ⋆ Required by worst-case hardness proofs
  - ⋆ There's an $\exp((\alpha q)^2)$-time attack! [AG'11]

SIS versus LWE

SIS	LWE
$\mathbf{A}\mathbf{z} = \mathbf{0}$, 'short' $\mathbf{z} \neq \mathbf{0}$	$(\mathbf{A}, \mathbf{b}^t = \mathbf{s}^t\mathbf{A} + \mathbf{e}^t)$ vs. $(\mathbf{A}, \mathbf{b}^t)$

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'CRYPTOMANIA'

SIS versus LWE

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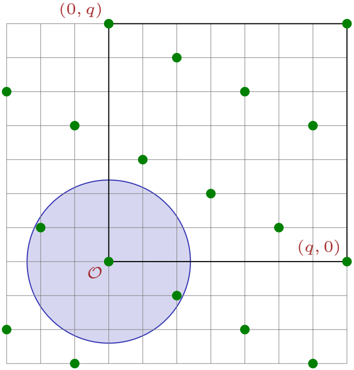
Average-case SVP:

$$\mathcal{L}^{\perp}(\mathbf{A}) = \{\mathbf{z} \in \mathbb{Z}^m \ : \ \mathbf{A}\mathbf{z} = \mathbf{0}\}$$



<u>LWE</u>

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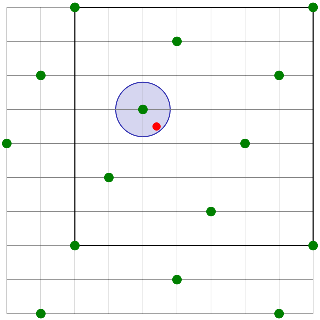
Average-case BDD:

$$\mathcal{L}(\mathbf{A}) = \{\mathbf{z}^t \equiv \mathbf{s}^t\mathbf{A} \bmod q\}$$



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   Lets us amplify success probabilities (both search & decision):

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3. Multiple secrets: $(\mathbf{a}, b_1 \approx \langle \mathbf{s}_1, \mathbf{a} \rangle, \ldots, b_t \approx \langle \mathbf{s}_t, \mathbf{a} \rangle)$ vs. $(\mathbf{a}, b_1, \ldots, b_t)$.

   Simple hybrid argument, since $\mathbf{a}$'s are *public*.

Search/Decision Equivalence [BFKL'94,R'05]

- Suppose $\mathcal{D}$ solves decision-LWE: it perfectly* distinguishes between pairs $(\mathbf{a}, b = \langle \mathbf{s}, \mathbf{a} \rangle + e)$ and $(\mathbf{a}, b)$.

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- (Don't actually need $\boxed{\text{prime } q = \text{poly}(n)}$.) [P'09,ACPS'09,MM'11,MP'12,BGV'12]

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Theorem [M'01,ACPS'09]

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- This maps $(\mathbf{a}, b)$ to $(\mathbf{a}', b')$, so it applies to decision-LWE too.

Public-Key Cryptosystem using LWE [Regev'05]

$\mathbf{s} \leftarrow \mathbb{Z}_q^n$

$\mathbf{A} \leftarrow \mathbb{Z}_q^{n \times m}$

(Images courtesy xkcd.org)

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(public key)

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(ciphertext 'preamble')

← $u' = \mathbf{b}^t \mathbf{x} + \text{bit} \cdot \frac{q}{2}$

('payload')

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$\mathbf{u} = \mathbf{A}\mathbf{x}$ ←

(ciphertext 'preamble')

$u' = \mathbf{b}^t \mathbf{x} + \text{bit} \cdot \frac{q}{2}$ ←

('payload')

$u' - \mathbf{s}^t \mathbf{u} \approx \text{bit} \cdot \frac{q}{2}$

(Images courtesy xkcd.org)

Public-Key Cryptosystem using LWE [Regev'05]

$\mathbf{s} \leftarrow \mathbb{Z}_q^n$

$\mathbf{A} \leftarrow \mathbb{Z}_q^{n \times m}$

$\mathbf{x} \leftarrow \{0,1\}^m$

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$(\mathbf{A}, \mathbf{b}^t), (\mathbf{u}, u')$

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$(\mathbf{A}, \mathbf{b}^t), (\mathbf{u}, u')$ by LWE

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$(\mathbf{A}, \mathbf{b}^t), (\mathbf{u}, u')$
by LWE and
by LHL when
$m \geq n \log q$

(Images courtesy xkcd.org)

"Dual" Cryptosystem [GPV'08]

$\mathbf{x} \leftarrow \{0,1\}^m$

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$(\mathbf{A}, \mathbf{u}), (\mathbf{b}, b')$

by LWE

Most Efficient Cryptosystem [A'03,LPS'10,LP'11]

$\mathbf{s} \leftarrow \chi^n$

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(public key)

$$\longleftarrow \mathbf{b} = \mathbf{A}\mathbf{r} + \mathbf{x}$$

(ciphertext 'preamble')

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(public key)

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(ciphertext 'preamble')

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$(\mathbf{A}, \mathbf{u}, \mathbf{b}, b')$

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by LWE (HNF)

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$(\mathbf{A}, \mathbf{u}, \mathbf{b}, b')$

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Wrapping Up

- Now you know all the basic techniques for working with SIS and LWE.
- We've covered a lot: do the exercises to reinforce your understanding!
- Tomorrow: more advanced applications, using "strong trapdoors."