Sequential product on effect logics

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Abstract

In categorical logic predicates on an object X are traditionally represented as subobjects. Jacobs proposes [9] an alternative in which the predicates on X are maps $p\colon X \to X + X$ with $[\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}$. If the coproduct of the category is well-behaved, the predicates form an effect algebra. So this approach is called *effect logic*.

In the three prime examples of effect logics, a sequential effect algebra arises naturally. These structures are studied by Greechie and others in quantum logic.

In this thesis we study several variations on effect logics, and prove that in these variations sequential effect algebras do not arise.

1 Introduction

1.1 Starting point

Given a category $\mathscr{C}$ with coproducts and an object X of $\mathscr{C}$. Following [9], define **the (internal) predicates on** X as

$$\mathrm{iPred}(X) = \{p\colon X \to X + X;\ [\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}\}.$$

In Set, the category of sets, the predicates on X correspond to subsets:

$$\mathrm{iPred}_{\mathsf{Set}}(X) = \Big\{p_U; \quad p_U(x) = \begin{cases} \kappa_1 x & x \in U \\ \kappa_2 x & x \notin U \end{cases}; \quad U \subseteq X\Big\}.$$

In $\mathcal{K}\ell(\mathcal{D})$, the Kleisli category of the distribution monad, the predicates on X correspond to maps $X \to [0,1]$:

$$\mathrm{iPred}_{\mathcal{K}\ell(\mathcal{D})}(X) = \big\{p_\psi; \quad p_\psi(x) = \psi(x)\kappa_1 x + (1 - \psi(x))\kappa_2 x; \quad \psi\colon X \to [0,1]\big\}.$$

In Hilb, the category of Hilbert spaces with (bounded linear) operators, the predicates on X correspond to operators on X:

$$\mathrm{iPred}_{\mathsf{Hilb}}(X) = \big\{p_A; \quad p_A(x) = (Ax, x - Ax); \quad A\colon X \to X\big\}.$$

If the category $\mathscr{C}$ is well-behaved, then $\mathrm{iPred}(X)$ carries an algebraic structure: it is an *effect module*. We will cover effect modules and related structures in Section 2. An effect module has (among other structure) a partially defined binary operation $\ovee$, a unary operation $(\ )^\perp$ and a selected element 1. In the previous examples:

$\mathscr{C}$	$p \ovee q$	defined whenever	$p^\perp$	1
Set	$U \cup V$	$U \cap V = \emptyset$	$X - U$	X
$\mathcal{K}\ell(\mathcal{D})$	$\psi + \chi$	$\psi + \chi \leq \mathbb{1}$	$\mathbb{1} - \psi$	$\mathbb{1}$

The predicates $\mathrm{iPred}(X)$ on a Hilbert space X do not form an effect module. However, the predicates that correspond to operators $0 \leq A \leq I$, called positive predicates $\mathrm{pPred}(X)$, do form an effect module.

$\mathscr{C}$	$p \ovee q$	defined whenever	$p^\perp$	1
Hilb	$A + B$	$A + B \leq I$	$I - A$	I

In Set and $\mathcal{K}\ell(\mathcal{D})$, there is an obvious way to extend the map $X \mapsto \mathrm{iPred}(X)$ to a functor $\mathrm{iPred}\colon \mathscr{C} \to \mathsf{EMod}^{\mathrm{op}}$. We write $(f)^*$ for $\mathrm{iPred}(f)$. In the case of Hilbert spaces, the map $X \mapsto \mathrm{pPred}(X)$ can be extended to a functor on the wide subcategory of Hilbert spaces with isometries: $\mathrm{pPred}\colon \mathsf{Hilb}_{\mathrm{isom}} \to \mathsf{EMod}^{\mathrm{op}}$. We have three functors:

$$\begin{array}{ccc} \mathsf{Set} & \mathcal{K}\ell(\mathcal{D}) & \mathsf{Hilb}_{\mathrm{isom}} \\ \big\downarrow{\scriptstyle \mathrm{iPred}} & \big\downarrow{\scriptstyle \mathrm{iPred}} & \big\downarrow{\scriptstyle \mathrm{pPred}} \\ \mathsf{EMod}^{\mathrm{op}} & \mathsf{EMod}^{\mathrm{op}} & \mathsf{EMod}^{\mathrm{op}} \end{array}$$

Assume we have any functor $\mathrm{Pred}\colon \mathscr{C} \to \mathsf{EMod}^{\mathrm{op}}$. Any effect module is also a partially ordered set — for each $\kappa_1\colon X \to X + Y$ we have an order preserving map $(\kappa_1)^*\colon \operatorname{Pred}(X+Y) \to \operatorname{Pred}(X)$. In our three examples, the map $(\kappa_1)^*$ has a left and right order adjoint: $\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1}$. Also, for each $p \in \operatorname{Pred}(X)$ there is an evident map $\operatorname{char}_p\colon X \to X + X$ such that $(\operatorname{char}_p)^* \coprod_{\kappa_1} 1 = p$.

On any $\operatorname{Pred}(X)$, we define $\langle p? \rangle\,(q)$, pronounced "*p andthen q*", by

$$\langle p? \rangle\,(q) = (\operatorname{char}_p)^* \textstyle\coprod_{\kappa_1} q.$$

In our examples, we have:

$\mathscr{C}$	$\langle p? \rangle\,(q)$
Set	$U \cap V$
$\mathcal{K}\ell(\mathcal{D})$	$\psi \cdot \chi$
$\mathsf{Hilb}_{\mathrm{isom}}$	$\sqrt{A}B\sqrt{A}$

These three operations are examples of Sequential Products as defined by Gudder and Greechie [7] to study Quantum Logic. In fact, these are their prime examples as well.

This leads to the following question, which is the starting point of this thesis: are there categorical axioms, which our examples obey, such that *andthen* is a sequential product as defined in [7]?

1.2 Overview

First, we will cover in Section 2 the basic theory of several algebraic structures related to effect modules. Also a few specialized results will be proven, for instance on the existence of certain effect monoids, which will be used in the study of effect logics later on. Then, in Subsection 2.4, we introduce the sequential product as defined by Gudder and Greechie in [7]. We conclude the preliminaries by recalling some basic topics, such as galois connections, monads and the Kleisli category.

We will assume the reader is familiar with Hilbert spaces and C^*-algebras. For an introductory text see [2]. To a lesser extend, we will assume familiarity with Category Theory. For an introductory text, see [1].

Before we will investigate effect logics in the line of [9], we will investigate, in Section 3, a weaker notion of effect logic, called weak effect logic. In a weak effect logic we start with a functor $\operatorname{Pred}\colon \mathscr{C} \to \mathsf{EMod}^{\mathrm{op}}$ of which we do *not* require that that $\operatorname{Pred}(X) = \operatorname{iPred}(X)$. First we look at several examples of weak effect logics. Then, we prove a representation theorem to characterize the possible *andthen* that occur in a weak effect logic.

In Section 4, we study effect logics. First we study internal predicates. Then we introduce the axioms of an effect logic and consider some examples. After that, we prove three representation theorems to partially characterize the *andthen* that occur in an effect logic.

Finally, in Section 5, we summarize our results and state the open problems.

We introduce some notions and prove some theorems that are not directly required for the results of this thesis. These are marked by a $*$. We include these for one of two reasons.

- Either it is a negative result to justify our approach. For instance, the non-commutative effect monoid we construct in Subsection 2.2.2 is rather complicated. One might expect that there is a finite example. That is why we include Proposition 40, that states every finite effect monoid is commutative.

- Or: the result is a worthwhile deviation. For instance, to show there is a non-commutative effect monoid (Corollary 51) we only require one direction of the equivalence between OAU-algebras and convex effect monoids. However, the full result is worth proving.

Contents

2 Preliminaries

Before we can fully state the axioms of (weak) effect logics, we need to introduce some algebraic structures of which the *effect algebra*, *effect module* and *effect monoid* are the most important. Also, we will use the notion of a *Galois connection* (or order adjunction).

One class of effect logics is based on the Kleisli category of a distribution monad generalized to an arbitrary *effect monoid*. This class of effect logics is used to prove Theorem 116. Due to this theorem, we will study examples of non-commutative effect monoids.

2.1 Effect algebras

In an effect logic, the set of predicates on an object will have (among others) the following algebraic structure.

Definition 1 ([3])**.** Given a structure $\langle E, \ovee, 0, 1\rangle$ where

- $\ovee \colon E \times E \to E$ is a partial binary operation: we write $a \perp b$ whenever $a \ovee b$ is defined *and*
- $0, 1 \in E$ are selected elements: 0 is called the **zero** and 1 the **unit**.

This structure is called an **effect algebra (EA)** if the following holds.

(E1) (partial commutativity) If $a \perp b$ then $b \perp a$ and $a \ovee b = b \ovee a$.

(E2) (partial associativity) If $a \perp b$ and $a \ovee b \perp c$, then $b \perp c$, $a \perp b \ovee c$ and $a \ovee (b \ovee c) = (a \ovee b) \ovee c$.

(E3) (unique orthocomplement) For every $a \in E$ there exists a unique $a^{\perp}$ such that $a \ovee a^{\perp} = 1$.

(E4) If $a \perp 1$, then $a = 0$.

Given two effect algebras E and F a map $f \colon E \to F$ is an **effect algebra homomorphism** if

1. f is **additive**, that is: when $a \perp b$ for $a, b \in E$, then also $f(a) \perp f(b)$ and furthermore: $f(a) \ovee f(b) = f(a \ovee b)$ *and*
2. f **preserves the unit**, that is: $f(1) = 1$.

The effect algebras along with their homomorphisms form a category called EA.

Example 2. The following are effect algebras.

1. $\langle [0,1], +, 0, 1\rangle$ where $[0,1] \subseteq \mathbb{R}$ is the unit interval and $+$ is the normal addition. $x \perp y$ whenever $x + y \leq 1$ and $x^{\perp} = 1 - x$.
2. $\langle \mathcal{P}(X), \dot\cup, \emptyset, X\rangle$ where $\mathcal{P}(X)$ is the set of subsets of X and $\dot\cup$ is the disjoint union: $A \perp B$ whenever $A \cap B = \emptyset$. The orthocomplement is the normal complement: $A^{\perp} = X - A$.
3. $\langle \mathrm{Eff}(\mathscr{H}), +, 0, I\rangle$ where $\mathrm{Eff}(\mathscr{H})$ are the positive operators on a Hilbert space below or equal to I; I is the unit operator and $+$ is addition of operators. $A \perp B$ whenever $A + B \leq I$ and $A^{\perp} = I - A$.

There is more structure on an effect algebra: there is a partial order $\leq$ and a difference $\ominus$. Before we define these, we need to derive some basic properties.

Proposition 3. *In any effect algebra, we have*

1. *(involution)* $a^{\perp\perp} = a$;
2. $1^{\perp} = 0$ *and* $0^{\perp} = 1$;
3. *(zero)* $a \perp 0$ *and* $a \ovee 0 = a$;
4. *(positivity) if* $a \ovee b = 0$ *then* $a = 0$ *and* $b = 0$ *and*
5. *(cancellation) if* $a \ovee b = a \ovee c$ *then* $b = c$.

Proof. One at a time.

1. By (E3), we have $a \ovee a^{\perp} = 1$. Then by (E1), we have $a^{\perp} \ovee a = 1$. And thus by (E3) again, we must have $a = a^{\perp\perp}$.
2. By (E3), we have $1 \ovee 1^{\perp} = 1$. Then by (E4), we have $1^{\perp} = 0$. By the previous $1 = 1^{\perp\perp} = 0^{\perp}$.
3. By (E3) and (E1), we have $a^{\perp} \ovee a = 1$. By the previous $(a^{\perp} \ovee a) \ovee 0 = 1 \ovee 0 = 1$. Thus by (E2) we know $a^{\perp} \ovee (a \ovee 0) = 1$. Hence by (E3) we have $a \ovee 0 = a$.
4. By (E3) and the previous, we have $(a \ovee b) \ovee 1 = 0 \ovee 1 = 1$. Then by (E2) we have $b \perp 1$. Hence by (E4) we know $b = 0$. And similarly, using (E1), we see $a = 0$.
5. Since $(a\ovee b)^{\perp} \ovee (a \ovee c) = (a \ovee b)^{\perp} \ovee (a \ovee b) = (((a\ovee b)^{\perp} \ovee a) \ovee b = 1$, we know by uniqueness of the orthocomplement that $b = ((a \ovee b)^{\perp} \ovee a)^{\perp} = c$. □

Thus: effect algebras are partial commutative monoids with the extra axioms (E3) and (E4).

Definition 4. We write $a \leq b$ if there exists a c such that $a \ovee c = b$.

Proposition 5. *For any effect algebra* E.

1. $\langle E, \leq \rangle$ *is a poset;*
2. $a \leq b$ *if and only if* $b^{\perp} \leq a^{\perp}$;
3. 0 *is the minimum and* 1 *is the maximum element;*
4. *if* $a \leq b$ *and* $b \perp c$*, then* $a \perp c$ *and* $a \ovee c \leq b \ovee c$ and
5. $a \perp b$ *if and only if* $a \leq b^{\perp}$.

Proof. One by one.

1. First, reflexivity: $a \ovee 0 = a$ thus $a \leq a$.

   Then, anti-symmetry: suppose $a \leq b$ and $b \leq a$. That is: there are $c, d \in E$ such that $a \ovee c = b$ and $b \ovee d = a$. Then $a \ovee 0 = a = b \ovee d = (a \ovee c) \ovee d = a \ovee (c \ovee d)$. Hence by cancellation $c \ovee d = 0$. Thus by positivity $c = d = 0$. Consequently $a = b \ovee d = b \ovee 0 = b$.

   Finally, transitivity: suppose $a \leq b$ and $b \leq c$. Then there are $d, e \in E$ such that $a \ovee d = b$ and $b \ovee e = c$. Hence $c = b \ovee e = (a \ovee d) \ovee e = a \ovee (d \ovee e)$. Thus $a \leq c$.

2. Suppose $a \leq b$. Then $a \ovee c = b$ for some c. Note that $(b^\perp \ovee c) \ovee a = b^\perp \ovee b = 1$. Thus $b^\perp \ovee c = a^\perp$. Hence $b^\perp \leq a^\perp$.

   Conversely, suppose $b^\perp \leq a^\perp$. Then by the previous $a = a^{\perp\perp} \leq b^{\perp\perp} = b$, as desired.

3. $0 \ovee a = a$ hence $0 \leq a$. Thus 0 is the minimum.

   In particular $0 \leq a^\perp$. Then by the previous: $a = a^{\perp\perp} \leq 0^\perp = 1$. Thus 1 is the maximum.

4. Suppose $a \leq b$ and $b \perp c$. There is a d such that $a \ovee d = b$. Hence $a \ovee c \ovee d = a \ovee d \ovee c = b \ovee c$. Thus $a \ovee c \leq b \ovee c$.

5. Suppose $a \perp b$. Then $(a \ovee b)^\perp \ovee a \ovee b = 1$. Hence by uniqueness of orthocomplement: $(a \ovee b)^\perp \ovee a = b^\perp$. And thus $a \leq b^\perp$.

   Conversely, suppose $a \ovee c = b^\perp$. Then $b \perp b^\perp = a \ovee c$ and thus by (E3) and (E2) in particular $a \perp b$. □

Note that if $a \ovee c = b = a \ovee c'$, then by cancellation $c = c'$.

Definition 6. Suppose $a \leq b$, let $b \ominus a$ be the unique element such that we have $a \ovee (b \ominus a) = b$.

Proposition 7. *For any effect algebra E.*

(D1) $x \ominus y$ *is defined if and only if* $y \leq x$*;*

(D2) $x \ominus y \leq x$*;*

(D3) $x \ominus (x \ominus y) = y$ and

(D4) if $x \leq y \leq z$*, then* $z \ominus y \leq z \ominus x$ *and* $(z \ominus x) \ominus (z \ominus y) = y \ominus x$*.*

Proof. One by one.

(D1) By definition.

(D2) $x \ovee (x \ominus y) = x$ hence $x \ominus y \leq x$.

(D3) $(x \ominus (x \ominus y)) \ovee (x \ominus y) = x = (x \ominus y) \ovee y$ and thus, by cancellation, we derive $x \ominus (x \ominus y) = y$.

(D4) $x \ovee (y \ominus x) \ovee (z \ominus y) = y \ovee (z \ominus y) = z$ and hence by uniqueness of the difference we have $(y \ominus x) \ovee (z \ominus y) = z \ominus x$ and thus $z \ominus y \leq z \ominus x$.

$$\begin{aligned}((z \ominus x) \ominus (z \ominus y)) \ovee z \ovee x &= ((z \ominus x) \ominus (z \ominus y)) \ovee (z \ominus y) \ovee y \ovee x \\ &= (z \ominus x) \ovee y \ovee x \\ &= z \ovee y\end{aligned}$$

and thus by cancellation $((z \ominus x) \ominus (z \ominus y)) \ovee x = y$ and finally by uniqueness of the difference: $y \ominus x = (z \ominus x) \ominus (z \ominus y)$. □

Remark 8. Any $\langle X, \leq, \ominus, 1\rangle$ in which (D1)–(D4) hold and has 1 as largest element is called a D-poset. Thus any effect algebra is a D-poset. Conversely: on any D-poset we can define $a \ovee b = c \Leftrightarrow c \ominus b = a$ and $a^\perp = 1 \ominus a$. This will form an effect algebra. Thus: D-posets are another way to look at effect algebras.

Proposition 9. *For any effect algebra E, we have*

1. $a^\perp = 1 \ominus a$;
2. *if* $b \le a$ *then* $(a \ominus b)^\perp = a^\perp \ovee b$;
3. *if* $b, c \le a$ *and* $a \ominus b = a \ominus c$, *then* $b = c$;
4. *if* $a \le b, c$ *and* $b \ominus a = c \ominus a$, *then* $b = c$;
5. *if* $a \perp b$, *then* $(a \ovee b) \ominus b = a$ and
6. $a \ominus (b \ominus c) = (a \ominus b) \ovee c$ *whenever they are both defined.*

Proof. One at a time.

1. $(1 \ominus a) \ovee a = 1$ and thus the unique orthocomplement $a^\perp = 1 \ominus a$.
2. Certainly $(a \ominus b) \ovee b = a$. Thus $(a \ominus b) \ovee b \ovee a^\perp = 1$. Hence by uniqueness of the orthocomplement we know $(a \ominus b)^\perp = a^\perp \ovee b$.
3. Suppose $a \ominus b = a \ominus c$. Then by the previous: $a^\perp \ovee b = (a \ominus b)^\perp = (a \ominus c)^\perp = a^\perp \ovee c$ and thus by cancellation $b = c$.
4. Suppose $b \ominus a = c \ominus a$. Then $b = (b \ominus a) \ovee a = (c \ominus a) \ovee a = c$.
5. Certainly $b \le a \ovee b$ and $((a \ovee b) \ominus b) \ovee b = a \ovee b$. Cancelling: $(a \ovee b) \ominus b = a$.
6. Certainly $a \le b \ovee b^\perp$. Thus $a \ominus b \le b^\perp$. Hence $(a \ominus b) \ovee c \le b^\perp \ovee c = (b \ominus c)^\perp$. Thus $(a \ominus b) \ovee c \perp b \ominus c$. Consequently $(a \ominus b) \ovee c \ovee (b \ominus c) = (a \ominus b) \ovee b = a$. By uniqueness of the difference $(a \ominus b) \ovee c = a \ominus (b \ominus c)$. □

Definition 10. Given $a^\perp \perp b^\perp$, define $a \owedge b = (a^\perp \ovee b^\perp)^\perp$.

Proposition 11. *For any effect algebra E.*

1. $(a \ovee b)^\perp = a^\perp \owedge b^\perp$;
2. $(a \owedge b)^\perp = a^\perp \ovee b^\perp$;
3. $a \ominus b = a \owedge b^\perp$ and
4. $E^\partial = \langle E, \owedge, 1, 0 \rangle$ *is as an effect algebra isomorphic to E.*

Proof. One by one.

1. $(a \ovee b)^\perp = (a^{\perp\perp} \ovee b^{\perp\perp})^\perp = a^\perp \owedge b^\perp$
2. $(a \owedge b)^\perp = (a^\perp \ovee b^\perp)^{\perp\perp} = a^\perp \ovee b^\perp$
3. By the previous: $a \ominus b = (a^\perp \ovee b)^\perp = a^{\perp\perp} \owedge b^\perp = a \owedge b^\perp$.
4. The map $x \mapsto x^\perp$ is its own inverse and thus a bijection. By the previous, $0^\perp = 1$ and $1^\perp = 0$ the effect algebra it induces is precisely E^∂. Hence $x \mapsto x^\perp$ is an isomorphism between E and E^∂. □

By definition, we only require an effect algebra homomorphism to preserve $\ovee$ and 1. This is enough for it to preserve the other structure as well

Proposition 12. *For any additive $f\colon E \to F$, we have the following.*

1. *(order preserving) If $x \leq y$, then $f(x) \leq f(y)$.*
2. *Whenever $y \ominus x$ is defined, we have: $f(y \ominus x) = f(y) \ominus f(x)$.*
3. *(preservers zero) $f(0) = 0$*

If additionally, f preserves the unit (and thus f is an effect algebra homomorphism), then also the following holds.

4. *(preserves orthocomplement) $f(x^\perp) = f(x)^\perp$*
5. *If $x^\perp \perp y^\perp$, then $f(x \owedge y) = f(x) \owedge f(y)$.*

Proof. One at a time.

1. Suppose $x \leq y$. Then there is a c such that $x \ovee c = y$. Thus $f(x) \ovee f(c) = f(x \ovee c) = f(y)$. Hence $f(x) \leq f(y)$, as desired.
2. Suppose $y \ominus x$ is defined. By definition $y = (y \ominus x) \ovee x$. Thus $f(y) = f((y \ominus x) \ovee x) = f(y \ominus x) \ovee f(x)$. By uniqueness of the difference: $f(y) \ominus f(x) = f(y \ominus x)$.
3. Certainly $0 = 0 \ovee 0$ and thus $0 \ovee f(0) = f(0) = f(0 \ovee 0) = f(0) \ovee f(0)$. Hence by cancelling: $0 = f(0)$, as desired.
4. By definition $1 = x \ovee x^\perp$. Thus $1 = f(1) = f(x \ovee x^\perp) = f(x) \ovee f(x^\perp)$. By uniqueness of the orthocomplement, we know $f(x^\perp) = f(x)^\perp$.
5. By definition $x \owedge y = (x^\perp \ovee y^\perp)^\perp$. Thus by the previous $f(x \owedge y) = f((x^\perp \ovee y^\perp)^\perp) = (f(x)^\perp \ovee f(y)^\perp)^\perp = f(x) \owedge f(y)$. □

2.1.1 ∗ Some results on infima and suprema

We will look at the order structure of an effect algebra in more detail. The results we prove, will be useful when we consider effect algebras that are lattice ordered in Subsection 2.1.5.

If we consider the partial binary relations $\ovee$, $\owedge$ and $\ominus$ with one argument fixed and restricted to its domain, we see they are either order isomorphisms or order antiisomorphisms.

Proposition 13. *Given an effect algebra E.*

1. *$b \mapsto a \ovee b$ is an order isomorphism from $\downarrow a^\perp$ to $\uparrow a$. Its inverse is the map $b \mapsto b \ominus a$, an order isomorphism from $\uparrow a$ to $\downarrow a^\perp$.*
2. *$b \mapsto a \owedge b$ is an order isomorphism from $\uparrow a^\perp$ to $\downarrow a$. Its inverse is the map $b \mapsto b \ovee a^\perp$, an order isomorphism from $\downarrow a$ to $\uparrow a^\perp$.*
3. *$b \mapsto a \ominus b$ is an order antiisomorphism from $\downarrow a$ to $\downarrow a$, which is its own inverse.*

Proof. We already saw that the maps are appropriately order preserving or order reversing. Also we saw that the cancellation law holds for all these operations, hence all maps are injective. It is left to show that each map is defined on the given domain; maps into the given codomain and is surjective.

1. $a \ovee b$ is defined whenever $b \leq a^\perp$. Thus $\downarrow a^\perp$ is indeed the domain. Furthermore $a \ovee b \geq a$. Given any $a \leq c$. Then $a \ovee (a \ominus c) = a$. Hence the map is surjective. This also show that $b \mapsto a \ominus b$ is its inverse.

2. Note that $b \ominus a^\perp = (b^\perp \ovee a^\perp)^\perp = b \owedge a$. Thus the previous pair of maps with $a^\perp$ for a are exactly the current maps. Hence these are also order isomorphisms.

3. $a \ominus b$ is defined whenever $b \leq a$. Thus $\downarrow a$ is indeed its domain. Furthermore $a \ominus b \leq a$. By (D3) the map is its own inverse. Thus in particular, it is surjective. □

A very useful corollary of the previous is that the operations, with some restriction due to their partial definition, either preserve or invert suprema and infima.

Corollary 14. *Given an effect algebra E and a subset $U \subseteq E$. Write $a \ovee U = \{a \ovee u;\ u \in U\}$. And similarly for the other operations. Write $a \leq U$ whenever $a \leq u$ for each $u \in U$.*

1. *Suppose $U \leq a^\perp$, then $\bigwedge a \ovee U$ exists if and only if $a \ovee \bigwedge U$ exists and we have $\bigwedge a \ovee U = a \ovee \bigwedge U$. Also $\bigvee a \ovee U$ exists if and only if $a \ovee \bigvee U$ exists and we have $\bigvee a \ovee U = a \ovee \bigvee U$.*

2. *Suppose $U \leq a$, then $\bigwedge a \ominus U$ exists if and only if $a \ominus \bigvee U$ exists and we have $\bigwedge a \ominus U = a \ominus \bigvee U$. Also $\bigvee a \ominus U$ exists if and only if $a \ominus \bigwedge U$ exists and we have $\bigvee a \ominus U = a \ominus \bigwedge U$.*

Proof. Note that if $U \leq a^\perp$, then also $\bigvee U, \bigwedge U \leq a^\perp$, whenever they exist. Thus the suprema and infima in the order restricted to $\downarrow a^\perp$, are the same as in the whole of E. Similarly for $\uparrow a$. The first part is now an easy consequence of the fact that $a \ovee (\ )$ is an order isomorphism, from $\downarrow a^\perp$ to $\uparrow a$, which preserves suprema and infima. The second part is similar. □

One of the applications is the following proposition.

Proposition 15. *Given an effect algebra E and $a, b \in E$.*
If $a \perp b$ and $a \vee b$ exists, then $a \wedge b = (a \ovee b) \ominus (a \vee b)$.

Proof. Certainly $a, b \leq a \ovee b$. And thus by the previous corollary, we have $(a \ovee b) \ominus (a \vee b) = ((a \ovee b) \ominus a) \wedge ((a \ovee b) \ominus b) = a \wedge b$. □

Corollary 16. *The previous proposition has some useful consequences.*

1. *If $a \wedge b = 0$ and $a \perp b$, then $a \ovee b = a \vee b$.*

2. *Whenever it is all defined: $(a \vee b) \ovee (a \wedge b) = a \ovee b$.*

2.1.2 $*$ Isotropic index

In this section we introduce terminology that we will use when we study finite effect algebras in Subsection 2.1.6 and lexicographically ordered vector spaces in Subsection 2.2.2.

Definition 17. Given an effect algebra E.

1. An element e is called **isotropic** if $e \perp e$.

2. Given $n \in \mathbb{N}$ and an $e \in E$. We can define $0e = 0$ and $(n+1)e = ne \ovee e$ whenever $ne \perp e$. That is: ne is e summed n times with itself.

3. If ne is defined, but $(n+1)e$ is not, then n is the **isotropic index** of e; in symbols: $\mathrm{ord}(e) = n$.

4. If ne is defined for all $n \in \mathbb{N}$, then e is called **infinitesimal** and we write $\mathrm{ord}(e) = \infty$.

5. If a is infinitesimal and for all $n \in \mathbb{N}$ we have $na \le b$, then a **is infinitely smaller than** b and we write $a \ll b$.

6. If 0 is the only infinitesimal, then we call E **Archimedean**.

2.1.3 Interval effect algebras

The effect algebras $[0, 1]$ and $\mathrm{Eff}(\mathscr{H})$ we saw before are examples of a more general class of effect algebras: those that are derived from partially ordered abelian groups.

Definition 18. A structure $\langle G, +, -, \le, 0\rangle$ is called a **(partially) ordered abelian group** provided

1. $\langle G, +, -, 0\rangle$ is an abelian group;

2. $\langle G, \le\rangle$ is a partial order *and*

3. if $a \le b$ then $a + c \le b + c$ for any $a, b, c \in G$.

An element $a \in G$ of a partially ordered group is called **positive** if $0 \le a$. We write $G^+ = \{g;\ 0 \le g\}$ for the positive elements.

Given two elements $a \le b$ in an ordered group G, we define the **(order) interval** with endpoints a and b as $[a, b] = \{c;\ c \in G;\ a \le c \le b\}$.

Example 19. The following are examples of ordered abelian groups.

1. $\langle \mathbb{R}, +, -, \le, 0\rangle$, the real line with addition.

2. $\langle \mathscr{B}(\mathscr{H})_{\mathbb{R}}, +, -, \le, 0\rangle$, the Hermitean operators on a Hilbert $\mathscr{H}$ space where the order is defined as follows. $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for all $x \in \mathscr{H}$.

Proposition 20. *Given any ordered group G and strictly positive element $0 < u$, the structure $\langle [0, u], +, 0, u\rangle$ is an effect algebra with $a^\perp = u - a$. Such an effect interval is called an **interval effect algebra**.*

Proof. Assume $0 \le a, b, c \le u$.

(E1) If $a + b \le u$, then $a + b = b + a \le u$, as desired.

(E2) If $(a + b) + c \le u$, then $(a + b) + c = a + (b + c) \le u$, as desired.

(E3) $0 \leq a$ thus $u \leq u + a$ hence $u - a \leq u + a - a = u$. Also $a \leq u$ thus $0 = a - a \leq u - a$. Thus $u - a$ is in $[0, u]$.

Clearly $a + (u - a) = u$, thus $u - a$ is an orthocomplement of a. Given any other $b \in [0, u]$ such that $a + b = u$. Then $b = u - a$. Thus the orthocomplement is unique.

(E4) Suppose $a + u \leq u$. Then $0 \leq a \leq 0$. Thus $a = 0$. □

2.1.4 Convex effect algebras

When we consider interval effect algebras derived from ordered vector spaces, the effect algebra inherits a scalar multiplication from the vector space. For a few proofs it is useful to introduce the notion of a scalar multiplication on any effect algebra. This definition will turn out to be equivalent to that of a $[0, 1]$-effect module.

Definition 21. An effect algebra E is called **convex** if for every $\lambda \in [0, 1]$ and $a \in E$ there exists a $\lambda \cdot a$ such that

(C1) $\alpha \cdot (\beta \cdot a) = (\alpha\beta) \cdot a$;

(C2) if $\alpha + \beta \leq 1$, then $\alpha a \perp \beta a$ and $(\alpha + \beta) \cdot a = \alpha \cdot a \ovee \beta \cdot a$;

(C3) if $\lambda \in [0, 1]$ and $a \perp b$ then $\lambda \cdot a \perp \lambda \cdot b$ and $\lambda \cdot a \ovee \lambda \cdot b = \lambda \cdot (a \ovee b)$ *and*

(C4) $1a = a$.

However, we did not discover anything new: any convex effect algebra is an interval effect algebra of some ordered vector space.

Definition 22. Given an ordered vector space V over $\mathbb{R}$ and a vector $u > 0$. We say $[0, u]$ **generates** V if for every $v \in V$ there are $r_1, r_2 \in \mathbb{R}$ and $v_1, v_2 \in [0, u]$ such that $v = r_1 v_1 - r_2 v_2$.

Theorem 23 (S. Gudder and S. Pulmannová [5])**.** *For any convex effect algebra E, there exists a unique ordered vector space V and an $u > 0$ such that $[0, u] \cong E$ and $[0, u]$ generates V.*

2.1.5 ∗ Lattice effect algebras

Another class of effect algebras are those derived from orthomodular lattices.

Definition 24. A **lattice** is a partial order for which each finite supremum and infimum exists. A **bounded lattice** is a lattice with a maximum element 1 and a minimum 0.

An **orthocomplemented lattice** is a bounded lattice together with a unary operation $(\ )^\perp$ such that

1. (complement) $a^\perp \vee a = 1$ and $a^\perp \wedge a = 0$;
2. (involution) $a^{\perp\perp} = a$ *and*
3. (order-reversing) if $a \leq b$ then $b^\perp \leq a^\perp$.

An **orthomodular lattice** L is a orthocomplemented lattice such that for any $a, b \in L$, we have: if $a \leq b$ then $a \vee (a^\perp \wedge b) = b$.

Example 25. The following are orthomodular lattices.

1. Any Boolean algebra is an orthomodular lattice with as orthocomplement the normal complement. In particular $\langle \mathcal{P}(X), \cap, \cup, \emptyset, X, X - (\ )\rangle$.

2. Given a Hilbert space, the partial order of its closed linear subspaces by inclusion is an orthomodular lattice.

We can extend any orthomodular lattice to an effect algebra. Before we prove this, we need a lemma.

Lemma 26. *In any orthocomplemented lattice, the laws of de Morgan are valid. That is: we have* $(a \vee b)^\perp = a^\perp \wedge b^\perp$ *and* $(a \wedge b)^\perp = a^\perp \vee b^\perp$.

Proof. $a \vee b \geq b$ thus $(a \vee b)^\perp \leq b^\perp$. Also $(a \vee b)^\perp \leq a^\perp$. By the definition of infimum $(a \vee b)^\perp \leq a^\perp \wedge b^\perp$. For the other inequality, we first note that clearly $a^\perp \wedge b^\perp \leq a^\perp$. Thus $a \leq (a^\perp \wedge b^\perp)^\perp$. Also $b \leq (a^\perp \wedge b^\perp)^\perp$. Hence $a \vee b \leq (a^\perp \wedge b^\perp)^\perp$. That is $(a \vee b)^\perp \geq a^\perp \wedge b^\perp$. We proved $(a \vee b)^\perp = a^\perp \wedge b^\perp$. The proof of the other equality is dual. □

Proposition 27. *Given any orthomodular lattice* $\left\langle L, \wedge, \vee, 0, 1, (\ )^\perp \right\rangle$. *Define* $a \perp b$ *if* $a \leq b^\perp$ *and then* $a \otimes b = a \vee b$. *The structure* $\langle L, \otimes, 0, 1\rangle$ *is an effect algebra. Furthermore, the order of the effect algebra is the same as the order on* L.

Proof. To prove that $\langle L, \otimes, 0, 1\rangle$ is an effect algebra.

(E1) Suppose $a \perp b$. Then $a \leq b^\perp$ hence $b \leq a^\perp$, since $(\ )^\perp$ is order reversing. Thus $b \perp a$ and $a \otimes b = a \vee b = b \vee a = b \otimes a$.

(E2) Suppose $a \perp b$ and $a \otimes b \perp c$. Then $a \leq b^\perp$ and $a \otimes b = a \vee b \leq c^\perp$. Certainly $b \leq a \vee b \leq c^\perp$. Thus $b \perp c$. Also $c \leq (a \vee b)^\perp = a^\perp \wedge b^\perp$. Thus $c \vee b \leq (a^\perp \wedge b^\perp) \vee c = a^\perp$, by the orthomodularity since $b \leq a^\perp$. Hence $a \perp b \vee c$.

Thus we are justified to write: $(a \otimes b) \otimes c = a \vee b \vee c = a \otimes (b \otimes c)$.

(E3) $a \leq a = a^{\perp\perp}$ thus $a \perp a^\perp$ hence $a \otimes a^\perp = a \vee a^\perp = 1$. Indeed: $a^\perp$ is an orthocomplement. Suppose $a \otimes b = 1$ for some b. Then $b \leq a^\perp$ and $a \vee b = 1$. Thus by orthomodularity $a^\perp = b \vee (b^\perp \wedge a^\perp) = b \vee (a \vee b)^\perp = b \vee 1^\perp = b \vee 0 = b$ — the orthocomplement is unique.

(E4) Suppose $a \perp 1$. Then $a \leq 1^\perp = 0$. Thus $a = 0$, as desired.

Now we prove that the order of the effect algebra on L is the same as the order of the lattice. Suppose that there is a c such that $a \otimes c = b$. By definition, we have $b = a \otimes c = a \vee c \geq a$. Conversely, suppose $a \leq b$. Then by orthomodularity we have $a \vee (a^\perp \wedge b) = b$. Certainly $a^\perp \geq a^\perp \wedge b$, thus $a \perp a^\perp \wedge b$. Hence $a \otimes (a^\perp \wedge b) = b$. □

We saw that every orthomodular lattice can be extended to an effect algebra. This extension is, in fact, unique.

Definition 28. If the order of an effect algebra is a lattice; that is: finite infima and suprema exist; then it is called a **lattice effect algebra**.

If the order of an effect algebra is an orthomodular lattice, then it is called an **orthomodular effect algebra**.

Proposition 29. *If E is an orthomodular effect algebra, then $a \varovee b = a \vee b$.*

Proof. Given $a \perp b$. Then $a \leq b^\perp$. Certainly $b \leq 1$. Thus by modularity $b \vee b^\perp = b \vee (b^\perp \wedge 1) = 1$. Hence $0 = 1^\perp = (b \vee b^\perp)^\perp = b^\perp \wedge b$. Consequently $a \wedge b \leq b^\perp \wedge b = 0$. By Corollary 16, we see $a \varovee b = (a \vee b) \ominus (a \wedge b) = (a \vee b) \ominus 0 = a \vee b$. □

2.1.6 ∗ Finite effect algebras

Another obvious class to investigate are the finite effect algebras.

Definition 30. An effect algebra E is called **finite** if it has a finite number of elements.

Definition 31. Given an effect algebra E. An element $a \in E$ is called an **atom** if $0 < a$ and for every $b < a$ we know $b = 0$.

Proposition 32. *Given a finite effect algebra E such that $0 \neq 1$ and $a_1, \ldots, a_n$ are its atoms. Then for each $e \in E$ there exist $e_1, \ldots, e_n \in \mathbb{N}$ such that $e = e_1 a_1 \varovee \cdots \varovee e_n a_n$.*

Proof. First we prove that for every $e > 0$, there exists an atom a such that $0 < a \leq e$. If e itself is an atom, we are done. If not, there must exist an e_1 such that $0 < e_1 < e$. Now we consider e_1. If it is atom, we are done. If not, there must exist an e_2 such that $0 < e_2 < e_1$. And so forth. Since E is finite, there cannot exist an infinite strictly decreasing sequence $0 < \ldots < e_3 < e_2 < e_1 < e$. Thus there must be an atom $a < e$.

Now we prove e is a sum of atoms. If e is an atom or equal 0, we are done. If not, there exists an atom a_1 such that $0 < a_1 < e$. Then $0 < e \ominus a_1 < e$. If $e \ominus a_1$ is an atom, we are done. If not: there exists an atom a_2 such that $0 < a_2 < e \ominus a_1$. Then $0 < e \ominus a_1 \ominus a_2 < e \ominus a_1 < e$. If $e \ominus a_1 \ominus a_2$ is an atom, we are done. If not: then we repeat with $e \ominus a_1 \ominus a_2$. This procedure must end, for otherwise we would find an infinite strictly decreasing sequence. □

Definition 33. Given a finite effect algebra E with atoms $a_1, \ldots, a_n$. A tuple $(t_1, \ldots, t_n) \in \mathbb{N}^n$ is called a **multiplicity vector** if $t_1 a_1 \varovee \cdots \varovee t_n a_n = 1$. Let $T(E)$ denote the set of multiplicity vectors.

The multiplicity vectors determine a finite effect algebra.

Definition 34. Define $\downarrow T = \{a;\ a \in \mathbb{N}^n;\ \exists t \in T(E).\ a_i \leq t_i \text{ for all } i\}$. For $a, b \in \downarrow T$, define $a + b$ pointwise. That is: $(a + b)_i = a_i + b_i$ for all i. We say $a \perp b$ if $a + b \in \downarrow T$. Then we define $a \varovee b = a + b$. We define an equivalence relation on $\downarrow T$ as follows: $a \sim b$ if there is a $c \in \mathbb{N}^n$ such that both $a + c, b + c \in T(E)$.

Theorem 35 ([4])**.** *Given a finite effect algebra E such that $0 \neq 1$.*
Then: E is isomorphic to $\langle {\downarrow T}/{\sim}, \varovee, [(0, \ldots, 0)]_\sim, T \rangle$.

Proof. We would like to prove $\langle {\downarrow T}/{\sim}, \varovee, [(0, \ldots, 0)]_\sim, T \rangle$ is an effect algebra and then prove it is isomorphic to E. However, for arbitrary T, it is not an effect algebra. That is why we first prove that there is an operation preserving bijection φ and then conclude the latter is an effect algebra and hence φ an isomorphism.

Given $b \in E$. Suppose c and c' are tuples of natural numbers such that

$$b = c_1 a_1 \ovee \cdots \ovee c_n a_n = c'_1 a_1 \ovee \cdots \ovee c'_n a_n.$$

Let d be such that

$$b^\perp = d_1 a_1 \ovee \cdots \ovee d_n a_n.$$

Then:

$$\begin{aligned} 1 = b \ovee b^\perp &= (c_1 + d_1) a_1 \ovee \cdots \ovee (c_n + d_n) a_n \\ &= (c'_1 + d_1) a_1 \ovee \cdots \ovee (c'_n + d_n) a_n. \end{aligned}$$

Hence: $c + d, c' + d \in T$. Thus: $c \sim c'$. Define $\varphi \colon E \to {\downarrow T}/{\sim}$ by $\varphi(b) = [c]_\sim$.

- Given $c \in {\downarrow} T$. Then $c \leq t$ for some $t \in T$. $b = c_1 a_1 \ovee \cdots \ovee c_n a_n$ is defined, since $c \leq t$ and t is a multiplicity vector. By definition: $\varphi(b) \sim c$. Thus φ is surjective.
- Given $b, b' \in E$ with $\varphi(b) = \varphi(b')$. Thus: there are tuples c, c' and d such that $b = c_1 a_1 \ovee \cdots \ovee c_n a_n$; $b' = c'_1 a_1 \ovee \cdots \ovee c'_n a_n$ and $b + d, b' + d \in T$. Note $d \leq t$ for some $t \in T$. Hence $d' = d_1 a_1 \ovee \cdots \ovee d_n a_n$ is defined. Furthermore: $d' \ovee b = d' \ovee b' = 1$. By canceling: $b = b'$. Thus φ is injective.
- Given tuples c, c' and d such that $c \sim c'$ and $c \perp d$. Then $c + d \in {\downarrow} T$ and:

  $$(c'_1 + d_1) a_1 \ovee \cdots \ovee (c'_n + d_n) a_n = (c_1 + d_1) a_1 \ovee \cdots \ovee (c_n + d_n) a_n \leq 1.$$

  Hence $c' \perp d$ and $c + d \sim c' + d$. Thus $\perp$ and $\ovee$ can be extended to ${\downarrow T}/{\sim}$.
- Suppose $b, b' \in E$ with $b \perp b'$. Let c and c' be such that:

  $$b = c_1 a_1 \ovee \cdots \ovee c_n a_n \qquad b' = c'_1 a_1 \ovee \cdots \ovee c'_n a_n.$$

  Then: $b \ovee b' = (c_1 + c'_1) a_1 \ovee \cdots \ovee (c_n + c'_n) a_n$. Hence $c + c' \in \varphi(b \ovee b')$. Thus $c + c' \in {\downarrow} T$. That is: $c \perp c'$. Also $\varphi(b) \ovee \varphi(b') = \varphi(b \ovee b')$.

  Conversely, suppose $\varphi(b) \perp \varphi(b')$. Again, find c and c' such that:

  $$b = c_1 a_1 \ovee \cdots \ovee c_n a_n \qquad b' = c'_1 a_1 \ovee \cdots \ovee c'_n a_n.$$

  Then $c \perp c'$. Thus $(c_1 + c'_1) a_1 \ovee \cdots \ovee (c_n + c'_n) a_n$ is defined. Thus $b \perp b'$.

  Finally, clearly $\varphi(1) = T$. The operations on ${\downarrow T}/{\sim}$ are preserved by the surjective φ. Hence it is an effect algebra. Furthermore: since φ is injective, we have ${\downarrow T}/{\sim} \cong E$. $\square$

2.2 Effect monoids

Various examples of effect algebras also carry a multiplication. We will consider *effect monoids*, which are effect algebras with an associative and distributive multiplication. They play an important rôle in one class of effect logics, see Subsection 4.3.2.

Definition 36. A structure $\langle E, \owedge, \odot, 0, 1\rangle$ is called an **effect monoid** if $\langle E, \owedge, 0, 1\rangle$ is an effect algebra and the (total) binary operation $\odot$ satisfies the following.

(M1) (unit) $a \odot 1 = 1 \odot a = a$.

(M2) (left distributivity) if $a \perp b$, then $c \odot a \perp c \odot b$ and $(c \odot a) \owedge (c \odot b) = c \odot (a \owedge b)$.

(M3) (right distributivity) if $a \perp b$, then $a \odot c \perp b \odot c$ and $(a \odot c) \owedge (b \odot c) = (a \owedge b) \odot c$.

(M4) (associativity) $a \odot (b \odot c) = (a \odot b) \odot c$.

Example 37. We can extend the first two effect algebras of Example 2 to effect monoids as follows.

1. $\langle [0, 1], +, \cdot, 0, 1\rangle$ where $\cdot$ is the standard multiplication on $\mathbb{R}$.
2. $\langle \mathcal{P}(X), \dot\cup, \cap, \emptyset, X\rangle$ where $\cap$ is the intersection.

Proposition 38. *Given an effect monoid E, we have*

1. $a \odot b \leq a$ *and* $a \odot b \leq b$ *for any* $a, b \in E$*;*
2. *if* $a \leq b$ *then* $c \odot a \leq c \odot b$ *and* $a \odot c \leq b \odot c$*;*
3. *if* $a \ll b$ *then* $c \odot a \ll c \odot b$ *and* $a \odot c \ll b \odot c$*;*
4. $a \odot b^{\perp} = a \ominus (a \odot b)$ *and* $a^{\perp} \odot b = b \ominus (a \odot b)$ and
5. *whenever* $c \leq b$*, we have* $a \odot (b \ominus c) = (a \odot b) \ominus (a \odot c)$.

Proof. One by one.

1. Certainly $b \perp b^{\perp}$. Thus $a \odot b \perp a \odot b^{\perp}$ and $(a \odot b) \owedge (a \odot b^{\perp}) = a \odot (b \owedge b^{\perp}) = a \odot 1 = a$. Hence $a \odot b \leq a$. The argument for the other statement is similar.
2. Suppose $a \leq b$. Then $a \owedge d = b$ for some d. Hence $(c \odot a) \owedge (c \odot d) = c \odot (a \owedge d) = c \odot b$. Consequently $c \odot a \leq c \odot b$. The argument for the other statement is similar.
3. For every $n \in \mathbb{N}$ we have $na \leq b$. Hence by the previous $n(c \odot a) = c \odot na \leq c \odot b$. Thus $c \odot a \ll c \odot b$. The proof of the other statement is similar.
4. Consider that $b^{\perp} \perp b$ and thus $(a \odot b^{\perp}) \owedge (a \odot b) = a \odot (b \owedge b^{\perp}) = a$. By uniqueness of the difference, we know $a \odot b^{\perp} = a \ominus (a \odot b)$. The other proof is similar.

5. If $c \leq b$, then $b \ominus c$ is defined and $b \ominus c \perp c$. Hence $(a \odot (b \ominus c)) \ovee (a \odot c) = a \odot ((b \ominus c) \ovee c) = a \odot b$ and thus by uniqueness of the difference we have $a \odot (b \ominus c) = (a \ominus b) \ovee (a \ominus c)$. □

Definition 39. An effect monoid E is called **commutative** if for every $a, b \in E$ we have $a \odot b = b \odot a$.

The previous two examples are both commutative. We will not find a finite effect monoid that is non-commutative.

Proposition 40 $(*)$**.** *If E is a finite effect monoid, then there exists a finite set X such that $E \cong \langle \mathcal{P}(X), \dot{\cup}, \cap, \emptyset, X \rangle$.*

Proof. Let $a_1, \ldots, a_n$ be the atoms of E. If $i \neq j$, then $a_i \odot a_j \leq a_i, a_j$ hence $a_i \odot a_j = 0$. Also note that $a_i \odot a_i = 0$ or $a_i \odot a_i = a_i$. Given a multiplicity vector $t_1, \ldots, t_n$, that is $t_1 a_1 \ovee \cdots \ovee t_n a_n = 1$, then

$$a_j = 1 \odot a_j = (t_1 a_i \ovee \cdots \ovee t_n a_n) \odot a_j = t_j (a_j \odot a_j) \leq a_j.$$

Thus $a_j \odot a_j = a_j$ and consequently $t_j = 1$ for any j. Hence: the only multiplicity vector of E is $(1, \ldots, 1)$. Furthermore, given any $b \in E$, we know $b = b_1 a_1 \ovee \cdots \ovee b_n a_n$ with $b_i \in \{0, 1\}$. Thus $b \odot b' = b \wedge b'$.

Pick any X with $|X| = n$. Then $\langle \mathcal{P}(X), \dot{\cup}, \cap, \emptyset, X \rangle$ also has n atoms and exactly one multiplicity vector: $(1, \ldots, 1)$. Thus by Theorem 35, it is, as an effect algebra, isomorphic to E. Also $a \odot b = a \wedge b$. Thus it is, as an effect monoid, isomorphic to E. □

The effect monoid structure on $[0, 1]$ is unique.

Proposition 41. *If $\odot$ makes $[0, 1] \subseteq \mathbb{R}$ an effect monoid, then $\odot$ is the standard multiplication on $\mathbb{R}$.*

Proof. First note that $n\frac{1}{n} = 1$ and thus $x \odot \frac{1}{n} = \frac{x}{n}$. Hence $x \odot \frac{m}{n} = \frac{m}{n} x$. Given any $x, y \in \mathbb{R}$. Let $q_1, q_2, \ldots \in \mathbb{Q}$ such that $y \leq \ldots \leq q_2 \leq q_1 \leq 1$ and $q_i \downarrow y$. Then $x \odot y \leq \ldots \leq x q_2 \leq x q_1$. And thus $x \odot y \leq xy$. Approximating y from the other side, we get $x \odot y \geq xy$. Consequently $x \odot y = xy$. □

The crux of the previous proof was to show that $\odot$ has to respect the scalar multiplication on the underlying effect algebra. Because we will encounter such effect monoids again, we define:

Definition 42. An effect monoid E is called **convex** if the underlying effect algebra is convex and $\lambda \cdot (a \odot b) = (\lambda \cdot a) \odot b = a \odot (\lambda \cdot b)$.

2.2.1 Convex effect monoids and OAU-algebras

We will prove a representation result for convex effect monoids, which we will use to simplify the study of a certain class of effect monoids.

Recall that given any ordered vector space V and vector $u > 0$, we know that the order interval $[0, u]$ is a convex effect algebra (Proposition 20). Conversely, any convex effect algebra is an interval effect algebra of some ordered vector space (Theorem 23).

Given any convex effect monoid E. Then in particular it is a convex effect algebra. And thus it is an interval effect algebra of some ordered vector space V.

We will show that we can extend the multiplication of the effect algebra to the whole vector space, which will form an ordered, associative and unitary algebra. Conversely, given any ordered, associative and unitary algebra with unit 1, we can restrict the algebra multiplication to the order interval $[0, 1]$, which will form an effect monoid.

Definition 43. A structure $\langle V, +, *, \cdot, \leq, 0, 1\rangle$ is called an **ordered associative unitary algebra (OAU-algebra)** if $\langle V, +, \cdot, \leq, 0\rangle$ is a vector space and $*$ is a binary operation that satisfies

1. (associativity) $a * (b * c) = (a * b) * c$;
2. (distributivity) $(b + c) * a = b * a + c * a$ and $a * (b + c) = a * b + a * c$;
3. (unit) $1 * a = a * 1 = a$;
4. ($*$ preserves order) if $c \geq 0$ and $a \leq b$, then $c * a \leq c * b$ and $a * c \leq b * c$ *and*
5. (homogeneity) $r \cdot (a * b) = (r \cdot a) * b = a * (r \cdot b)$.

Proposition 44. *Given an OAU-algebra V. For $a, b \in [0, 1]$ with $a + b \leq 1$, define $a \varovee b = a + b$. Then $E = \langle [0, 1], \varovee, *, 0, 1\rangle$ is a convex effect monoid.*

Proof. Given $a, b \in [0, 1]$. Then $0 \leq a$ and thus $0 = 0 \cdot (0 * b) = 0 * b \leq a * b$. Also $a \leq 1$ and thus $a * b \leq 1 * b = b \leq 1$. Thus $a * b$ is a total binary operation on $[0, 1]$. By Proposition 20, E is a convex effect algebra. Now, we check the convex effect monoid axioms. (M1), (M4) and convexity are immediate from the OAU-algebra axioms. To prove (M2), assume $a, b \in [0, 1]$ with $a + b \leq 1$. Then $c * a + c * b = c * (a + b) \leq c * 1 = c \leq 1$, as desired. The proof of (M3) similar. □

Before we prove that any convex interval effect monoid can be extended to an OAU-algebra, we need a small result on ordered vector spaces.

Lemma 45 ($*$)**.** *Given an ordered vectorspace V and an element $u > 0$. The following are equivalent.*

1. *$[0, u]$ generates V, see Definition 22.*
2. *Given $v \in V$, we have $v = r(v_1 - v_2)$ for some $v_1, v_2 \in [0, u]$ and $r \in [1, \infty)$.*
3. *u is a **strong unit**, that is: for every $v \in V$, there is a $n \in \mathbb{N}$ such that $v \leq nu$.*

Proof. 1. First we prove (i) implies (ii). Thus suppose $[0, u]$ generates V. Given $v = r_1 v_1 - r_2 v_2$. Suppose $r_1, r_2 < 0$. Then $v = r_2 v_2 - r_1 v_1$. Thus we may assume not both: $r_1, r_2 < 0$. Suppose only $r_1 < 0$. Note $\frac{1}{2}(v_1 + v_2) \in [0, u]$. Thus $2(r_2 - r_1)\frac{1}{2}(v_1 + v_2) - 0 \cdot 0 = r_1 v_1 - r_2 v_2$. Note $r_2 - r_1 \geq 0$. Thus we may assume $r_1 \geq 0$. Similarly, we may assume $r_2 \geq 0$.

Suppose both $r_1, r_2 \in [0, 1]$. Then $r_1 v_1, r_2 v_2 \in [0, u]$ and hence $v = 1(r_1 v_1 - r_2 v_2)$, which is of the right form. Suppose $r_1 \in [0, 1]$ and $r_2 \in (1, \infty)$. Then $\frac{r_1}{r_2} v_1 \in [0, u]$ and $v = r_2(\frac{r_1}{r_2} v_1 - v_2)$, thus we are done. We are also done if $r_1 \in (1, \infty)$ and $r_2 \in [0, 1]$. Finally, suppose $r_1, r_2 \in (1, \infty)$. Then $\frac{1}{r_2} v_1, \frac{1}{r_1} v_2 \in [0, u]$ and $v = r_1 r_2(\frac{1}{r_2} v_1 - \frac{1}{r_1} v_2)$, which is of the desired form.

2. Now we prove (ii) implies (iii). Thus suppose (ii). Given $v \in V$. Then $v = r(v_1 - v_2)$ for some $r \in [1, \infty)$ and $0 \leq v_1, v_2 \leq u$. Note $v_1 - v_2 \leq u$. Hence $v = r(v_1 - v_2) \leq ru \leq \lceil r \rceil u$ as desired.

3. Finally, we prove (iii) implies (i). Thus suppose u is a strong unit. In particular, for some $n \geq 1$, we have $-nu \leq v \leq nu$. Thus:

$$0 \leq \frac{1}{2}\big(u + \frac{v}{n}\big) \leq u \quad \text{and} \quad 0 \leq \frac{1}{2}\big(u - \frac{v}{n}\big) \leq u.$$

Now define $v_1 = \frac{1}{2}(u + \frac{v}{n})$; and $v_2 = \frac{1}{2}(u - \frac{v}{n})$. We have $v = nv_1 - nv_2$ as desired. □

Theorem 46 ($*$)**.** *Given an ordered vector space V and a vector $u > 0$ such that $[0, u]$ generates V. Suppose $\langle [0, u], \oslash, \odot, \cdot, 0, u \rangle$ is a convex effect monoid. Then there is a unique extension $*$ of $\odot$ to V such that $\langle V, +, *, \cdot, \leq, 0, u \rangle$ is an OAU-algebra.*

Proof. Suppose $*$ is an extension of $\odot$ to V such that $\langle V, +, *, \cdot, \leq, 0, u \rangle$ is an OAU-algebra. Write $U = [0, u]$. Given $a, a' \in V$. Then $a = r(v - w)$ and $a' = r'(v' - w')$ for some $r, r' \in (1, \infty)$ and $v, v', w, w' \in U$. Hence

$$\begin{aligned} a * a' &= r(v - w) * r'(v' - w') \\ &= rr'(v * v' + w * w' - w * v' - v * w') \\ &= rr'(v \odot v' + w \odot w' - w \odot v' - v \odot w'). \end{aligned} \tag{1}$$

Thus the extension, if it exists, is unique. It also suggests a definition for $a * a'$. However, we need to show that the choice of r, r', v, v', w and w' does not effect the value of (1). We do this in two steps. First we define $*$ on $U \times V$. Then extend it to $V \times V$.

Given $r, r' \in [1, \infty)$ and $x, v, v', w, w' \in U$. Without loss of generality, we may assume $r' \leq r$. Suppose $r(v - w) = r'(v' - w')$. We want to show that $r(x \odot v - x \odot w) = r'(x \odot v' - x \odot w')$. From the assumption

$$rv + r'w' = r'v' + rw$$

and thus by dividing by $2r$ gives

$$\tfrac{1}{2}v + \tfrac{r'}{2r}w' = \tfrac{r'}{2r}v' + \tfrac{1}{2}w.$$

Note that $\frac{r'}{r} \in [0, 1]$ and thus $\frac{r'}{r}w', \frac{r'}{r}v' \in U$. Furthermore, if $u, u' \in U$, then also $\frac{1}{2}u + \frac{1}{2}u' \in U$. Thus $\frac{1}{2}v + \frac{r'}{2r}w' = \frac{r'}{2r}v' + \frac{1}{2}w \in U$. And consequently

$$\begin{aligned} \tfrac{1}{2}(x \odot v) + \tfrac{r'}{2r}(x \odot w') &= x \odot (\tfrac{1}{2}v + \tfrac{r'}{2r}w') \\ &= x \odot (\tfrac{r'}{2r}v' + \tfrac{1}{2}w) \\ &= \tfrac{r'}{2r}(x \odot v') + \tfrac{1}{2}(x \odot w). \end{aligned}$$

Rearranging and multiplying by $2r$, yields the desired

$$r(x \odot v - x \odot w) = r'(x \odot v' - x \odot w').$$

And thus we can define $x * a = r(x \odot v - x \odot w)$ if $a = r(v - w)$ for $r \in [1, \infty)$ and $x, v, w \in U$. We want to repeat this argument to define $x * y$ for $x, y \in V$, by $x * y = r(v * y - w * y)$. If we review the argument, we see we need to check whether $(a + b) * y = a * y + b * y$ and $s(a * y) = (sa) * y$ for $s \in [0, 1]$ and $a, b, a + b \in U$.

We check the latter first. Suppose $s \in [0, 1]$, $a \in U$ and $y \in V$ with $y = r(u - v)$ for some $r \in [0, 1]$ and $u, v \in U$. Then

$$\begin{aligned} s(a * y) &= sr(a \odot v - a \odot w) \\ &= r(a \odot (sv) - a \odot (sw)) \\ &= a * (sy). \end{aligned}$$

Now, for the partial distributivity, additionally assume $a, b, a + b \in U$. Then

$$\begin{aligned} (a + b) * y &= r((a + b) \odot v - (a + b) \odot w) \\ &= r(a \odot v + b \odot v - (a \odot w + b \odot w)) \\ &= r(a \odot v - a \odot w) + r(b \odot v - b \odot w) \\ &= a * y + b * y. \end{aligned}$$

Thus we can indeed repeat to previous argument and define $x*y = r(v*y-w*y)$, which is the same as (1). Finally, we need to check whether $*$ obeys the axioms of an OAU-algebra. We do this in a convenient order.

Suppose $a, b, c \in V$; $a = r_a(v_a - w_a)$; $b = r_b(v_b - w_b)$ and $c = r_c(v_c - w_c)$.

1. (distributivity) Assume, without loss of generality that $r_b \leq r_c$. Then $\frac{r_b}{r_c} \in [0, 1]$ and we have $a + b = 2r_c\big((\frac{r_b}{2r_c} v_b + \frac{1}{2} v_c) - (\frac{r_b}{2r_c} w_b + \frac{1}{2} w_c)\big)$. Hence

$$\begin{aligned} a * (b + c) &= 2r_a r_c\big(v_a \odot (\tfrac{r_b}{2r_c} v_b + \tfrac{1}{2} v_c) + w_a \odot (\tfrac{r_b}{2r_c} w_b + \tfrac{1}{2} w_c) \\ &\quad - v_a \odot (\tfrac{r_b}{2r_c} w_b + \tfrac{1}{2} w_c) - w_a \odot (\tfrac{r_b}{2r_c} v_b + \tfrac{1}{2} v_c)\big) \\ &= 2r_a r_c\big(\tfrac{r_b}{2r_c}(v_a \odot v_b) + \tfrac{1}{2}(v_a \odot v_c) + \tfrac{r_b}{2r_c}(w_a \odot w_b) + \tfrac{1}{2}(w_a \odot w_c) \\ &\quad - \tfrac{r_b}{2r_c}(v_a \odot w_b) - \tfrac{1}{2}(v_a \odot w_c) - \tfrac{r_b}{2r_c}(w_a \odot v_b) - \tfrac{1}{2}(w_a \odot v_c)\big) \\ &= r_a r_b(v_a \odot v_b + w_a \odot w_b - v_a \odot w_b - w_a \odot v_b) \\ &\quad + r_a r_c(v_a \odot v_c + w_a \odot w_c - v_a \odot w_c - w_a \odot v_c) \\ &= a * b + a * c. \end{aligned}$$

   The argument for right distributivity is similar.

2. (homogeneity) Suppose $r \in \mathbb{R}$. We distinguish cases. If $r \in [1, \infty)$, then $ra = rr_a(v_a - w_a)$ with $rr_a \in [1, \infty)$ and thus

$$\begin{aligned} (ra) * b &= rr_a r_b(v_a \odot v_b + w_a \odot w_b - v_a \odot w_b - w_a \odot v_b) \\ &= r(a * b). \end{aligned}$$

   Suppose $r \in [0, 1]$. Then $ra = r_a(rv_a - rw_a)$ with $rv_a, rw_a \in U$ and thus

$$\begin{aligned} (ra) * b) &= r_a r_b((rv_a) \odot v_b + (rw_a) \odot w_b - (rv_a) \odot w_b - (rw_a) \odot v_b) \\ &= r_a r_b r(v_a \odot v_b + w_a \odot w_b - v_a \odot w_b - w_a \odot v_b) \\ &= r(a * b). \end{aligned}$$

Suppose $r = -1$. Then $-a = -r_a(v_a - w_a) = r_a(w_a - v_a)$. And thus

$$\begin{aligned}(-a) * b &= r_a r_b(w_a \odot v_b + v_a \odot w_b - v_a \odot v_b - w_a \odot w_b) \\ &= -r_a r_b(v_a \odot v_b + w_a \odot w_b - v_a \odot w_b - w_a \odot v_b) \\ &= -(a * b) = r(a * b).\end{aligned}$$

For the remaining case, suppose $r < 0$. We can reduce it to the previous cases: $(ra) * b = (-r \cdot (-a)) * b = -((-ra) * b) = -(-r)(a * b) = r(a * b)$.

The argument for $r(a * b) = a * (rb)$ is similar.

3. (associativity) Using the homogeneity and distributivity we just demonstrated, we can reduce the associativity of $*$ to that of $\odot$, as follows.

$$\begin{aligned}a * (b * c) &= a * (r_b r_c(v_b \odot v_c + w_b \odot w_c - v_b \odot w_c - w_b \odot v_c)) \\ &= r_b r_c\big(a * (v_b \odot v_c) + a * (w_b \odot w_c) \\ &\quad - a * (v_b \odot w_c) - a * (w_b \odot v_c)\big) \\ &= r_b r_c\big((r_a(v_a - w_a)) * (v_b \odot v_c) + (r_a(v_a - w_a)) * (w_b \odot w_c) \\ &\quad - (r_a(v_a - w_a)) * (v_b \odot w_c) - (r_a(v_a - w_a)) * (w_b \odot v_c)\big) \\ &= r_a r_b r_c\big(v_a \odot (v_b \odot v_c) + v_a \odot (w_b \odot w_c) + w_a \odot (v_b \odot w_c) \\ &\quad + w_a \odot (w_b \odot v_c) - v_a \odot (v_b \odot w_c) + v_a \odot (w_b \odot v_c) \\ &\quad - w_a \odot (v_b \odot v_c) + w_a \odot (w_b \odot w_c)\big) \\ &= r_a r_b r_c\big((v_a \odot v_b) \odot v_c + (v_a \odot w_b) \odot w_c + (w_a \odot v_b) \odot w_c \\ &\quad + (w_a \odot w_b) \odot v_c - (v_a \odot v_b) \odot w_c + (v_a \odot w_b) \odot v_c \\ &\quad - (w_a \odot v_b) \odot v_c + (w_a \odot w_b) \odot w_c\big) \\ &= r_a r_b\big((v_a \odot v_b) * c + (w_a \odot w_b) * c \\ &\quad - (v_a \odot w_b) * c - (w_a \odot v_b) * c\big) \\ &= r_a r_b(v_a \odot v_b + w_a \odot w_b - v_a \odot w_b - w_a \odot v_b) * c \\ &= (a * b) * c\end{aligned}$$

4. (unit) $u * a = r_a(u \odot v_a - u \odot w_a) = r_a(v_a - w_a) = a$. Similarly $a * u = a$.

5. ($*$ preserves order) First note that if $a \geq 0$, then $r_a(v_a - w_a) \geq 0$ and thus $v_a - w_a \geq 0$. Also $v_a - w_a \leq u$, thus $v_a - w_a \in U$. Thus $a = rv$ for some $v \in U$ viz $v = v_a - w_a$.

   Next, suppose $c, a \geq 0$. With the previous we may assume $c = r_c v_c$ and $a = r_a v_a$. Thus $c * a = (r_c v_c) * (r_a v_a) = r_c r_a(v_c * v_a) = r_c r_a(v_c \odot v_a)$. We know $v_c \odot v_a \in U$. In particular $v_c \odot v_a \geq 0$. Also $r_c r_a \geq 0$. Thus $c * a \geq 0$.

   Finally, suppose $c \geq 0$ and $a \leq b$. Then $b - a \geq 0$. Hence $0 \leq c * (b - a) = c * b - c * a$. Thus $c * a \leq c * b$, as desired. The other case is similar. □

2.2.2 Effect monoids on finite dimensional lexicographically ordered vector spaces

We want to study effect monoids that are not commutative. Suppose we find a non-commutative OAU-algebra. By Proposition 44, its unit interval is an effect monoid. It is not hard to see it must be non-commutative too.

In this section we will study the class of effect monoids derived from OAU-algebras on lexicographically ordered vector spaces. This will give us examples of non-commutative effect monoids.

For this section, assume $n \in \mathbb{N}$ and $n \geq 1$. We write $e_1, \ldots, e_n$ for the standard basis of the real vector space $\mathbb{R}^n$. Given a vector $v \in \mathbb{R}^n$, we assume $v_1, \ldots, v_n \in \mathbb{R}$ are the components; that is: $\sum_i v_i e_i = v$.

We can totally order $\mathbb{R}^n$ as if it were words in a dictionary.

Definition 47. Given $n \in \mathbb{N}$. Given $v, w \in \mathbb{R}^n$, we say $v < w$ if there is an i such that $v_i < w_i$ and for all $j < i$ we have $v_j = w_j$. $\mathbb{R}^n$ with this order is an ordered vector space, which is called **lexicographically ordered**.

Since the order is total, we can familiarly define

$$|v| = \begin{cases} v & v \geq 0 \\ -v & v \leq 0. \end{cases}$$

We write $v \ll w$ if for all $n \in \mathbb{N}$ we have $n|v| \leq |w|$. Note that

$$0 \ll e_n \ll e_{n-1} \ll \cdots \ll e_2 \ll e_1.$$

The order interval $[0, e_1]$ generates $\mathbb{R}^n$ and thus $[0, e_1]$ is a convex effect monoid. See Section 2.1.4. Call it E^n_{lex}. We are interested in the effect monoids on E^n_{lex}. First, we will prove that any effect monoid on E^n_{lex} is convex. Then by Theorem 46 we know that an effect monoid on E^n_{lex} extends uniquely to an OAU-algebra on $\mathbb{R}^n$. We will show that an OAU-algebra on $\mathbb{R}^n$ is fixed by $e_i * e_j$. Then we will give necessary and sufficient conditions to extend a multiplication defined on the standard basis to an OAU-algebra.

Lemma 48. *Any effect monoid on E^n_{lex} (the unit interval effect algebra of the n-dimensional lexicographically ordered vector space) is convex.*

Proof. Given $n \in N$ and $a, b \in E^n_{\text{lex}}$. Certainly $n(a \odot \frac{1}{n} \cdot b) = a \odot n(\frac{1}{n} \cdot b) = a \odot b$ and thus $a \odot (\frac{1}{n} \cdot b) = \frac{1}{n} \cdot (a \odot b)$. Then also for any $0 \leq m \leq n$, we have $\frac{m}{n} \cdot (a \odot b) = a \odot (\frac{m}{n}) \cdot b$. Similarly $\frac{m}{n} \cdot (a \odot b) = (\frac{m}{n} \cdot a) \odot b$. For any $r \in [0, 1]$ we can find $q_i, q'_i \in \mathbb{Q} \cap [0, 1]$ such that r is the uniquely defined by $q_i \leq r \leq q'_i$ for all i.

Suppose $a \odot b = 0$. Then certainly $a \odot (r \cdot b) \leq a \odot b = 0 = r(a \odot b)$. Suppose $a \odot b \neq 0$. Then $a \odot b > 0$ and if $q_i \cdot (a \odot b) \leq r \cdot (a \odot b)$, then $q_i \leq r$. Thus $r \cdot (a \odot b)$ is uniquely defined by $q_i \cdot (a \odot b) \leq r \cdot (a \odot b) \leq q'_i \cdot (a \odot b)$ for all i. Now note that $q_i \cdot (a \odot b) = a \odot q_i \cdot b \leq a \odot (r \cdot b) \leq a \odot q'_i \cdot b = q'_i \cdot (a \odot b)$. Thus $a \odot (r \cdot b) = r \cdot (a \odot b)$. Similarly $(r \cdot a) \odot b = r \cdot (a \odot b)$. □

Given any OAU-algebra on the lexicographically ordered $\mathbb{R}^n$. Note that by the homogeneity and distributivity of $*$, that is: its bilinearity, we have

$$v * w = \big(\sum_i v_i e_i\big) * \big(\sum_j v_j e_j\big) = \sum_{i,j} v_i v_j (e_i * e_j).$$

Write $e^{ij} = e_i * e_j$. We see $*$ is fixed by the vectors e^{ij}. Conversely, given vectors e^{ij}, we can define a multiplication by

$$v * w = \sum_{i,j} v_i v_j e^{ij}.$$

However, this does not necessarily form an OAU-algebra. The following are necessary and sufficient conditions.

Proposition 49. *Given vectors* e^{ij}. *Write* $v * w = \sum_{i,j} v_i w_j e^{ij}$. *The following are equivalent.*

- $\langle \mathbb{R}^n, +, *, \cdot, \leq, 0, e_1 \rangle$ *is an OAU-algebra.*
- *The following four conditions hold.*
  - $e^{1j} = e_j$ *and* $e^{i1} = e_i$
  - $e_i * (e_j * e_k) = (e_i * e_j) * e_k$
  - $e^{ij} \geq 0$
  - *If* $i < j$, *then* $e^{ik} \gg e^{jk}$ *and* $e^{ki} \gg e^{kj}$.

Proof. The necessity is clear. To prove sufficiency of the four conditions, we check the axioms of an OAU-algebra in a convenient order.

1. (distributivity) Given $a, b, c \in \mathbb{R}^n$. Then

$$\begin{aligned} a * (b + c) &= \sum_{i,j} a_i (b_j + c_j) e^{ij} \\ &= \sum_{i,j} a_i b_j e^{ij} + \sum_{i,j} a_i c_j e^{ij} \\ &= a * b + a * c. \end{aligned}$$

   Right distributivity is proven similarly.

2. (homogeneity) Given $a, b \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Then $r(a*b) = r \sum_{i,j} a_i b_j e^{ij} = \sum_{i,j} (r a_i) b_j e^{ij} = (ra) * b$. The other case is proven in the same way.

3. (associativity) Given $a, b, c \in \mathbb{R}^n$. By the second condition and the distributivity and homogeneity just proven, we have

$$\begin{aligned} a * (b * c) &= a * \sum_{j,k} b_j c_k e^{jk} \\ &= \sum_{j,k} b_j c_k (a * e^{jk}) \\ &= \sum_{j,k} b_j c_k \big(\sum_i a_i e_i\big) * (e_j * e_k) \\ &= \sum_{i,j,k} a_i b_j c_k (e_i * (e_j * e_k)) \\ &= \sum_{i,j,k} a_i b_j c_k ((e_i * e_j) * e_k) \\ &= (a * b) * c. \end{aligned}$$

4. (unit) From the first condition and the definition of $*$ we get $e_1 * a = \sum_j a_j e^{1j} = \sum_j a_j e_j = a$ and similarly $a * e_1 = \sum_i a_i e^{i1} = \sum_i a_i e_i = a$.

5. ($*$ preserves order) If we can prove that $a * b \geq 0$, whenever $a, b \geq 0$, then we are done. For suppose $c \geq 0$ and $a \leq b$. Then $b - a \geq 0$. Thus $0 \leq c * (b - a) = c * b - c * a$. And thus $c * a \leq c * b$, as desired. The other case is similar.

   Thus, suppose $a, b \geq 0$. If $a = 0$ or $b = 0$, then $a * b = 0 \geq 0$. Thus, we may assume $a, b > 0$. Then there are α and β such that $a_\alpha, b_\beta > 0$ *and* $a_i = 0$ for all $i < \alpha$ and $b_i = 0$ for all $i < \beta$.

   Consider $a_\alpha b_\beta e^{\alpha\beta}$. By the third condition and the current assumptions, we know $a_\alpha b_\beta e^{\alpha\beta} \geq 0$. Given i and j with $(i, j) \neq (\alpha, \beta)$. We will show $a_\alpha b_\beta e^{\alpha\beta} \gg a_i b_j e^{ij}$, by distinguishing cases.

   - Suppose $i < \alpha$. Then $a_i = 0$ and thus $a_i b_j e^{ij} = 0 \ll a_\alpha b_\beta e^{\alpha\beta}$. The same argument covers the case $j < \beta$.
   - Suppose $i > \alpha$ and $j \geq \beta$. By the fourth condition, we know $e^{\alpha\beta} \gg e^{i\beta} \geq e^{ij}$. And since $a_\alpha b_\beta > 0$, we also have $a_\alpha b_\beta e^{\alpha\beta} \gg a_i b_j e^{ij}$. The case $i \geq \alpha$ and $j > \beta$ is similar.

   Note that if $0 \leq v$ then for any $w \ll v$ we have $0 \leq v + w$. And thus

   $$a * b = \sum_{i,j} a_i b_j e^{ij} = a_\alpha b_\beta e^{\alpha\beta} + \sum_{(i,j)\neq(\alpha,\beta)} a_i b_j e^{ij} \geq 0. \qquad \square$$

Corollary 50 ($*$)**.** *The unique effect monoid on E^2_{lex} is:*

$\odot$	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

Corollary 51. *There is a non-commutative effect monoid on E^5_{lex}, fixed by:*

$\odot$	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5
e_2	e_2	e_4	e_5	0	0
e_3	e_3	0	0	0	0
e_4	e_4	0	0	0	0
e_5	e_5	0	0	0	0

2.3 Effect modules

Recall that a convex effect algebra is an effect algebra equipped with a scalar multiplication with $[0, 1]$. See Definition 2.1.4. Effect modules are a generalization of convex effect algebras, where the scalars can come from any effect monoid.

Definition 52. Given an effect monoid M. An M**-effect module** is an effect algebra E together with an operation $(\ )\cdot(\ ) \colon M \times E \to E$ such that

(V1) $\alpha \cdot (\beta \cdot a) = (\alpha \odot \beta) \cdot a$;

(V2) if $\alpha \perp \beta$ then $\alpha a \perp \beta a$ and $(\alpha \ovee \beta) \cdot a = \alpha \cdot a \ovee \beta \cdot a$;

(V3) if $a \perp b$ then $\lambda \cdot a \perp \lambda \cdot b$ and $\lambda \cdot a \ovee \lambda \cdot b = \lambda \cdot (a \ovee b)$ *and*

(V4) $1 \cdot a = a$.

Example 53.

1. Every convex effect algebra is a $[0, 1]$-effect module.
2. Every effect algebra is a 2-effect module with $0 \cdot a = 0$ and $1 \cdot a = a$.
3. Given any effect monoid M and $n \in \mathbb{N}$, the set M^n is a M-effect module with pointwise operations.
4. A bit more general: given any effect monoid M and set X, the set M^X of functions from X to M is an M-effect module with pointwise operations.

Definition 54. Given a map between M-effect modules $f \colon E_1 \to E_2$. f is an **effect module homomorphism** if f is an effect algebra homomorphism and furthermore $f(\lambda \cdot a) = \lambda \cdot f(a)$ for all $\lambda \in M$ and $a \in E$.

We write EMod_M for the category of M-effect modules with effect module homomorphisms.

2.4 Sequential effect modules

Recall that the starting point of this thesis, was the observation that in the examples of effect logics initially studied by Jacobs, a *sequential effect algebra* arises. We did not define this notion, yet. See Subsection 1.1.

Definition 55 ([7]). A **sequential effect algebra** is an effect algebra E together with a binary multiplication $*$ such that

(S1) $a * (b \ovee c) = (a * b) \ovee (a * c)$

(S2) $1 * a = a$

(S3) If $a * b = 0$, then $a * b = b * a$.

(S4) If $a * b = b * a$, then $a * b^{\perp} = b^{\perp} * a$ and $a * (b * c) = (a * b) * c$.

(S5) If $c * a = a * c$ and $c * b = b * c$.
Then: $c * (a * b) = (a * b) * c$ and $c * (a \ovee b) = (a \ovee b) * c$.

Definition 56. A **sequential effect module** is a sequential effect algebra, where the underlying effect algebra is an effect module and

(SM) $\lambda(a * b) = (\lambda a) * b = a * (\lambda b)$ for any scalar λ.

Definition 57. A sequential effect algebra E is called **commutative** if for any $a, b \in E$, we have $a * b = b * a$.

2.4.1 Examples

We have seen commutative sequential effect algebras already, in disguise.

Proposition 58. *Every commutative effect monoid is a commutative sequential effect algebra. And, conversely, every commutative sequential effect algebra is a commutative effect monoid.*

Furthermore, the commutative effect monoid is convex if and only if the commutative sequential effect algebra is a $[0, 1]$*-effect module.*

Proof.

1. Given a commutative effect monoid E. The axioms (S1) and (S2) are satisfied directly by definition. The axioms (S3), (S4) and (S5) are implications of which the conclusions are directly satisfied by definition.

   If the effect monoid is convex, then (SM) follows by definition.

2. Conversely, given a commutative sequential effect algebra. (M3) is the same as (S1). Since everything commutes, (M3) implies (M2) and (S4) implies the associativity of $*$; that is: (M4). We are left to prove (M1).

   By (S1), we have $(a * 0) \ovee (a * 1) = a * 1$. By cancellation: $a * 0 = 0$. Thus by (S3), we have $0 * a = a * 0 = 0$ and by (S4) and (S2) also $1 * a = a * 1 = a$. That is: we have shown (M1).

   If the sequential effect algebra is a sequential $[0, 1]$-effect monoid, then the underlying effect algebra is convex and the multiplication is bi-homogeneous, hence (SM). $\square$

Now, the prime example of a sequential effect module:

Theorem 59 ([6])**.** *Given any Hilbert space $\mathscr{H}$. $\langle \mathrm{Eff}(\mathscr{H}), *\rangle$ with $A * B = \sqrt{A}B\sqrt{A}$ is a $[0,1]$-sequential effect module.*

We will need some lemmas before we can prove this Theorem. Assume $\mathscr{H}$ is a Hilbert space and write $\mathscr{B}(\mathscr{H})^+$ for its positive bounded operators.

Lemma 60. *Given $A, B \in \mathscr{B}(\mathscr{H})^+$. If $AB = BA$, then $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$.*

Proof. There are polynomials $p_1, p_2, \ldots$ such that $p_n(A)$ converges uniformly to $\sqrt{A}$. Clearly $p_n(A)B = Bp_n(A)$. Thus all $p_n(A)$ are in the set of commutants of B, which is strongly closed. Hence $\sqrt{A}B = B\sqrt{A}$. Repeating the argument, yields $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$, as desired. □

Lemma 61. *Given $A, B \in \mathscr{B}(\mathscr{H})^+$. If $AB = BA$, then $\sqrt{AB} = \sqrt{A}\sqrt{B}$.*

Proof. By the previous Lemma:

$$\sqrt{A}\sqrt{B}(\sqrt{A}\sqrt{B})^\dagger = \sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A} = \sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B} = AB.$$

Thus AB is positive. Furthermore $(AB)A = A(AB)$ and $(AB)B = B(AB)$. Consider the commutative C^*-algebra $\mathscr{A}$ generated by A, B and AB. As they can be approximated uniformly by polynomials in A, B and AB, we have: $\sqrt{A}, \sqrt{B}, \sqrt{AB} \in \mathscr{A}$. By Gelfand's Theorem: $\mathscr{A} \cong C(X)$ for some compact Hausdorff space X. Thus: $\sqrt{A}\sqrt{B} = \sqrt{AB}$, since it holds in $C(X)$. □

For the next Lemma, we will need a Theorem from functional analysis, which is easy to state, but rather hard to prove.

Theorem 62 (Fuglede-Putnam-Rosenblum)**.** *Given $A, B, C \in \mathscr{B}(\mathscr{H})$. If A and B are normal and $AC = CB$, then $A^\dagger C = CB^\dagger$.*

Proof. Note that for any $z \in \mathbb{C}$, the operator $e^{i\overline{z}A}$ can be approximated uniformly by polynomials in A. Hence, as before $e^{i\overline{z}A}C = Ce^{i\overline{z}B}$. And thus we have $C = e^{-i\overline{z}A}Ce^{i\overline{z}B}$. Define $f(z) = e^{-izA^\dagger}Ce^{izB^\dagger}$. Recall that $e^{X+Y} = e^X e^Y$ whenever X and Y commute. Observe that X and $X^\dagger$ commute, if X is normal. Thus:

$$\begin{aligned} f(z) &= e^{-izA^\dagger}Ce^{izB^\dagger} \\ &= e^{-izA^\dagger}e^{-i\overline{z}A}Ce^{i\overline{z}B}e^{izB^\dagger} \\ &= e^{-i(zA^\dagger + \overline{z}A)}Ce^{i(\overline{z}B + zB^\dagger)}. \end{aligned}$$

The operators $zA^\dagger + \overline{z}A$ and $\overline{z}B + zB^\dagger$ are self-adjoint for any $z \in \mathbb{C}$. Thus both $e^{-i(zA^\dagger + \overline{z}A)}$ and $e^{i(\overline{z}B + zB^\dagger)}$ are unitary. Hence $\|f(z)\| \le \|B\|$.

There are $A_0, A_1, A_2, \ldots \in \mathscr{B}(\mathscr{H})$ such that $\sum_{n=0}^N z^n A_n$ converges uniformly to $f(z)$ as $N \to \infty$. Given a linear continuous functional $\varphi\colon \mathscr{B}(\mathscr{H}) \to \mathbb{C}$. Then $\varphi(f(z)) = \sum_n z^n \varphi(A_n)$ and $|\varphi(f(z))| \le \|\varphi\|\|f(z)\| \le \|\varphi\|\|B\|$. Thus $\varphi \circ f$ is analytic and bounded. Hence, by Liouville's Theorem, it is constant. Thus $f(z)$ is constant.

Consequently:

$$\begin{aligned} 0 = f'(0) &= -iA^\dagger e^{-i0A^\dagger}Ce^{i0B^\dagger} + ie^{-i0A^\dagger}CB^\dagger e^{i0B^\dagger} \\ &= -iA^\dagger C + iCB^\dagger \end{aligned}$$

and hence $A^\dagger C = CB^\dagger$, as desired. □

Lemma 63 ([6])**.** *Given $A, B \in \mathrm{Eff}(\mathscr{H})$. $AB = BA$ if and only if $A * B = B * A$.*

Proof. Suppose $AB = BA$. Then $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$. Thus:

$$A * B = \sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A} = \sqrt{B}\sqrt{A}\sqrt{A}\sqrt{B} = B * A.$$

The proof of the converse is more involved. Suppose $A * B = B * A$. Hence $\sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A} = \sqrt{B}\sqrt{A}\sqrt{A}\sqrt{B}$. Thus $\sqrt{A}\sqrt{B}$ and $\sqrt{B}\sqrt{A}$ are normal. Note that:

$$(\sqrt{A}\sqrt{B})\sqrt{A} = \sqrt{A}(\sqrt{B}\sqrt{A})$$

and hence, by the Fuglede-Putnam-Rosenblum Theorem:

$$\sqrt{B}A = (\sqrt{A}\sqrt{B})^{\dagger}\sqrt{A} = \sqrt{A}(\sqrt{B}\sqrt{A})^{\dagger} = A\sqrt{B}.$$

And thus $BA = AB$, as desired. □

Lemma 64. *Given $A, B \in \mathrm{Eff}(\mathscr{H})$. If $A * B = B * A$, then $A * B = AB$.*

Proof. Suppose $A * B = B * A$. By the previous Lemma $AB = BA$. Hence by Lemma 60, also $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$. Thus:

$$A * B = \sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A} = \sqrt{B}\sqrt{A}\sqrt{B}\sqrt{A} = \sqrt{B}\sqrt{B}\sqrt{A}\sqrt{A} = BA = AB.$$ □

Now we are ready to prove that $\mathrm{Eff}(\mathscr{H})$ is a sequential $[0, 1]$-effect module.

Proof of Theorem 59. One at a time.

(S1)
$$\begin{aligned} A * (B \ovee C) &= \sqrt{A}(B + C)\sqrt{A} \\ &= \sqrt{A}B\sqrt{A} + \sqrt{A}C\sqrt{A} \\ &= (A * B) \ovee (A * C) \end{aligned}$$

(S2)
$$\begin{aligned} 1 * A &= \sqrt{I}A\sqrt{I} \\ &= IAI \\ &= A \end{aligned}$$

(S3) Suppose $A * B = 0$. Then $\sqrt{A}B\sqrt{A} = 0$. That is, for all $v \in \mathscr{H}$:

$$\begin{aligned} 0 &= \langle \sqrt{A}B\sqrt{A}v, v \rangle \\ &= \langle \sqrt{B}\sqrt{A}v, \sqrt{B}\sqrt{A}v \rangle \\ &= \|\sqrt{B}\sqrt{A}\|^2. \end{aligned}$$

and thus $\sqrt{B}\sqrt{A} = 0$, hence

$$\begin{aligned} 0 &= \langle \sqrt{B}\sqrt{A}v, v \rangle \\ &= \langle \sqrt{A}v, \sqrt{B}v \rangle \\ &= \overline{\langle \sqrt{B}v, \sqrt{A}v \rangle} \\ &= \langle \sqrt{B}v, \sqrt{A}v \rangle \\ &= \langle \sqrt{A}\sqrt{B}v, v \rangle \end{aligned}$$

and thus $\sqrt{A}\sqrt{B} = \sqrt{B}\sqrt{A}$. But then:

$$\begin{aligned} A * B &= \sqrt{A}B\sqrt{A} \\ &= \sqrt{A}\sqrt{B}\sqrt{B}\sqrt{A} \\ &= \sqrt{B}\sqrt{A}\sqrt{A}\sqrt{B} \\ &= \sqrt{B}A\sqrt{B} \\ &= B * A. \end{aligned}$$

(S4) Suppose $A*B = B*A$. First we prove that $A*B^{\perp} = B^{\perp}*A$. By Lemma 63, we have $AB = BA$. Also: it is sufficient to prove that $B^{\perp}A = AB^{\perp}$, which is easily checked:

$$B^{\perp}A = (I - B)A = A - BA = A - AB = A(I - B) = AB^{\perp}.$$

Now, to check $A * (B * C) = (A * B) * C$:

$$\begin{aligned} (A * B) * C &= (AB) * C && \text{by Lemma 63} \\ &= \sqrt{AB}C\sqrt{AB} \\ &= \sqrt{A}\sqrt{B}C\sqrt{A}\sqrt{B} && \text{by Lemma 61} \\ &= \sqrt{A}\sqrt{B}C\sqrt{B}\sqrt{A} && \text{by Lemma 60} \\ &= A * (B * C). \end{aligned}$$

(S5) Suppose $C * A = A * C$ and $C * B = B * C$. Using Lemma 63, we may assume $CA = AC$ and $CB = BC$ *and* it is sufficient to prove that $CAB = ABC$ and $C(A + B) = (A + B)C$. The first:

$$CAB = ACB = ABC.$$

The second:

$$C(A + B) = CA + CB = AC + BC = (A + B)C. \qquad \square$$

2.4.2 Counterexamples

In this subsection we will study some basic properties that sequential effect modules *do not* have.

Proposition 65. *Not every sequential effect module is commutative.*

Proof. Consider the Hilbert space $\mathbb{C}^2$ with the projections

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then:

$$A * B = \sqrt{A}B\sqrt{A} = ABA = \frac{1}{2}A \neq \frac{1}{2}B = BAB = B * A.$$

Alternatively: note $AB \neq BA$ and apply Lemma 63. $\square$

Proposition 66. *Not every sequential effect module is left-additive.*

Proof. Again, consider the Hilbert space $\mathbb{C}^2$ with the projections

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, we have:

$$A * B = ABA = \frac{1}{2}A \qquad (I - A) * B = (I - A)B(I - A) = \frac{1}{2}(I - A).$$

Consequently:

$$A * B + (I - A) * B = \frac{1}{2}I \neq B = I * B = (A + I - A) * B.$$ □

Proposition 67. *Not every convex effect monoid is a sequential effect algebra.*

Proof. Consider the convex effect monoid on E^5_{lex} from Example 51. We have $e_3 \odot e_2 = 0$, but $e_2 \odot e_3 = e_5 \neq e_3 \odot e_2$. Thus it does not obey sequential effect algebra axiom (S3). □

2.5 Galois connections

In the axioms of a (weak) effect logic, we use the notion of a Galois connection, which is also called an order adjunction.

Definition 68. Given two posets P and Q and two maps between them in opposite direction $f : P \leftrightarrows Q : g$. We say (f, g) is a **Galois connection** (or **order adjunction**), in symbols $f \dashv g$, if for all $p \in P$ and $q \in Q$ we have:

$$f(p) \leq q \iff p \leq g(q).$$

f is called the left (or lower) adjoint of g and g is called the right (or upper) adjoint of f.

Example 69.

1. For any map $f\colon X \to Y$, the forward and inverse image are adjoint. That is: $f_* : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y) : f^*$ given by $f_*(U) = \{f(x);\ x \in U\}$ and $f^*(U) = \{x;\ x \in X;\ f(x) \in U\}$ is a Galois connection, since

$$f_*(U) \subseteq V \iff U \subseteq f^*(V).$$

   In symbols: $f_* \dashv f^*$. Actually, there is also a f_{**} such that $f^* \dashv f_{**}$, which is given by $f_{**}(U) = \{y;\ y \in Y;\ f^*(\{y\}) \subseteq U\}$. This is called the direct image.

2. If f and g are each others inverse, then $f \dashv g$ and $g \dashv f$. For instance, in an effect algebra E for some selected element a we have $a \oslash (\ ) \dashv (\ ) \ominus a \dashv a \oslash (\ )$ as maps between $\downarrow a^\perp$ and $\uparrow a$.

Proposition 70. *For any maps $f : P \leftrightarrows Q : g$ with $f \dashv g$, we have*

1. *$p \leq g(f(p))$ for all $p \in P$ and $q \geq f(g(q))$ for all $q \in Q$;*
2. *f and g are order-preserving* and
3. *f preserves suprema and g infima.*

Proof. One at a time.

1. Certainly $f(p) \leq f(p)$. Thus $p \leq g(f(p))$. Similarly $q \geq f(g(q))$.
2. Suppose $p \leq p'$. Then with the previous: $p \leq p' \leq g(f(p'))$. Thus $f(p) \leq f(p')$, as desired. g is proven order-preserving in the same way.
3. Suppose $X \subseteq P$ with $\sup X$ exists. We will show $f(\sup X) = \sup f(X)$.

   First, we show it is an upper bound. Suppose $x \in X$. Certainly $x \leq \sup X$. Thus $f(x) \leq f(\sup X)$, since f is order-preserving. Hence $f(X) \leq f(\sup X)$.

   Now we show its the least upper bound. Suppose $u \in X$ with $f(X) \leq u$. Given any $x \in X$. Then $f(x) \leq u$. Thus $x \leq g(u)$. Hence $\sup X \leq g(u)$. Finally: $f(\sup X) \leq u$ as desired.

   The preservation of infima by g is demonstrated in the same way. □

Proposition 71. *For any maps $f : P \leftrightarrows Q : g$ the following are equivalent:*

1. *$f \dashv g$;*

2. $f(p) = \min\{q;\ p \leq g(q)\}$ *and* g *is order-preserving* and

3. $g(q) = \max\{p;\ f(p) \leq q\}$ *and* f *is order-preserving.*

Proof. We will prove that the first two are equivalent. The equivalence between the first and last is very similar.

Suppose $f \dashv g$. Then $p \leq g(f(p))$. Thus: $f(p) \in \{q;\ p \leq g(q)\}$. Suppose there is another q such that $p \leq g(q)$. Then $f(p) \leq q$. Thus indeed: $f(p) = \min\{q;\ p \leq g(q)\}$.

Suppose $f(p) = \min\{q;\ p \leq g(q)\}$ and g is order-preserving. Assume $f(p) \leq q$. By definition of f and since g is order-preserving, we have $p \leq g(f(q)) \leq g(q)$, as desired. Conversely, assume $p \leq g(q)$. Since $f(p)$ is by definition the minimal such q, we have $f(p) \leq q$, as desired. Thus $f \dashv g$. □

Corollary 72. *If a map has a left/right adjoint, this adjoint is unique.*

Lemma 73. *Given maps* $f : P \leftrightarrows Q : g$ *with* $f \dashv g$. *Then:* $f \circ g \circ f = f$ *and* $g \circ f \circ g = g$.

Proof. We knew already $p \leq g(f(p))$ and $f(g(q)) \leq q$ for all $p \in P$ and $q \in Q$. Thus, since f is order-preserving: $f(x) \leq f(g(f(x)))$. And with $g(f(x)) = g(f(x))$, we see $f(g(f(x))) \leq f(x)$. Thus $f(g(f(x))) = f(x)$. The proof of the other statement is similar. □

Definition 74. Given maps $f : P \leftrightarrows Q : g$. We say (f, g) is an (order) **coreflection** if $f \dashv g$ and $g \circ f = \mathrm{id}$.

Proposition 75. *Given maps* $f : P \leftrightarrows Q : g$ *with* $f \dashv g$. *The following are equivalent:*

1. (f, g) *is a coreflection;*

2. g *is surjective;*

3. $g \circ f = \mathrm{id}$ *and*

4. f *is injective.*

Proof. By definition $3 \Leftrightarrow 1$. We will prove: $2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2$.

Suppose g is surjective. Given $p \in P$. By surjectivity, there is a q such that $p = g(q)$. By the previous Lemma: $g(f(p)) = g(f(g(q))) = g(q) = p$, as desired.

For the second implication, suppose $g \circ f = \mathrm{id}$. Given $p, p' \in P$ with $f(p) = f(p')$. Then $p = g(f(p)) = g(f(p')) = p'$, as desired.

Suppose f is injective. Given $p \in P$. By our Lemma $f(g(f(p))) = f(p)$. By injectivity: $g(f(p)) = p$. Thus g is surjective. □

2.6 Kleisli category

If $f \dashv g$ are order adjoints, then $g \circ f$ is a closure operator. Similarly, if $F \dashv G$ are adjoint functors, then $G \circ F$ is a monad.

Definition 76. A triplet $\langle T, \mu, \eta\rangle$ of an endofunctor $T\colon \mathscr{C} \to \mathscr{C}$ and natural transformations $\eta\colon 1 \Rightarrow T$ and $\mu\colon T^2 \Rightarrow T$ is called a **monad** if $\mu \circ T\mu = \mu \circ \mu$ and $\mu \circ \eta = \mu \circ T\eta = \mathrm{id}$.

Example 77. The **distribution monad** $\mathcal{D}\colon \mathsf{Set} \to \mathsf{Set}$ is defined as follows. To a set X, we assign the set of convex formal sums

$$\mathcal{D}(X) = \{\varphi;\ \varphi\colon X \to [0,1];\ |\operatorname{supp}\varphi| < \infty \text{ and } \sum_x \varphi(x) = 1\}.$$

Given $f\colon X \to Y$, we define $\mathcal{D}(f)$ by

$$\mathcal{D}(f)(\varphi)(x) = \sum_{y;\ f(y)=x} \varphi(y).$$

The unit and multiplication are given by

$$\eta_X(x)(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \qquad \mu_X(\Phi)(x) = \sum_\varphi \Phi(\varphi)\varphi(x).$$

Monads are not just a generalization of closure operators: they have varied applications without order-theoretic analogue.

Definition 78. Given a monad $\langle T, \mu, \eta\rangle$ over a category $\mathscr{C}$, we will define the **Kleisli category of** T, in symbols: $\mathscr{K}\ell(T)$. Its objects are the objects of $\mathscr{C}$. An arrow $f\colon X \to Y$ in $\mathscr{K}\ell(T)$ is an arrow $\hat{f}\colon X \to T(Y)$ in $\mathscr{C}$.

Given arrows $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathscr{K}\ell(T)$, $g \circ f$ is given by

$$X \xrightarrow{\hat{f}} TY \xrightarrow{T\hat{g}} TTZ \xrightarrow{\mu_Z} TZ,$$

that is: $\widehat{g \circ f} = \mu_Z \circ T\hat{g} \circ \hat{f}$. The identity map is the unit: $\widehat{\mathrm{id}}_X = \eta_X$.

Example 79. The objects of $\mathscr{K}\ell(\mathcal{D})$ are the sets. A map $X \to Y$ in $\mathscr{K}\ell(\mathcal{D})$, assigns to each X a convex sum of elements in Y. Thus a map $m\colon X \to X$ in $\mathscr{K}\ell(\mathcal{D})$ is a markov chain on X. The composition $m \circ m$ is then the derived markov chain associated with taking two steps in the original. The identity maps x to the singleton convex sum $1x$.

2.6.1 The distribution monad for an effect monoid

A useful application of effect monoids (Definition 36) is that we can generalize the distribution monad to any effect monoid. In this definition, we can replace the effect monoid $[0,1]$ with any other effect monoid M:

Definition 80 ([11])**.** For any effect monoid M define $\mathcal{D}_M\colon \mathsf{Set} \to \mathsf{Set}$ by

$$\mathcal{D}_M(X) = \{\varphi;\ \varphi\colon X \to M;\ |\operatorname{supp}\varphi| < \infty \text{ and } \bigovee_x \varphi(x) = 1\}.$$

And similar to the normal distribution monad, given $f\colon X \to Y$ in Set, we define $\mathcal{D}_M(f)$ by

$$\mathcal{D}_M(f)(\varphi)(x) = \bigovee_{y;\ f(y)=x} \varphi(y).$$

There is no surprise in the monad structure either. We define η and μ as follows.

$$\eta_X(x)(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} \qquad \mu_X(\Phi)(x) = \bigovee_{\varphi} \Phi(\varphi) \odot \varphi(x)$$

Proposition 81. *$\mathcal{D}_M$ is a monad.*

Proof. First we prove that $\mathcal{D}_M$ is a functor. Then we prove that μ and η are natural transformations. Finally, we prove they obey the monad laws.

- Clearly $\mathcal{D}_M(1)(\varphi)(x) = \varphi(x)$. Thus $\mathcal{D}_M(1) = 1$.

- Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ in Set.

$$\begin{aligned} \mathcal{D}_M(gf)(\varphi)(x) &= \bigovee_{y;\ gf(y)=x} \varphi(y) \\ &= \bigovee_{z;\ g(z)=x}\ \bigovee_{y;\ f(y)=z} \varphi(y) \\ &= \bigovee_{z;\ g(z)=x} \mathcal{D}_M(f)(\varphi)(z) \\ &= \mathcal{D}_M(g)(\mathcal{D}_M(f)(\varphi))(x) \end{aligned}$$

  Thus $\mathcal{D}_M(gf) = \mathcal{D}_M(g)\mathcal{D}_M(f)$.

- Given $X \xrightarrow{f} Y$ in Set.

$$\begin{aligned} \mathcal{D}_M(f)\eta_X(x)(y) &= \bigovee_{x';\ f(x')=y} \eta_X(x)(x') \\ &= \eta_Y(f(x'))(y) \\ &= \eta_Y f(x)(y) \end{aligned}$$

  and thus $\eta\colon 1 \Rightarrow \mathcal{D}_M$.

- Given $X \xrightarrow{f} Y$ in Set.

$$\begin{aligned}
\mathcal{D}_M(f)\mu_X(\Phi)(y) &= \bigvee_{x;\ f(x)=y} \mu_X(\Phi)(x) \\
&= \bigvee_{x;\ f(x)=y} \bigvee_{\varphi} \Phi(\varphi) \odot \varphi(x) \\
&= \bigvee_{\varphi} \bigvee_{x;\ f(x)=y} \Phi(\varphi) \odot \varphi(x) \\
&= \bigvee_{\varphi} \Phi(\varphi) \odot \bigvee_{x;\ f(x)=y} \varphi(x) \\
&= \bigvee_{\varphi} \Phi(\varphi) \odot \mathcal{D}_M(f)(\varphi)(y) \\
&= \bigvee_{\psi} \bigvee_{\varphi;\ \mathcal{D}_M(f)(\varphi)=\psi} \Phi(\varphi) \odot \psi(y) \\
&= \bigvee_{\psi} \big(\bigvee_{\varphi;\ \mathcal{D}_M(f)(\varphi)=\psi} \Phi(\varphi)\big) \odot \psi(y) \\
&= \bigvee_{\psi} \mathcal{D}_M(\mathcal{D}_M(f))(\Phi)(\psi) \odot \psi(y) \\
&= \mu_Y(\mathcal{D}_M(\mathcal{D}_M(f))(\Phi))(y)
\end{aligned}$$

  and thus $\mu \colon \mathcal{D}_M\mathcal{D}_M \Rightarrow \mathcal{D}_M$.

- For any $X \in \mathsf{Set}$, we have

$$\begin{aligned}
\mu_X \eta_{\mathcal{D}_M X}(\psi)(x) &= \bigvee_{\varphi} \eta_{\mathcal{D}_M X}(\psi)(\varphi) \odot \varphi(x) \\
&= \psi(x)
\end{aligned}$$

  and thus $\mu\eta_{\mathcal{D}_M} = 1$.

- For any $X \in \mathsf{Set}$, we have

$$\begin{aligned}
\mu_X \mathcal{D}_M(\eta_X)(\psi)(x) &= \bigvee_{\varphi} \mathcal{D}_M(\eta_X)(\psi)(\varphi) \odot \varphi(x) \\
&= \bigvee_{\varphi} (\bigvee_{y;\ \eta_X(y)=\varphi} \psi(y)) \odot \varphi(x) \\
&= \bigvee_{y} \psi(y) \odot \eta_X(y)(x) \\
&= \psi(x)
\end{aligned}$$

  and thus $\mu_X \mathcal{D}_M(\eta_X) = 1$.

- For any $X \in \mathsf{Set}$, we have

$$\begin{aligned}
\mu_X \mathcal{D}_M(\mu_X)(\aleph)(x) &= \bigvee_{\varphi} \mathcal{D}_M(\mu_X)(\aleph)(\varphi) \odot \varphi(x) \\
&= \bigvee_{\varphi} \Big(\bigvee_{\Phi;\ \mu_X(\Phi)=\varphi} \aleph(\Phi)\Big) \odot \varphi(x) \\
&= \bigvee_{\varphi} \bigvee_{\Phi;\ \mu_X(\Phi)=\varphi} \aleph(\Phi) \odot \varphi(x) \\
&= \bigvee_{\varphi} \bigvee_{\Phi;\ \mu_X(\Phi)=\varphi} \aleph(\Phi) \odot \mu_X(\Phi)(x) \\
&= \bigvee_{\Phi} \aleph(\Phi) \odot \mu_X(\Phi)(x) \\
&= \bigvee_{\Phi} \aleph(\Phi) \odot \Big(\bigvee_{\varphi} \Phi(\varphi) \odot \varphi(x)\Big) \\
&= \bigvee_{\Phi} \bigvee_{\varphi} \aleph(\Phi) \odot \Phi(\varphi) \odot \varphi(x) \\
&= \bigvee_{\varphi} \bigvee_{\Phi} \aleph(\Phi) \odot \Phi(\varphi) \odot \varphi(x) \\
&= \bigvee_{\varphi} \mu_{\mathcal{D}_M(X)}(\aleph)(\varphi) \odot \varphi(x) \\
&= \mu_X(\mu_{\mathcal{D}_M(X)}(\aleph))(x)
\end{aligned}$$

and thus $\mu_X \mathcal{D}_M(\mu_X) = \mu_X(\mu_{\mathcal{D}_M(X)})$. □

2.6.2 Coproducts and split monos

In this subsection, we will prove two basic results on coproducts and split monos in a Kleisli category.

Proposition 82. *Given a monad T on $\mathscr{C}$. If $A \xrightarrow{\kappa_1} A+B \xleftarrow{\kappa_2} B$ is a coproduct in $\mathscr{C}$, then $A \xrightarrow{\kappa_1} A+B \xleftarrow{\kappa_2} B$ with $\hat{\kappa}_1 = \eta\kappa_1$ and $\hat{\kappa}_2 = \eta\kappa_2$ is a coproduct in $\mathscr{K}\ell(T)$.*

Proof. Given $f\colon A \to Z$ and $g\colon B \to Z$ in $\mathscr{K}\ell(T)$. Then $\hat{f}\colon A \to TZ$ and $\hat{g}\colon B \to Z$ in $\mathscr{C}$. Let $h\colon A+B \to Z$ in $\mathscr{K}\ell(T)$ be given by $\hat{h} = [\hat{f}, \hat{g}]$. We will show that h is the unique map in $\mathscr{K}\ell(T)$ with $h\kappa_1 = f$ and $h\kappa_2 = g$.

First to show the equality holds for h: $h\kappa_1 = f$ in $\mathscr{K}\ell(T)$ holds whenever $\mu T[\hat{f}, \hat{g}]\eta\kappa_1 = \hat{f}$ in $\mathscr{C}$. Indeed:

$$\begin{aligned}
\mu T[\hat{f}, \hat{g}]\eta\kappa_1 &= \mu T[\hat{f}, \hat{g}]T\kappa_1\eta \\
&= \mu T([\hat{f}, \hat{g}]\kappa_1)\eta \\
&= \mu T\hat{f}\eta \\
&= \mu\eta\hat{f} \\
&= \hat{f}.
\end{aligned}$$

The reasoning for $h\kappa_2 = g$ is similar.

Suppose there is another map h' with $h'\kappa_1 = f$ and $h'\kappa_2 = g$. Then

$$\begin{aligned} h &= [\hat{f}, \hat{g}] \\ &= [\mu T\hat{h}'\eta\kappa_1, \mu T\hat{h}'\eta\kappa_2] \\ &= \mu T\hat{h}'\eta \\ &= \mu\eta\hat{h}' \\ &= \hat{h}'. \end{aligned}$$

□

Proposition 83. *Given a monad T on $\mathscr{C}$. Given a map $f\colon X \to Y$ in $\mathscr{K}\ell(T)$ such that $\hat{f} = \eta \circ g$ for some split mono g. Then: f is split mono.*

Proof. Let $h\colon Y \to X$ be a map in $\mathscr{C}$ such that $h \circ g = \mathrm{id}$. Let k be an arrow in $\mathscr{K}\ell(T)$ with $\hat{k} = \eta \circ h$. Then:

$$\begin{aligned} \widehat{k \circ f} &= \mu \circ T\hat{k} \circ \hat{f} \\ &= \mu \circ T\eta \circ Th \circ \eta \circ g \\ &= Th \circ \eta \circ g \\ &= \eta \circ h \circ g \\ &= \eta \\ &= \widehat{\mathrm{id}}. \end{aligned}$$

□

3 Weak effect logics

Before we turn to the more complicated effect logics, we consider just any functor $\mathscr{C} \to \mathsf{EMod}^{\mathrm{op}}$ for which we can define a reasonable *andthen*.

Definition 84. A **weak effect logic** consists of

1. a category $\mathscr{C}$ with (finite) coproducts;
2. a wide subcategory $\mathscr{D} \subseteq \mathscr{C}$ that contains the coprojections of $\mathscr{C}$;
3. a functor $\mathrm{Pred}\colon \mathscr{D} \to \mathsf{EMod}_M^{\mathrm{op}}$ for some effect monoid M, written $X \mapsto \mathrm{Pred}(X)$ and $f \mapsto (f)^*$ *and*
4. for each $X \in \mathscr{D}$ and $p \in \mathrm{Pred}(X)$, an arrow $\mathrm{char}_p\colon X \to X +_{\mathscr{C}} X$ in $\mathscr{D}$

such that

(WEL1) $\mathrm{char}_1 = \kappa_1$ and $\mathrm{char}_0 = \kappa_2$;

(WEL2) $(\kappa_1)^*$ is surjective and has a left and right order adjoint: $\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1}$ for each coprojection $\kappa_1\colon X \to X + Y$ in $\mathscr{C}$;

(WEL3) $(\mathrm{char}_p)^* \coprod_{\kappa_1} 1 = p$ for each $X \in \mathscr{C}$ and $p \in \mathrm{Pred}(X)$.

For any $X \in \mathscr{C}$, we define the following two binary operations on $\mathrm{Pred}(X)$.

$$\langle p? \rangle (q) = (\mathrm{char}_p)^* \textstyle\coprod_{\kappa_1} q \qquad [p?]\,(q) = (\mathrm{char}_p)^* \prod_{\kappa_1} q.$$

If $\mathscr{D} = \mathscr{C}$ the weak effect logic is called **full** .

As $\mathscr{D}$ might not have coproducts itself, $X + Y$ will always denote the coproduct of X and Y in $\mathscr{C}$.

3.1 Examples

We will briefly discuss some examples of weak effect logics. Later on we will return to most of these categories in as examples of effect logics. The first three examples and the last example are derived respectively from [9] and [10].

Example 85. Set is a category with finite coproducts.

For every set X, the powerset $\mathcal{P}(X)$ is an effect algebra. See Example 2. Thus it is a 2-effect module. Let $f\colon X \to Y$ be a map of Set. Consider its preimage $f^{-1}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$. It preserves the unit: $f^{-1}(Y) = X$. Furthermore, it is additive: $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Thus f^{-1} is an effect algebra homomorphism. Hence, we have a functor

$$\mathrm{Pred}\colon \mathsf{Set} \to \mathsf{EMod}_2^{\mathrm{op}} \quad X \mapsto \mathcal{P}(X) \quad f \mapsto f^{-1}$$

Given a coprojection $\kappa_1\colon X \to X+Y$. $(\kappa_1)^*$ is surjective: for any $U \in \mathcal{P}(X)$, we have $(\kappa_1)^*(\kappa_1(U)) = U$. Furthermore it has left and right adjoints: the forward image and the direct image. See Example 69. This shows (WEL2).

For $X \in \mathsf{Set}$ and $p \in \mathrm{Pred}(X) = \mathcal{P}(X)$, define $\mathrm{char}_p\colon X \to X + X$ by

$$\mathrm{char}_p(x) = \begin{cases} \kappa_1 x & x \in p \\ \kappa_2 x & x \notin p. \end{cases}$$

Then $(\mathrm{char}_p)^*(\coprod_{\kappa_1} 1) = (\mathrm{char}_p)^{-1}\kappa_1(X) = p$, hence (WEL3). Also: $\mathrm{char}_0 = \mathrm{char}_\emptyset = \kappa_2$ and $\mathrm{char}_1 = \mathrm{char}_X = \kappa_1$, hence (WEL1).

$$\begin{aligned}\langle p?\rangle(q) &= (\mathrm{char}_p)^* \textstyle\coprod_{\kappa_1} q\\ &= (\mathrm{char}_p)^{-1}(\kappa_1(q))\\ &= p \cap q.\end{aligned}$$

Example 86. Recall $\mathcal{K}\ell(\mathcal{D})$ from Example 79. Set has finite coproducts, thus so has $\mathcal{K}\ell(\mathcal{D})$ by Proposition 82.

For each set X, the set $[0,1]^X$ is a $[0,1]$-effect module, see Example 53. Define $\mathrm{Pred}(X) = [0,1]^X$. Given a map $f\colon X \to Y$ in $\mathcal{K}\ell(\mathcal{D})$. That is: a map $\hat{f}\colon X \to \mathcal{D}Y$ in Set. Define $(f)^*\colon [0,1]^Y \to [0,1]^X$ by

$$(f)^*(\varphi)(x) = \sum_y \hat{f}(x)(y)\varphi(y).$$

This is an effect module homomorphism:

$$\begin{aligned}(f)^*(1)(x) &= \sum_y \hat{f}(x)(y)1(y)\\ &= \sum_y \hat{f}(x)(y) = 1\\ (f)^*(\varphi \ovee \psi)(x) &= \sum_y \hat{f}(x)(y)(\varphi+\psi)(y)\\ &= \sum_y \hat{f}(x)(y)\varphi(y) + \sum_y \hat{f}(x)(y)\psi(y)\\ &= ((f)^*\varphi \ovee (f)^*\psi)(x)\\ (f)^*(\lambda\varphi)(x) &= \sum_y \hat{f}(x)(y)\lambda\varphi(y)\\ &= \lambda\sum_y \hat{f}(x)(y)\varphi(y)\\ &= (\lambda(f)^*(\varphi))(x).\end{aligned}$$

Hence we have a functor

$$\mathrm{Pred}\colon \mathcal{K}\ell(\mathcal{D}) \to \mathsf{EMod}_{[0,1]}^{\mathrm{op}} \quad X \mapsto [0,1]^X \quad f \mapsto f^*.$$

Given $\varphi \in \mathrm{Pred}(X+Y) = [0,1]^{X+Y}$. Then $\varphi = \psi + \chi$ for some $\psi \in [0,1]^X$ and $\chi \in [0,1]^Y$. Given a coprojection $\kappa_1\colon X \to X+Y$. Then $(\kappa_1)^*(\psi+\chi) = \psi$. Thus $(\kappa_1)^*$ is surjective: for any $\varphi \in [0,1]^X$ we have $(\kappa_1)^*(\varphi + 0) = \varphi$. Furthermore, it has left and right order-adjoints:

$$\textstyle\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1} \quad \text{where} \quad \coprod_{\kappa_1}\varphi = \varphi + 0 \text{ and } \prod_{\kappa_1}\varphi = \varphi + 1.$$

Thus (WEL2). For a $\varphi \in \mathrm{Pred}(X)$, define $\mathrm{char}_\varphi\colon X \to X+X$ in $\mathcal{K}\ell(\mathcal{D})$ by

$$\widehat{\mathrm{char}}_\varphi(x)(y) = \begin{cases}\varphi(x) & y = \kappa_1 x\\ 1-\varphi(x) & y = \kappa_2 x\\ 0 & \text{otherwise}\end{cases}$$

That is: x is mapped to the convex sum $\varphi(x) + (1 - \varphi(x))x$. Clearly $\mathrm{char}_1 = \kappa_1$ and $\mathrm{char}_0 = \kappa_2$, hence (WEL1). Finally, to show (WEL3):

$$\begin{aligned}
(\mathrm{char}_\varphi)^* \textstyle\coprod_{\kappa_1}(1)(x) &= (\mathrm{char}_\varphi)^*(1+0)(x) \\
&= \sum_y \widehat{\mathrm{char}_\varphi}(x)(y) \cdot (1+0)(y) \\
&= \sum_y \widehat{\mathrm{char}_\varphi}(x)(\kappa_1 y) \\
&= \sum_y \begin{cases} \varphi(x) & y = x \\ 0 & \text{else} \end{cases} \\
&= \varphi(x).
\end{aligned}$$

Now similarly, given $\varphi, \psi \in \mathrm{Pred}(X)$:

$$\begin{aligned}
\langle \varphi ? \rangle (\psi) &= (\mathrm{char}_\varphi)^* \textstyle\coprod_{\kappa_1} \psi \\
&= (\mathrm{char}_\varphi)^*(\psi + 0) \\
&= \varphi \cdot \psi.
\end{aligned}$$

Example 87. Hilb, the category of Hilbert spaces with (bounded linear) operators has finite coproducts. $\mathsf{Hilb}_{\mathrm{isom}}$, the category of Hilbert spaces with isometries does *not* have finite coproducts, but it is a wide subcategory of Hilb.

Given a Hilbert space $\mathscr{H}$. Consider the bounded linear operators $\mathscr{B}(\mathscr{H})$. They form an ordered vector space over $\mathbb{R}$. Thus, the interval $[0, I]_{\mathscr{B}(\mathscr{H})}$ is a convex effect algebra and thus a $[0,1]$-effect module. The operators in this interval are called the **effects** on $\mathscr{H}$, in symbols $\mathrm{Eff}(\mathscr{H})$. Let $\mathrm{Pred}(H) = \mathrm{Eff}(\mathscr{H})$.

For any isometry $f \colon \mathscr{H}_1 \to \mathscr{H}_2$, define $(f)^* \colon \mathrm{Eff}(\mathscr{H}_2) \to \mathrm{Eff}(\mathscr{H}_1)$ by

$$(f)^* A = f^\dagger A f.$$

Now note that for any $v \in \mathscr{H}_1$

$$\left\langle f^\dagger A f v, v \right\rangle = \langle A f v, f v \rangle \geq 0,$$

hence $(f)^* A \geq 0$. Also for any $v \in \mathscr{H}_1$

$$\left\langle f^\dagger A f v, v \right\rangle = \langle A f v, f v \rangle \leq \langle f v, f v \rangle = \langle v, v \rangle$$

and thus $(f)^* A \leq I$. Together: it is indeed an effect on $\mathscr{H}_2$. Now to check $(f)^*$ is an effect module homomorphism:

$$\begin{aligned}
(f)^*(A \ovee B) &= f^\dagger (A + B) f \\
&= f^\dagger A f + f^\dagger B f \\
&= ((f)^* A) \ovee ((f)^* B) \\
(f)^*(\lambda A) &= f^\dagger (\lambda A) f \\
&= \lambda f^\dagger A f \\
&= \lambda (f)^* A \\
(f)^*(I) &= f^\dagger I f \\
&= f^\dagger f \\
&= I.
\end{aligned}$$

Hence we have a functor

$$\mathrm{Pred}\colon \mathsf{Hilb}_{\mathrm{isom}} \to \mathsf{EMod}_{[0,1]}^{\mathrm{op}} \qquad \mathscr{H} \mapsto \mathrm{Eff}(\mathscr{H}) \qquad f \mapsto f^*.$$

An effect on $\mathscr{H}_1 \oplus \mathscr{H}_2$ is of the form $\left(\begin{smallmatrix} A & S \\ S^\dagger & B \end{smallmatrix}\right)$, where $A \in \mathrm{Eff}(\mathscr{H}_1)$, $B \in \mathrm{Eff}(\mathscr{H}_2)$ and $S \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$.

$$\begin{aligned}(\kappa_1)^* \begin{pmatrix} A & S \\ S^\dagger & B \end{pmatrix} &= \kappa_1^\dagger \begin{pmatrix} A & S \\ S^\dagger & B \end{pmatrix} \kappa_1 \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} A & S \\ S^\dagger & B \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= A.\end{aligned}$$

Clearly $(\kappa_1)^*$ is surjective: for any $A \in \mathrm{Eff}(\mathscr{H}_1)$, we have $(\kappa_1)^* \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) = A$. Furthermore, it has left and right order-adjoints:

$$\textstyle\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1} \quad \text{where} \quad \coprod_{\kappa_1} A = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \prod_{\kappa_1} A = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}.$$

Thus (WEL2). For an effect $A \in \mathrm{Eff}(\mathscr{H})$, define $\mathrm{char}_A \colon \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ by

$$\mathrm{char}_A = \begin{pmatrix} \sqrt{A} \\ \sqrt{I - A} \end{pmatrix}.$$

It is an isometry:

$$\mathrm{char}_A^\dagger \mathrm{char}_A = \begin{pmatrix} \sqrt{A} & \sqrt{I-A} \end{pmatrix} \begin{pmatrix} \sqrt{A} \\ \sqrt{I - A} \end{pmatrix} = \sqrt{A}\sqrt{A} + \sqrt{I-A}\sqrt{I-A} = I.$$

Note that $\mathrm{char}_0 = \begin{pmatrix} 0 & I \end{pmatrix} = \kappa_2$ and $\mathrm{char}_1 = \begin{pmatrix} I & 0 \end{pmatrix} = \kappa_1$, hence (WEL1). Finally, to show (WEL3):

$$\begin{aligned}(\mathrm{char}_A)^* \textstyle\coprod_{\kappa_1} I &= \begin{pmatrix} \sqrt{A} & \sqrt{I-A} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{A} \\ \sqrt{I - A} \end{pmatrix} \\ &= A.\end{aligned}$$

Observe

$$\begin{aligned}\langle A? \rangle (B) &= (\mathrm{char}_A)^* \textstyle\coprod_{\kappa_1} B \\ &= \begin{pmatrix} \sqrt{A} & \sqrt{I-A} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{A} \\ \sqrt{I - A} \end{pmatrix} \\ &= \sqrt{A} B \sqrt{A}.\end{aligned}$$

Example 88. Let $\mathsf{CStar}_{\mathrm{PU}}$ denote the category of C^*-algebras with unit together with positive, linear and unit preserving maps. We are interested in $\mathsf{CStar}_{\mathrm{PU}}^{\mathrm{op}}$. Coproducts in $\mathsf{CStar}_{\mathrm{PU}}^{\mathrm{op}}$ are products in $\mathsf{CStar}_{\mathrm{PU}}$, which exist. A C^*-algebra is an ordered vectorspace. Hence, given a C^*-algebra $\mathscr{A}$, the unit interval $[0,1]_{\mathscr{A}}$ is a $[0,1]$-effect module. Let $\mathrm{Pred}(\mathscr{A}) = [0,1]_{\mathscr{A}}$.

A map $f \colon \mathscr{A}_1 \to \mathscr{A}_2$ in $\mathsf{CStar}_{\mathrm{PU}}^{\mathrm{op}}$ is a map $f \colon \mathscr{A}_2 \to \mathscr{A}_1$ in $\mathsf{CStar}_{\mathrm{PU}}$. Given an $a \in [0,1]_{\mathscr{A}_2}$, we have $0 \leq a \leq 1$ and thus $0 = f(0) \leq f(a) \leq f(1) = 1$, hence: $f(a) \in [0,1]_{\mathscr{A}_1}$. Thus we can restrict f to a map $(f)^*\colon [0,1]_{\mathscr{A}_2} \to [0,1]_{\mathscr{A}_1}$. This map is an effect module homomorphism: it is additive and homogeneous, because f is linear and it preserves the unit, because f does.

Hence we have a functor

$$\mathrm{Pred}\colon \mathsf{CStar}_{\mathrm{PU}}^{\mathrm{op}} \to \mathsf{EMod}_{[0,1]}^{\mathrm{op}} \quad \mathscr{A} \mapsto [0,1]_{\mathscr{A}} \quad f \mapsto f^*.$$

The coprojection κ_1 in $\mathsf{CStar}_{\mathrm{PU}}^{\mathrm{op}}$ is the projection π_1 in $\mathsf{CStar}_{\mathrm{PU}}$. Hence $(\kappa_1)^*(a_1,a_2) = a_1$. This map is clearly surjective. Furthermore, it has left and right order-adjoints:

$$\textstyle\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1} \quad \text{where} \quad \coprod_{\kappa_1} a = (a,0) \text{ and } \prod_{\kappa_1} a = (a,1).$$

This shows (WEL2). Given $a \in [0,1]_{\mathscr{A}}$, define $\mathrm{char}_a\colon A \to A+A$ in $\mathsf{CStar}_{\mathrm{PU}}^{\mathrm{op}}$ by

$$\mathrm{char}_a(b_1,b_2) = \sqrt{a}b_1\sqrt{a} + \sqrt{1-a}b_2\sqrt{1-a}.$$

Clearly char_a is linear. In a C^*-algebra, xyx is positive if x and y are positive — thus char_a is positive. char_a preserves the unit:

$$\mathrm{char}_a(1,1) = \sqrt{a}\sqrt{a} + \sqrt{1-a}\sqrt{1-a} = 1.$$

Similarly $\mathrm{char}_1 = \kappa_1$ and $\mathrm{char}_0 = \kappa_2$. That is: (WEL1). Now, observe

$$(\mathrm{char}_a)^* \textstyle\coprod_{\kappa_1} a = \sqrt{a}1\sqrt{a} + \sqrt{1-a}0\sqrt{1-a} = a,$$

which demonstrates (WEL3). Note

$$\begin{aligned} \langle a? \rangle (b) &= (\mathrm{char}_a)^* \textstyle\coprod_{\kappa_1} b \\ &= \sqrt{a}b\sqrt{a} + \sqrt{1-a}0\sqrt{1-a} \\ &= \sqrt{a}b\sqrt{a}. \end{aligned}$$

3.2 Basic theory

We have seen various examples of weak effect logics. Lets see what we can derive from the axioms. The second, (WEL2) has some surprising consequences.

Proposition 89. *1.* $(\kappa_1)^* \coprod_{\kappa_1} = \mathrm{id}$*;*

2. $\coprod_{\kappa_1}$ *is an order embedding;*

3. $\coprod_{\kappa_1} a \ovee b = (\coprod_{\kappa_1} a) \ovee (\coprod_{\kappa_1} b)$ *whenever* $a \perp b$ and

4. $\lambda \coprod_{\kappa_1} a = \coprod_{\kappa_1} \lambda a$ *for any scalar* $\lambda \in M$.

Proof. One at a time.

1. Given any p. Let q be such that $(\kappa_1)^* q = p$. Since $\coprod_{\kappa_1} \dashv (\kappa_1)^*$, we have

$$\textstyle\coprod_{\kappa_1} (\kappa_1)^* \coprod_{\kappa_1} (\kappa_1)^* q \leq \coprod_{\kappa_1} (\kappa_1)^* q \leq q$$

and thus, as desired:

$$(\kappa_1)^* \textstyle\coprod_{\kappa_1} p \leq (\kappa_1)^* \coprod_{\kappa_1} (\kappa_1)^* q \leq (\kappa_1)^* q = p \leq (\kappa_1)^* \coprod_{\kappa_1} p.$$

2. Suppose $\coprod_{\kappa_1} p \le \coprod_{\kappa_1} q$. Then $p \le (\kappa_1)^* \coprod_{\kappa_1} q = q$. Thus $\coprod_{\kappa_1}$ is an order embedding.

3. First note that $\coprod_{\kappa_1} c = \min\{z;\ c \le (\kappa_1)^* z\}$, and $a \le (\kappa_1)^* \coprod_{\kappa_1} a$. Thus $a \owedge b \le ((\kappa_1)^* \coprod_{\kappa_1} a) \owedge ((\kappa_1)^* \coprod_{\kappa_1} b) = (\kappa_1)^*(\coprod_{\kappa_1} a) \owedge (\coprod_{\kappa_1} b)$. Suppose $a \owedge b \le (\kappa_1)^* z$, for some other z. Then

$$\begin{aligned} a &\le ((\kappa_1)^* z) \ominus b \\ &= ((\kappa_1)^* z) \ominus (\kappa_1)^* \textstyle\coprod_{\kappa_1} b \\ &= (\kappa_1)^*(z \ominus \textstyle\coprod_{\kappa_1} b). \end{aligned}$$

   And thus $\coprod_{\kappa_1} a \le z \ominus \coprod_{\kappa_1} b$. Hence $(\coprod_{\kappa_1} a) \owedge (\coprod_{\kappa_1} b) \le z$. And consequently $(\coprod_{\kappa_1} a) \owedge (\coprod_{\kappa_1} b) = \min\{z;\ a \owedge b \le (\kappa_1)^* z\} = \coprod_{\kappa_1} a \owedge b$.

4. We have $a \le (\kappa_1)^* \coprod_{\kappa_1} a$ and thus $\lambda a \le \lambda(\kappa_1)^* \coprod_{\kappa_1} a = (\kappa_1)^* \lambda \coprod_{\kappa_1} a$. Consequently $\coprod_{\kappa_1} \lambda a \le \lambda \coprod_{\kappa_1} a$. From this follows

$$\begin{aligned} \textstyle\coprod_{\kappa_1} a &= \textstyle\coprod_{\kappa_1} \lambda^\perp a \owedge \lambda a \\ &= (\textstyle\coprod_{\kappa_1} \lambda^\perp a) \owedge (\textstyle\coprod_{\kappa_1} \lambda a) \\ &\le (\textstyle\coprod_{\kappa_1} \lambda^\perp a) \owedge \lambda(\textstyle\coprod_{\kappa_1} a) \\ &\le \lambda^\perp(\textstyle\coprod_{\kappa_1} a) \owedge \lambda(\textstyle\coprod_{\kappa_1} a) = \textstyle\coprod_{\kappa_1} a. \end{aligned}$$

   Thus all in between are equal, hence

$$(\textstyle\coprod_{\kappa_1} \lambda^\perp a) \owedge (\textstyle\coprod_{\kappa_1} \lambda a) = (\textstyle\coprod_{\kappa_1} \lambda^\perp a) \owedge \lambda(\textstyle\coprod_{\kappa_1} a)$$

   and with cancellation $\coprod_{\kappa_1} \lambda a = \lambda \coprod_{\kappa_1} a$. □

From this, we can derive some properties of *andthen*.

Definition 90. A **weak sequential effect module** is an effect module with a right-linear binary operation with unit 1.

Proposition 91. *For* Pred(X) *in a weak effect logic,* andthen *makes* Pred(X) *a weak sequential effect module. That is:*

1. $\langle p? \rangle (1) = p$,
2. $\langle 1? \rangle (p) = p$,
3. $\langle p? \rangle (q_1 \owedge q_2) = \langle p? \rangle (q_1) \owedge \langle p? \rangle (q_2)$ *and*
4. $\langle p? \rangle (\lambda q) = \lambda \langle p? \rangle (q)$.

Proof. One at a time.

1. $\langle p? \rangle (1) = (\mathrm{char}_p)^* \coprod_{\kappa_1} 1 = p$ by (WEL3).
2. $\langle 1? \rangle (p) = (\mathrm{char}_1)^* \coprod_{\kappa_1} p = (\kappa_1)^* \coprod_{\kappa_1} p = p$.
3. $(\mathrm{char}_p)^*$ and $\coprod_{\kappa_1}$ are both additive. Thus so is its composition $\langle p? \rangle (\ )$.
4. $(\mathrm{char}_p)^*$ and $\coprod_{\kappa_1}$ are both homogeneous. Thus so is its composition $\langle p? \rangle (\ )$. □

3.3 Representation of weak sequential effect modules

One may wonder: can we deduce more properties of *andthen* than that it is a weak sequential effect module? No: the following representation theorem entails that any general theorem about *andthen* is a consequence of the weak sequential effect module axioms.

Definition 92. Given an effect module E with a binary operation $*$ and a weak effect logic. We say E **is represented in the weak effect logic** if $E = \mathrm{Pred}(X)$ for some $X \in \mathscr{D}$ and $p * q = \langle p? \rangle\,(q)$ for all $p, q \in E$.

Theorem 93. *Any weak sequential effect module is represented in a full weak effect logic.*

Proof. We will first define a category $\mathscr{C} = \mathscr{D}$. Then we construct a functor $\mathrm{Pred}\colon \mathscr{C} \to \mathsf{EMod}_M^{\mathrm{op}}$. Finally, we show they form a weak effect logic.

As objects of our category $\mathscr{C}$ we will use the natural number $\mathbb{N}$. For the most part, the construction is fixed by the requirements. One important choice we made in this construction is the following. We will set $\mathrm{Pred}(n) = E^n$. Thus $\mathrm{Pred}(0) = 2$, $\mathrm{Pred}(1) = E$ and $\mathrm{Pred}(2) = E \times E$. Also note $\mathrm{Pred}(n+m) \cong E^n \times E^m$.

1. The arrows of $\mathscr{C}$ are given syntactically. We specify which arrows exist and which should be considered equal.

   (a) For each $n \in \mathbb{N}$ and $p \in E^n$, there is an arrow $\mathrm{char}_p\colon n \to 2n$.

   (b) For each $n, m \in \mathbb{N}$ there are $\kappa_1\colon n \to n + m$ and $\kappa_2\colon m \to n + m$.

   (c) Given arrows $f\colon n \to l$ and $g\colon m \to l$, there is an arrow $[f, g]\colon n + m \to l$.

   (d) For every n, there is an arrow $\mathrm{id}\colon n \to n$.

   (e) Given arrows $f\colon n \to m$ and $g\colon m \to l$, there is an arrow $g \circ f\colon n \to l$.

   The equality is given by the following rules.

   (a) For any $n \in N$, if $\kappa_1\colon n \to n+n$, then $\kappa_1 = \mathrm{char}_1$ and if $\kappa_2\colon n \to n+n$, then $\kappa_2 = \mathrm{char}_0$.

   (b) $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$

   (c) $[h \circ \kappa_1, h \circ \kappa_2] = h$

   (d) $f \circ \mathrm{id} = \mathrm{id} \circ f = f$

   (e) $(f \circ g) \circ h = f \circ (g \circ h)$

   (f) If $f = f'$ and $g = g'$ then $[f, g] = [f', g']$ and $f \circ g = f' \circ g'$.

2. We want Pred to act on arrows as follows.

   (a) The construction hinges on this requirement:

   $$(\mathrm{char}_p)^*(q_1, q_2) = (p * q_1) \otimes (p^\perp * q_2).$$

   (b) $(\kappa_1)^* = \pi_1$ and $(\kappa_2)^* = \pi_2$

   (c) $([f, g])^* = \langle (f)^*, (g)^* \rangle$

(d) $(\mathrm{id})^* = \mathrm{id}$

(e) $(f \circ g)^* = (g)^* \circ (f)^*$

Before we call this the inductive definition of Pred, we need to check that it respects the equality we forced and it determines effect module homomorphisms. The latter first. The only non trivial case, is the first case.

(a) First we check whether $p * q_1 \perp p^\perp * q_2$. Note that $p * q \leq (p * q) \ovee (p * q^\perp) = p * (q \ovee q^\perp) = p * 1 = p$. Thus $p * q_1 \leq p$ and $p^\perp * q_2 \leq p^\perp$. Because $p \perp p^\perp$, also $p \perp p^\perp * q_2$. And thus $p * q_1 \perp p^\perp * q_2$.

Now we check whether $h(q_1, q_2) = (p * q_1) \ovee (p^\perp * q_2)$ is an effect module homomorphism. First additivity:

$$\begin{aligned} h(q_1 \ovee q_1', q_2 \ovee q_2') &= (p * (q_1 \ovee q_1')) \ovee (p^\perp * (q_2 \ovee q_2')) \\ &= (p * q_1) \ovee (p * q_1') \ovee (p^\perp * q_2) \ovee (p^\perp * q_2') \\ &= h(q_1, q_2) \ovee h(q_1', q_2'). \end{aligned}$$

Secondly, preservation of unit: $h(1,1) = (p*1) \ovee (p^\perp * 1) = p \ovee p^\perp = 1$. Finally, homogeneity:

$$\begin{aligned} h(\lambda q_1, \lambda q_2) &= (p * (\lambda q_1)) \ovee (p^\perp * (\lambda q_2)) \\ &= \lambda(p * q_1) \ovee \lambda(p^\perp * q_2) \\ &= \lambda(p * q_1 \ovee p^\perp * q_2) \\ &= \lambda h(q_1, q_2). \end{aligned}$$

Now, to check the preservation of equality.

(a) We check whether $(\mathrm{char}_1)^* = (\kappa_1)^* = \pi_1$. First note that $0 * p \leq 0 * 1 = 0$ and thus:

$$\begin{aligned} (\mathrm{char}_1)^*(q_1, q_2) &= (1 * q_1) \ovee (0 * q_2) \\ &= q_1 = \pi_1(q_1, q_2). \end{aligned}$$

(b) The first: $([f,g] \circ \kappa_1)^* = (\kappa_1)^* \circ ([f,g])^* = \pi_1 \circ \langle (f)^*, (g)^* \rangle = (f)^*$. The second equality with κ_2 is just as easy.

(c) The first: $([h \circ \kappa_1, h \circ \kappa_2])^* \langle \pi_1 \circ (h)^*, \pi_2 \circ (h)^* \rangle = (h)^*$. And again, the second equality is just as easy.

(d) $(f \circ \mathrm{id})^* = (\mathrm{id})^* \circ (f)^* = \mathrm{id} \circ (f)^* = (f)^* = (f)^* \circ (\mathrm{id})^* = (\mathrm{id} \circ f)^*$

(e) $(f \circ (g \circ h))^* = (g \circ h)^* \circ (f)^* = (h)^* \circ (g)^* \circ (f)^* = (h)^* \circ (f \circ g)^* = ((f \circ g) \circ h)^*$

(f) Suppose $f = f'$ and $g = g'$. Proving inductively, we may assume $(f)^* = (f')^*$ and $(g)^* = (g')^*$. Then $(f \circ g)^* = (g)^* \circ (f)^* = (g')^* \circ (f')^* = (f' \circ g')^*$ and similarly $([f,g])^* = \langle (g)^*, (f)^* \rangle = \langle (g')^*, (f')^* \rangle = ([f', g'])^*$.

3. Our category $\mathscr{C}$ has finite coproducts. Given $n, m \in \mathscr{C}$. Their coproduct in $\mathscr{C}$ is $n + m$, the sum of natural numbers, with coprojections κ_1 and κ_2.

To prove this, assume $f: n \to l$ and $g: m \to l$. We ensured $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$. Now, given any other $h: n+m \to l$ such that $h \circ \kappa_1 = f$ and $h \circ \kappa_2 = g$. Then $[f, g] = [h \circ \kappa_1, h \circ \kappa_2] = h$. Thus indeed, $n + m$ is the coproduct in $\mathscr{C}$.

4. Now, we check the axioms of a weak effect logic. Concerning (WEL1): $\mathrm{char}_1 = \kappa_1$ and $\mathrm{char}_0 = \kappa_2$, is satisfied by construction.

   To show (WEL2), given a coprojection κ_1, define $\coprod_{\kappa_1}(p) = (p, 0)$ and $\prod_{\kappa_1}(q) = (q, 1)$. Then

   $$\coprod_{\kappa_1}(p) = (p, 0) \leq (q_1, q_2) \iff p \leq q_1 = \pi_1(q_1, q_2) = (\kappa_1)^*(q_1, q_2)$$

   and thus $\coprod_{\kappa_1} \dashv (\kappa_1)^*$. It is also easy to see $(\kappa_1)^* \dashv \prod_{\kappa_1}$. Furthermore, $(\kappa_1)^* = \pi_1$ is surjective.

   Note that $(a*0) \ovee (a*0) = a*(0 \ovee 0) = a*0$ and thus by cancellation $a*0 = 0$. Thus $(\mathrm{char}_p)^* \coprod_{\kappa_1} 1 = (p * 1) \ovee (p^\perp * 0) = p * 1 = p$, which shows (WEL3).

5. Finally, we check whether $p * q = \langle p? \rangle (q)$ in $\mathrm{Pred}(1) = E$:

   $$\langle p? \rangle (q) = (\mathrm{char}_p)^* \coprod_{\kappa_1} q = (\mathrm{char}_p)^*(q, 0) = (p * q) \ovee (p^\perp * 0) = p * q.$$

□

4 Effect logics

4.1 Internal predicates

Definition 94. Recall that in the introduction we defined for a category $\mathscr{C}$ with coproducts and an object $X \in \mathscr{C}$, the set of internal predicates on X as follows.

$$\mathrm{iPred}(X) = \{p\colon X \to X + X;\ [\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}\}.$$

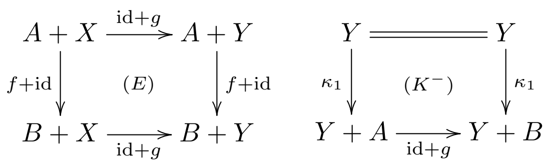
We will define 1, 0, $(\ )^{\perp}$ and $\owedge$ on $\mathrm{iPred}(X)$.

1. Define $1 = \kappa_1$ and $0 = \kappa_2$.
2. For $p \in \mathrm{iPred}(X)$, let $p^{\perp} = [\kappa_2, \kappa_1] \circ p$.
3. Given $p, q \in \mathrm{iPred}(X)$, write $p \perp q$ if there is a unique map $b\colon X \to X + X + X$, called the **bound**, such that $[\kappa_1, \kappa_2, \kappa_2] \circ b = p$ and $[\kappa_2, \kappa_1, \kappa_2] \circ b = q$. In that case, define $p \owedge q = [\kappa_1, \kappa_1, \kappa_2] \circ b$.

The structure $\langle \mathrm{iPred}(X), \owedge, 0, 1\rangle$ is in general not an an effect algebra. However, the following assumptions, a slight variation on those of [9], are sufficient.

Proposition 95 $(*)$**.** *Given a category $\mathscr{C}$ with (finite) coproducts. Suppose:*

1. *The following two diagrams are pullbacks.*



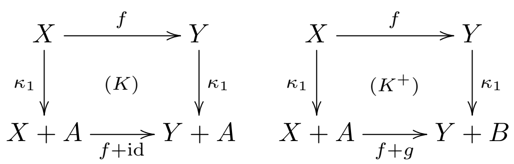
2. *The maps $[\kappa_1, \kappa_2, \kappa_2]$ and $[\kappa_2, \kappa_1, \kappa_2]$ are jointly monic.*

Then:

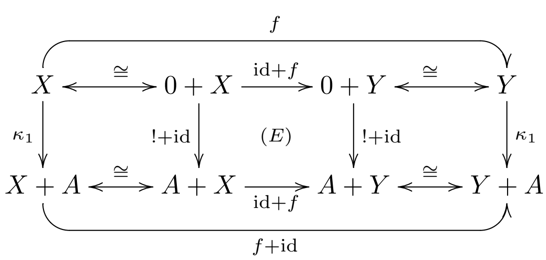
1. *$\langle \mathrm{iPred}(X), 0, 1, \owedge\rangle$ is an effect algebra.*
2. *coprojections are monic.*

Proof. First, two lemmas:

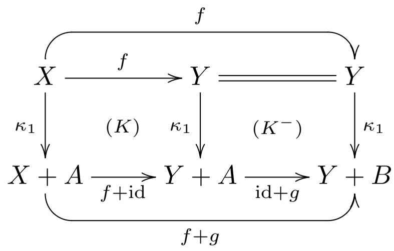
1. First, we will prove that the following two diagrams are pullbacks.



To see (K) is a pullback diagram, we note it is a special case of (E):



And (K^+) is, by the pullback lemma applied to (K) and (K^-):



2. Given $p, q\colon X \to X + X$ and $b, b'\colon X \to X + X + X$ such that both

$$[\kappa_1, \kappa_2, \kappa_2] \circ b = p = [\kappa_1, \kappa_2, \kappa_2] \circ b'$$
$$[\kappa_2, \kappa_1, \kappa_2] \circ b = q = [\kappa_2, \kappa_1, \kappa_2] \circ b'$$

then since $[\kappa_1, \kappa_2, \kappa_2]$ and $[\kappa_2, \kappa_1, \kappa_2]$ are jointly monic, we have $b = b'$. Thus to show $p \perp q$, we only have to give a map b which obeys the the two equalities: the uniqueness follows from this lemma.

Now, to prove the two consequences:

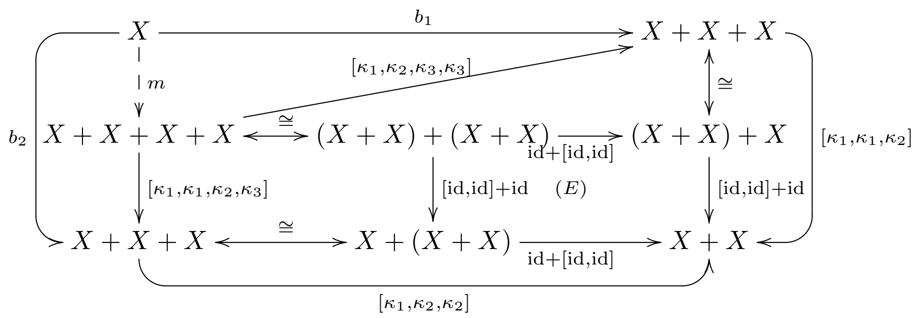
1. (E1) Suppose $p \perp q$ with bound b. Then:

$$[\kappa_1, \kappa_2, \kappa_2] \circ [\kappa_2, \kappa_1, \kappa_3] \circ b = [\kappa_2, \kappa_1, \kappa_2] \circ b = q$$
$$[\kappa_2, \kappa_1, \kappa_2] \circ [\kappa_2, \kappa_1, \kappa_3] \circ b = [\kappa_1, \kappa_2, \kappa_2] \circ b = p.$$

Thus $[\kappa_2, \kappa_1, \kappa_2] \circ b$ is a bound for q and p. Hence $q \perp p$. Furthermore

$$p \ovee q = [\kappa_1, \kappa_1, \kappa_2] \circ b = [\kappa_1, \kappa_1, \kappa_2] \circ [\kappa_2, \kappa_1, \kappa_3] \circ b = q \ovee p.$$

(E2) Suppose $p \perp q$ with bound b_1 and $p \ovee q \perp r$ with bound b_2. Then by (E) there exists an arrow $m\colon X \to X + X + X + X$ such that the following diagram commutes.



Note that

$$\begin{aligned}
[\kappa_1, \kappa_2, \kappa_2] \circ [\kappa_3, \kappa_1, \kappa_2, \kappa_3] \circ m &= [\kappa_2, \kappa_1, \kappa_2, \kappa_2] \circ m \\
&= [\kappa_2, \kappa_1, \kappa_2] \circ [\kappa_1, \kappa_2, \kappa_3, \kappa_3] \circ m \\
&= [\kappa_2, \kappa_1, \kappa_2] \circ b_1 \\
&= q
\end{aligned}$$

and

$$
\begin{aligned}
[\kappa_2,\kappa_1,\kappa_2]\circ[\kappa_3,\kappa_1,\kappa_2,\kappa_3]\circ m &= [\kappa_2,\kappa_2,\kappa_1,\kappa_2]\circ m\\
&= [\kappa_2,\kappa_1,\kappa_2]\circ[\kappa_1,\kappa_1,\kappa_2,\kappa_3]\circ m\\
&= [\kappa_2,\kappa_1,\kappa_2]\circ b_2\\
&= r
\end{aligned}
$$

thus $[\kappa_3,\kappa_1,\kappa_2,\kappa_2]\circ m$ is a bound for $q\perp r$. Also observe

$$
\begin{aligned}
[\kappa_1,\kappa_2,\kappa_2]\circ[\kappa_1,\kappa_2,\kappa_2,\kappa_3]\circ m &= [\kappa_1,\kappa_2,\kappa_2,\kappa_2]\circ m\\
&= [\kappa_1,\kappa_2,\kappa_2]\circ[\kappa_1,\kappa_2,\kappa_3,\kappa_3]\circ m\\
&= [\kappa_1,\kappa_2,\kappa_2]\circ b_1\\
&= p
\end{aligned}
$$

and

$$
\begin{aligned}
[\kappa_2,\kappa_1,\kappa_2]\circ[\kappa_1,\kappa_2,\kappa_2,\kappa_3]\circ m &= [\kappa_2,\kappa_1,\kappa_1,\kappa_2]\circ m\\
&= [\kappa_1,\kappa_1,\kappa_2]\circ[\kappa_3,\kappa_1,\kappa_2,\kappa_3]\circ m\\
&= q\ovee r
\end{aligned}
$$

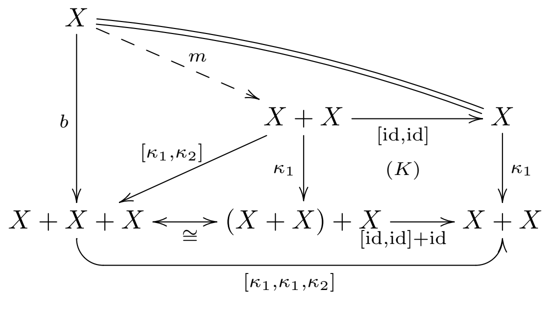
thus $[\kappa_1,\kappa_2,\kappa_2,\kappa_3]\circ m$ is a bound for $p\perp q\ovee r$. Finally,

$$
\begin{aligned}
p\ovee(q\ovee r) &= [\kappa_1,\kappa_1,\kappa_2]\circ[\kappa_1,\kappa_2,\kappa_2,\kappa_3]\circ m\\
&= [\kappa_1,\kappa_1,\kappa_1,\kappa_2]\circ m\\
&= [\kappa_1,\kappa_1,\kappa_2]\circ[\kappa_1,\kappa_1,\kappa_2,\kappa_3]\circ m\\
&= [\kappa_1,\kappa_1,\kappa_2]\circ b_2\\
&= (p\ovee q)\ovee r.
\end{aligned}
$$

(E3) Observe that

$$
\begin{aligned}
[\kappa_1,\kappa_2,\kappa_2]\circ[\kappa_1,\kappa_2]\circ p &= [\kappa_1,\kappa_2]\circ p = p\\
[\kappa_2,\kappa_1,\kappa_2]\circ[\kappa_1,\kappa_2]\circ p &= [\kappa_2,\kappa_1]\circ p = p^\perp\\
[\kappa_1,\kappa_1,\kappa_2]\circ[\kappa_1,\kappa_2]\circ p &= [\kappa_1,\kappa_1]\circ p\\
&= \kappa_1\circ[\mathrm{id},\mathrm{id}]\circ p\\
&= \kappa_1 = 1.
\end{aligned}
$$

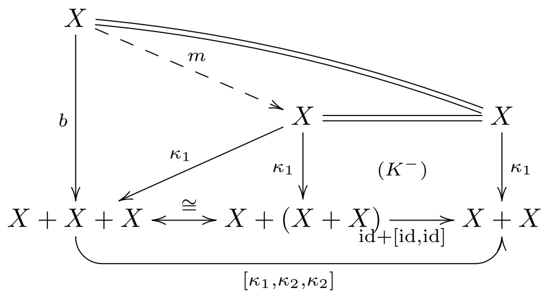
So $[\kappa_1,\kappa_2]\circ p$ is a bound for p and $p^\perp$ and $p\ovee p^\perp = 1$.
Suppose $p\ovee q = 1$ with bound b. Then, with (K), there is a map $m\colon X\to X+X$ such that the following diagram commutes.



But then $b = [\kappa_1, \kappa_2] \circ m$ and thus

$$\begin{aligned} p^\perp &= [\kappa_2, \kappa_1] \circ p \\ &= [\kappa_2, \kappa_1] \circ [\kappa_1, \kappa_2, \kappa_2] \circ b \\ &= [\kappa_2, \kappa_1, \kappa_1] \circ b \\ &= [\kappa_2, \kappa_1, \kappa_1] \circ [\kappa_1, \kappa_2] \circ m \\ &= [\kappa_2, \kappa_1] \circ m \\ &= [\kappa_2, \kappa_1, \kappa_2] \circ [\kappa_1, \kappa_2] \circ m \\ &= [\kappa_2, \kappa_1, \kappa_2] \circ b \\ &= q. \end{aligned}$$

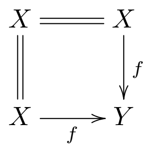
(E4) Suppose $1 \perp p$ with bound b. Then with (K^-) we see that there is a unique map $m\colon X \to X$ such that the following diagram commutes:



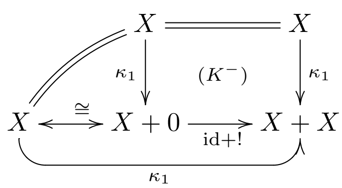
Thus $m = \text{id}$ and consequently $b = \kappa_1$. Hence

$$\begin{aligned} p &= [\kappa_2, \kappa_1, \kappa_1] \circ b \\ &= [\kappa_2, \kappa_1, \kappa_1] \circ \kappa_1 \\ &= \kappa_2 = 0. \end{aligned}$$

2. Note that a map $f\colon X \to Y$ is a monomorphism if and only if the following diagram is a pullback.



The diagram for κ_1 is a pullback, as it is a case of (K^-):



□

4.2 Axioms

Definition 96. An **effect logic** consists of

1. a category $\mathscr{C}$ with (finite) coproducts;
2. a wide subcategory $\mathscr{D} \subseteq \mathscr{C}$, which contains the coprojections of $\mathscr{C}$;
3. a functor $\mathrm{Pred}\colon \mathscr{D} \to \mathsf{EMod}_M^{\mathrm{op}}$ for some effect monoid M, written $X \mapsto \mathrm{Pred}(X)$ and $f \mapsto (f)^*$ *and*
4. for each $X \in \mathscr{C}$ and $p \in \mathrm{Pred}(X)$, an arrow $\mathrm{char}_p\colon X \to X +_{\mathscr{C}} X$ in $\mathscr{D}$

such that

(EL1) each $p \in \mathrm{Pred}(X)$ is a map $X \to X + X$ in $\mathscr{C}$ with $[\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}$;

(EL2) (a) for each $p, q \in \mathrm{Pred}(X)$, we have $p \perp q$ if and only if there is a map $b\colon X + X \to X + X + X$ in $\mathscr{C}$, called the **bound**, such that $[\kappa_1, \kappa_2, \kappa_2] \circ b = p$; $[\kappa_2, \kappa_1, \kappa_2] \circ b = q$ and $[\kappa_1, \kappa_1, \kappa_2] \circ b \in \mathrm{Pred}(X)$ and then: $p \ovee q = [\kappa_1, \kappa_1, \kappa_2] \circ b$;

(b) for each $p \in \mathrm{Pred}(X)$, we have $p^\perp = [\kappa_2, \kappa_1] \circ p$ *and*

(c) for $1 \in \mathrm{Pred}(X)$, we have $1 = \kappa_1$.

(EL3) for every coprojection $\kappa_1\colon X \to X + Y$, we have $\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1}$, where $\coprod_{\kappa_1} p = [(\kappa_1 + \kappa_1) \circ p, \kappa_2 \circ \kappa_2]$ and $\prod_{\kappa_1} p = [(\kappa_1 + \kappa_1) \circ p, \kappa_1 \circ \kappa_2]$;

(EL4) coprojections are monic in $\mathscr{C}$;

(EL5) $\mathrm{char}_1 = \kappa_1$ and $\mathrm{char}_0 = \kappa_2$ *and*

(EL6) $(\mathrm{char}_p)^* \coprod_{\kappa_1} 1 = p$.

For any $X \in \mathscr{C}$, we define the following two binary operations on $\mathrm{Pred}(X)$:

$$\langle p? \rangle (q) = (\mathrm{char}_p)^* \textstyle\coprod_{\kappa_1} q \qquad [p?] (q) = (\mathrm{char}_p)^* \textstyle\prod_{\kappa_1} q.$$

If for every $p\colon X \to X + X$ in $\mathscr{C}$ with $[\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}$, we have $p \in \mathrm{Pred}(X)$, then the effect logic is called **internal**. If $\mathscr{C} = \mathscr{D}$, then the effect logic is called **full**.

Proposition 97. *Every (full) effect logic is a (full) weak effect logic.*

Proof. Almost all weak effect logic axioms are satisfied by definition. The only thing left to prove is that $(\kappa_1)^*$ is surjective. By Proposition 75, it is sufficient to prove that $\coprod_{\kappa_1}$ is injective. First, note that for every $p \in \mathrm{Pred}(X)$ we have

$$\begin{aligned}
[\kappa_1 + \kappa_1, \kappa_2 + \kappa_2](\textstyle\coprod_{\kappa_1} p)\kappa_1 &= [\kappa_1 + \kappa_1, \kappa_2 + \kappa_2][(\kappa_1 + \kappa_1)p, \kappa_2\kappa_2]\kappa_1 \\
&= [\kappa_1 + \kappa_1, \kappa_2 + \kappa_2](\kappa_1 + \kappa_1)p \\
&= [(\kappa_1 + \kappa_1)\kappa_1, (\kappa_2 + \kappa_2)\kappa_1]p \\
&= [\kappa_1\kappa_1, \kappa_1\kappa_2]p \\
&= \kappa_1[\kappa_1, \kappa_2]p \\
&= \kappa_1 p.
\end{aligned}$$

And thus if $\coprod_{\kappa_1} p = \coprod_{\kappa_1} p'$, then also $\kappa_1 p = [\kappa_1 + \kappa_1, \kappa_2 + \kappa_2](\coprod_{\kappa_1} p)\kappa_1 = [\kappa_1 + \kappa_1, \kappa_2 + \kappa_2](\coprod_{\kappa_1} p')\kappa_1 = \kappa_1 p'$. Now, since coprojections are monic, we have: $p = p'$, as desired. $\square$

4.3 Examples

Now, we will look at several examples of effect logics.

4.3.1 Set

The first example is the classical case: Set, the category of sets.

Definition 98. Given a set X and a subset $U \subseteq X$. Write p_U for the map $X \to X + X$ in Set given by

$$p_U(x) = \begin{cases} \kappa_1 x & x \in U \\ \kappa_2 x & x \notin U. \end{cases}$$

Proposition 99. *A map $X \to X + X$ in Set is an internal predicate on X if and only if $p = p_U$ for some $U \subseteq X$.*

Proof. $p\colon X \to X + X$ is an internal predicate if and only if $[\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}$. This is the case if and only if for every $x \in X$, either $p(x) = \kappa_1 x$ or $p(x) = \kappa_2 x$. Clearly, $p = p_U$ with $U = \{x;\ p(x) = \kappa_1 x\}$. □

$\mathcal{P}(X)$, the set of subsets of X, is an effect algebra. See Example 2. Its effect algebra operations are compatible with the corresponding operations on the internal predicates.

Proposition 100. *Given a set X. For any $U, V \in \mathcal{P}(X)$:*

1. $p_U \perp p_V$ *if and only if* $U \perp V$*;*
2. $p_{U \varovee V} = p_U \varovee p_V$*;*
3. $p_U^\perp = p_{U^\perp}$ and
4. $1 = p_1$.

Proof. 3 and 4 follow directly from the definition of p_U. Given $U, V \subseteq X$.

1. $p_U \perp p_V$ if and only if there is a bound b for p_U and p_V. This is the case if and only if for every x:

$$b(x) = \begin{cases} \kappa_1 x & x \in U \\ \kappa_2 x & x \in V \\ \kappa_3 x & \text{otherwise.} \end{cases}$$

   Such a b is unique whenever it exists and it exists if and only if $U \perp V$.

2. Let b be the bound of U and V. Then:

$$\begin{aligned} p_U \varovee p_V(x) &= [\kappa_1, \kappa_1, \kappa_2] \circ b(x) \\ &= \begin{cases} \kappa_1 x & x \in U \text{ or } x \in V \\ \kappa_2 x & \text{otherwise} \end{cases} \\ &= p_{U \varovee V}. \end{aligned}$$

□

Definition 101. Set $\mathscr{D} = \mathscr{C} = \mathsf{Set}$. Let $\mathrm{Pred}\colon \mathsf{Set} \to \mathsf{EA}^{\mathrm{op}} = \mathsf{EMod}_2^{\mathrm{op}}$ map

$$X \mapsto \mathcal{P}(X) \cong \mathrm{iPred}(X) \qquad f \mapsto f^{-1}.$$

Set $\mathrm{char}_U = p_U$.

Proposition 102. $\mathrm{Pred}\colon \mathsf{Set} \to \mathsf{EMod}_2^{\mathrm{op}}$ *is a full and internal effect logic. Furthermore:* $\langle U?\rangle\,(V) = U \cap V$*.*

Proof. We already saw that Pred is indeed a functor in Example 85. Also: Set is a category with coproducts.

(EL1) Shown in Proposition 99.

(EL2) Shown in Proposition 100.

(EL3) We first note that for a coprojection $\kappa_1\colon X \to X + Y$, we have

$$(\kappa_1)^* U = \kappa_1^{-1}(U) \qquad \textstyle\coprod_{\kappa_1} U = \kappa_1(U) \qquad \prod_{\kappa_1} U = \kappa_1(U) \cup Y = (\kappa_1)_{**}(U),$$

where $(\kappa_1)_{**}$ is the direct image. See Example 69. There it is also demonstrated that we have the order adjunction

$$(\kappa_1)_* = \textstyle\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1} = (\kappa_1)_{**}.$$

(EL4) Coprojections in Set are injective, hence monic.

(EL5) Shown in Proposition 100.

Before we continue with the demonstration of (EL6), observe

$$\begin{aligned}\langle U?\rangle\,(V) &= (\mathrm{char}_U)^* \textstyle\coprod_{\kappa_1} V \\ &= (p_U)^* \kappa_1(V) \\ &= (p_U)^{-1}(\kappa_1(V)) \\ &= U \cap V.\end{aligned}$$

(EL6) Given $U \in \mathrm{Pred}(X)$. Then: $(\mathrm{char}_U)^* \coprod_{\kappa_1} 1 = \langle U?\rangle\,(1) = U \cap X = U$.

Finally, note that Pred is full and internal by definition. ∎

4.3.2 $\mathscr{K}\ell(\mathcal{D}_M)$

The second example is a generalization of the probabilistic case. For this subsection, assume M is an effect monoid (Definition 36). If $M = [0,1]$, we have the probabilistic case. We will investigate the category $\mathscr{K}\ell(\mathcal{D}_M)$ — the Kleisli category (Definition 78) of the M-distribution monad (Definition 80).

Definition 103. Given a set X and a map $\psi\colon X \to M$ (in Set). Write p_ψ for the arrow $X \to X + X$ in $\mathscr{K}\ell(\mathcal{D}_M)$ given by

$$p_\psi(x)(y) = \begin{cases} \psi(x) & y = \kappa_1 x \\ \psi(x)^\perp & y = \kappa_2 x \\ 0 & \text{otherwise.}\end{cases}$$

Proposition 104. *Given a set X and an arrow $p\colon X \to X + X$ in $\mathscr{K}\ell(\mathcal{D}_M)$. Then: p is an internal predicate if and only if $p = p_\psi$ for some $\psi\colon X \to M$.*

Proof. Given an internal predicate p. Then $[\mathrm{id}, \mathrm{id}] \circ p = \mathrm{id}$. Thus, for every $x \in X$ we have $p(x)(\kappa_1 x) \ovee p(x)(\kappa_2 x) = 1$ and $p(x)(y) = 0$ if $y \notin \{\kappa_1 x, \kappa_2 x\}$. Define $\psi(x) = p(x)(\kappa_1 x)$. Then $p = p_\psi$.

Conversely, suppose $\psi\colon X \to M$. Given $x \in X$. Then:

$$\begin{aligned}[\mathrm{id}, \mathrm{id}] \circ p_\psi(x)(y) &= p_\psi(x)(\kappa_1 y) \ovee p_\psi(x)(\kappa_2 y). \\ &= \begin{cases} 0 \ovee 0 = 0 & y \neq x \\ \psi(x) \ovee \psi(x)^\perp = 1 & y = x \end{cases}\end{aligned}$$

and thus $[\mathrm{id}, \mathrm{id}] \circ p_\psi = \mathrm{id}$. □

Write M^X for the maps from X to M in Set. This is an M-effect module with pointwise operations. See Example 53. It is also an effect monoid with pointwise multiplication. Its effect algebra disjoint sum is compatible with the disjoint sum on $\mathrm{iPred}(X)$:

Proposition 105. *Given $\psi, \chi \in M^X$.*

1. *$p_\psi \perp p_\chi$ if and only if $\psi \perp \chi$;*
2. *$p_{\psi \ovee \chi} = p_\psi \ovee p_\chi$;*
3. *$p_\psi^\perp = p_{\psi^\perp}$* and
4. *$1 = p_1$.*

Proof. 3 and 4 follow directly from the definitions. Given $\psi, \chi \in M^X$.

1. $p_\psi \perp p_\chi$ if and only if there is a bound b for p_ψ and p_χ. Obvserve this is the case if and only if for every $x \in X$:

   $$b(x)(y) = \begin{cases} \psi(x) & y = \kappa_1 x \\ \chi(x) & y = \kappa_2 x \\ (\psi(x) \ovee \chi(x))^\perp & y = \kappa_3 x \\ 0 & \text{otherwise} \end{cases}$$

   Such a b is unique if it exists and it exists if and only if $\psi \perp \chi$.

2. Let b be the bound of ψ and χ. Then:

   $$\begin{aligned} p_\psi \ovee p_\chi(x)(y) &= [\kappa_1, \kappa_1, \kappa_2] \circ b(x)(y) \\ &= \begin{cases} \psi(x) \ovee \chi(x) & y = \kappa_1 x \\ (\psi(x) \ovee \chi(x))^\perp & y = \kappa_2 x. \end{cases} \\ &= p_{\psi \ovee \chi}(x)(y). \end{aligned}$$ □

Definition 106. Set $\mathscr{D} = \mathscr{C} = \mathscr{K}\ell(\mathcal{D}_M)$. Let $\mathrm{Pred}\colon \mathscr{K}\ell(\mathcal{D}_M) \to \mathsf{EMod}_M^{\mathrm{op}}$ map

$$X \mapsto M^X \cong \mathrm{iPred}(X) \qquad f \mapsto (f)^* \qquad (f)^*(\psi)(x) = \bigovee_y f(x)(y) \odot \psi(y).$$

Set $\mathrm{char}_\psi = p_\psi$.

Proposition 107. Pred: $\mathscr{K}\ell(\mathcal{D}_M) \to \mathsf{EMod}_M^{\mathrm{op}}$ *is a full and internal effect logic. Furthermore:* $\langle \psi? \rangle (\chi) = \psi \odot \chi$.

Proof. For the sake of presentation, we prematurely stated Pred is a functor. We need to convince ourselves $(f)^*$ is an effect module homomorphism. The argument to show this, is the same as in Example 79. Now, for the axioms of an effect logic:

(EL1) Shown in Proposition 104.

(EL2) Shown in Proposition 105.

(EL3) We first observe that for a coprojection $\kappa_1 \colon X \to X + Y$, we have

$$(\kappa_1)^*\psi = \psi \restriction X \qquad \textstyle\coprod_{\kappa_1} \psi = \psi + 0 \qquad \prod_{\kappa_1} \psi = \psi + 1.$$

and thus $\coprod_{\kappa_1} \dashv (\kappa_1)^* \dashv \prod_{\kappa_1}$.

(EL4) Given a coprojection $\kappa_1 \colon X \to X + Y$ in $\mathscr{K}\ell(\mathcal{D}_M)$. Then $\hat{\kappa_1} = \eta \circ \kappa_1$. In Set, we know κ_1 is a split monomorphism and thus, by Proposition 83, κ_1 in $\mathscr{K}\ell(\mathcal{D}_M)$ is a (split) monomorphism.

(EL5) Shown in Proposition 105.

Before we prove (EL6), we note:

$$\begin{aligned}
\langle \psi? \rangle (\chi)(x) &= (\mathrm{char}_\psi)^* \textstyle\coprod_{\kappa_1} \chi(x) \\
&= (p_\psi)^*(\chi + 0)(x) \\
&= \bigovee_y p_\psi(x)(y) \odot (\chi + 0)(y) \\
&= (\psi(x) \odot \chi(x)) \ovee (\psi(x)^\perp \odot 0) \\
&= (\psi \odot \chi)(x).
\end{aligned}$$

Thus $\langle \psi? \rangle (\chi) = \psi \odot \chi$. In particular:

(EL6) $(\mathrm{char}_\psi)^* \coprod_{\kappa_1} 1 = \langle \psi? \rangle (1) = \psi \odot 1 = \psi$.

Finally, note that Pred is full and internal by definition. □

4.3.3 Hilb

The third example is a quantum mechanical case. Let Hilb denote the category of Hilbert spaces with (bounded linear) operators as arrows.

Definition 108. Given a Hilbert space $\mathscr{H}$ and an operator $A \colon \mathscr{H} \to \mathscr{H}$. Write p_A for the operator $\mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ in Hilb given by

$$p_A = \begin{pmatrix} A \\ I - A \end{pmatrix}.$$

Proposition 109. *Given a Hilbert space* $\mathscr{H}$ *and an operator* $p \colon \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ *in* Hilb. *Then:* p *is an internal predicate on* $\mathscr{H}$ *if and only if* $p = p_A$ *for some* $A \colon \mathscr{H} \to \mathscr{H}$.

Proof. An operator $B = \binom{B_1}{B_2}$ is an internal predicate if and only if $[\mathrm{id}, \mathrm{id}] \circ B = \mathrm{id}$. That is: if and only if $B_1 + B_2 = I$. Thus clearly, p_A is an internal predicate for any $A \colon \mathscr{H} \to \mathscr{H}$. And: given an internal predicate $p = \binom{p_1}{p_2}$. Then $p = p_{p_1}$. □

The internal predicates on a Hilbert space do *not* carry a compatible effect algebra structure: $\begin{pmatrix} I \\ A \\ -A \end{pmatrix}$ is a bound for $1 = \kappa_1$ and p_A. Thus $A \perp 1$, but not in general $A = 0$. Hence effect algebra axiom (E4) fails.

Definition 110. An internal predicate $p = \binom{A}{I-A}$ is called **positive** if both A and $I - A$ are positive. Let $\mathrm{pPred}(\mathscr{H})$ denote the set of positive predicates on $\mathscr{H}$.

Proposition 111. *A map $p \colon \mathscr{H} \to \mathscr{H} \oplus \mathscr{H}$ is a positive internal predicate if and only if $p = p_A$ for some $0 \leq A \leq I$.*

Proof. $0 \leq I - A$ hence $I = I - 0 \geq I - (I - A) = A \geq 0$. □

As we already saw in Example 87, the operators A with $0 \leq A \leq I$ form a $[0,1]$-effect module called $\mathrm{Eff}(\mathscr{H})$. Note that $\mathrm{pPred}(\mathscr{H})$ is not an effect algebra with the operations of Definition 94: the map $\begin{pmatrix} I \\ I \\ -I \end{pmatrix}$ is a bound for $1 \perp 1$. This is not a problem for axiom (EL2) as we will see lateron. For now, we will regard $\mathrm{pPred}(\mathscr{H})$ as a $[0,1]$-effect module with the operations inherited from $\mathrm{Eff}(\mathscr{H})$.

Definition 112. Set $\mathscr{D} = \mathsf{Hilb}_{\mathrm{isom}}$, the category of Hilbert spaces with isometries. Let $\mathscr{C} = \mathsf{Hilb}$. Now, define $\mathrm{Pred} \colon \mathsf{Hilb}_{\mathrm{isom}} \to \mathsf{EMod}_{[0,1]}^{\mathrm{op}}$ by

$$\mathscr{H} \mapsto \mathrm{Eff}(\mathscr{H}) \cong \mathrm{pPred}(\mathscr{H}) \qquad f \mapsto (f)^* \qquad (f)^* A = f^\dagger A f.$$

Let $\mathrm{char}_A = \binom{\sqrt{A}}{\sqrt{I-A}}$.

Proposition 113. $\mathrm{Pred} \colon \mathsf{Hilb}_{isom} \to \mathsf{EMod}_{[0,1]}^{\mathrm{op}}$ *is an effect logic. Furthermore:* $\langle A? \rangle (B) = \sqrt{A} B \sqrt{A}$.

Proof. In Example 87, we have shown $(f)^*$ is an effect module homomorphism and thus, indeed: Pred is a functor of the promised type.

(EL1) Shown in Proposition 111.

(EL2) (a) Given $\mathscr{H}$ in $\mathscr{C}$. Given $A, B \in \mathrm{Eff}(\mathscr{H})$.
Suppose $A \perp B$. Then $b = \begin{pmatrix} A \\ B \\ I-A-B \end{pmatrix}$ is a bound for A and B *and* $[\kappa_1, \kappa_1, \kappa_2] \circ b = p_{A \ovee B} = p_A \ovee p_B \in \mathrm{Pred}(\mathscr{H})$.
Conversely, suppose there is a bound b for A and B. Then $\binom{A+B}{I-A-B} = [\kappa_1, \kappa_1, \kappa_2] \circ b \in \mathrm{Pred}(\mathscr{H})$. Hence $A \perp B$.

(b) Given $\mathscr{H}$ in $\mathscr{C}$ and $A \in \mathrm{Eff}(\mathscr{H})$. Then $[\kappa_2, \kappa_1] \circ p_A = \binom{I-A}{A} = p_{A^\perp}$.

(c) $p_1 = \binom{I}{0} = \kappa_1$.

(EL3) First, we observe that

$$\coprod_{\kappa_1} A = \begin{pmatrix} A & 0 \\ 0 & 0 \\ I-A & 0 \\ 0 & I \end{pmatrix} = p_{\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}} \qquad \prod_{\kappa_1} A = \begin{pmatrix} A & 0 \\ 0 & I \\ I-A & 0 \\ 0 & 0 \end{pmatrix} = p_{\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}.$$

Thus $\coprod_{\kappa_1}$ and $\prod_{\kappa_1}$ are the same as in Example 87, where we saw they are the left and right adjoint of $(\kappa_1)^*$.

(EL4) Coprojections are injective, hence monic.

(EL5) First $\text{char}_1 = \begin{pmatrix} \sqrt{I} \\ \sqrt{0} \end{pmatrix} = \begin{pmatrix} I \\ 0 \end{pmatrix} = \kappa_1$ and secondly $\text{char}_0 = \begin{pmatrix} \sqrt{0} \\ \sqrt{I} \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} = \kappa_2$.

(EL6) As $\coprod_{\kappa_1}$ is the same as in Example 87, we also may use its result: $\langle A? \rangle (B) = \sqrt{A}B\sqrt{B}$. Hence $(\text{char}_p)^* \coprod_{\kappa_1} 1 = \langle p? \rangle (1) = \sqrt{p}I\sqrt{p} = p$, as desired. □

4.4 Representation theorems

Theorem 114. *Any left-additive weak sequential effect module is represented in a full effect logic.*

Proof. We will extend the category and functor defined in the proof of Theorem 93 such that it is becomes an effect logic.

1. The objects of $\mathscr{C}$ are $\mathbb{N}$. We want a full effect logic, thus $\mathscr{D} = \mathscr{C}$. Again, $\text{Pred}(n) = E^n$.

   The arrows of $\mathscr{C}$ are given syntactically. We will specify which arrows exist and which are considered equal. From the original construction:

   (a) For each $n \in \mathbb{N}$ and $p \in E^n$, there is an arrow $\text{char}_p \colon n \to 2n$.
   (b) For each $n, m \in \mathbb{N}$ there are $\kappa_1 \colon n \to n + m$ and $\kappa_2 \colon m \to n + m$.
   (c) Given arrows $f \colon n \to l$ and $g \colon m \to l$, there is an arrow $[f, g] \colon n + m \to l$.
   (d) For every n, there is an arrow $\text{id} \colon n \to n$.
   (e) Given arrows $f \colon n \to m$ and $g \colon m \to l$, there is an arrow $g \circ f \colon n \to l$.

   We add the following arrows.

   (f) For each $n \in \mathbb{N}$ and $p, q \in E^n$ with $p \perp q$, there is an arrow $b_{p,q} \colon n \to 3n$.

   The equality is given by the following rules. From the original construction:

   (a) For any $n \in N$, if $\kappa_1 \colon n \to n+n$, then $\kappa_1 = \text{char}_1$ and if $\kappa_2 \colon n \to n+n$, then $\kappa_2 = \text{char}_0$.
   (b) $[f, g] \circ \kappa_1 = f$ and $[f, g] \circ \kappa_2 = g$
   (c) $[h \circ \kappa_1, h \circ \kappa_2] = h$
   (d) $f \circ \text{id} = \text{id} \circ f = f$
   (e) $(f \circ g) \circ h = f \circ (g \circ h)$
   (f) If $f = f'$ and $g = g'$ then $[f, g] = [f', g']$ and $f \circ g = f' \circ g'$.

And additionally:

(g) i. $[[\kappa_1, \kappa_1], \kappa_2] \circ b_{p,q} = \text{char}_{p \ovee q}$
   ii. $[[\kappa_1, \kappa_2], \kappa_2] \circ b_{p,q} = \text{char}_p$
   iii. $[[\kappa_2, \kappa_1], \kappa_2] \circ b_{p,q} = \text{char}_q$

(h) For each $n \in \mathbb{N}$ and any $b\colon n \to 3n$, if
   i. $[[\kappa_1, \kappa_2], \kappa_2] \circ b = \text{char}_p$ and
   ii. $[[\kappa_2, \kappa_1], \kappa_2] \circ b = \text{char}_q$,

   then $b = b_{p,q}$.

(i) $[\text{id}, \text{id}] \circ \text{char}_p = \text{id}$

(j) $\text{char}_{(p,q)} = [(\kappa_1 + \kappa_1) \circ \text{char}_p, (\kappa_2 + \kappa_2) \circ \text{char}_q]$ for $(p, q) \in E^{n+m}$.

(k) i. If $\kappa_1 \circ f = \kappa_1 \circ g$, then $f = g$.
   ii. If $\kappa_2 \circ f = \kappa_2 \circ g$, then $f = g$.

(l) $\text{char}_{p^\perp} = [\kappa_2, \kappa_1] \circ \text{char}_p$

2. Again, we want Pred to act on the arrows of the original construction as follows.

   (a) $(\text{char}_p)^*(q_1, q_2) = (p * q_1) \ovee (p^\perp * q_2)$
   (b) $(\kappa_1)^* = \pi_1$ and $(\kappa_2)^* = \pi_2$
   (c) $([f, g])^* = \langle (f)^*, (g)^* \rangle$
   (d) $(\text{id})^* = \text{id}$
   (e) $(f \circ g)^* = (g)^* \circ (f)^*$

   And for the new arrows:

   (f) $(b_{p,q})^*(q_1, q_2, q_3) = (p * q_1) \ovee (q * q_2) \ovee ((p \ovee q)^\perp * q_3)$

   Before we call this the inductive definition of Pred, we once again check that it respects the forced equalities and determines effect module homomorphisms. The latter first. The cases (a)–(e) are the same as in Theorem 93.

   (f) First we check whether the expression for $(b_{p,q})^*$ is defined. Given p and q with $p \perp q$. Then $p * q_1 \leq p * 1 = p$ and $q * q_2 \leq q^\perp * 1 = q$. Thus: $(p * q_1) \ovee (q * q_2) \leq p \ovee q$. Also $(p \ovee q)^\perp * q_3 \leq (p \ovee q)^\perp$ and hence $(p * q_1) \ovee (q * q_2) \ovee ((p \ovee q)^\perp * q_3)$ is defined.
   Note that $(b_{p,q})^*(1, 1, 1) = (p * 1) \ovee (q * 1) \ovee ((p \ovee q)^\perp * 1) = p \ovee q \ovee (p \ovee q)^\perp = 1$, thus $(b_{p,q})^*$ preserves the unit. Linearity follows from the linearity of $\ovee$ and right-linearity of $*$, just like in the proof of linearity of $(\text{char}_p)^*$.

   Now, we check the preservation of equality. The cases (a)-(f) are the same as in Theorem 93.

   (g) i.
$$\begin{aligned}
[[\kappa_1, \kappa_1], \kappa_2] \circ (b_{p,q})^*(q_1, q_2) &= (b_{p,q})^* \circ \langle \langle \pi_1, \pi_1 \rangle, \pi_2 \rangle (q_1, q_2) \\
&= (b_{p,q})^*(q_1, q_1, q_2) \\
&= (p * q_1) \ovee (q * q_1) \ovee ((p \ovee q)^\perp * q_2) \\
&= ((p \ovee q) * q_1) \ovee ((p \ovee q)^\perp * q_2) \\
&= (\text{char}_{p \ovee q})^*(q_1, q_2)
\end{aligned}$$

ii. $[[\kappa_1, \kappa_2], \kappa_2] \circ (b_{p,q})^*(q_1, q_2) = (b_{p,q})^* \circ \langle\langle \pi_1, \pi_2\rangle, \pi_2\rangle (q_1, q_2)$
$= (b_{p,q})^*(q_1, q_2, q_2)$
$= (p * q_1) \otimes (q * q_2) \otimes ((p \otimes q)^{\perp} * q_2)$
$= (p * q_1) \otimes (q \otimes (p \otimes q)^{\perp} * q_2)$
$= (p * q_1) \otimes (p^{\perp} * q_2)$
$= (\mathrm{char}_p)^*(q_1, q_2)$

iii. $[[\kappa_2, \kappa_1], \kappa_2] \circ (b_{p,q})^*(q_1, q_2) = (b_{p,q})^* \circ \langle\langle \pi_1, \pi_2\rangle, \pi_2\rangle (q_1, q_2)$
$= (b_{p,q})^*(q_2, q_1, q_2)$
$= (p * q_2) \otimes (q * q_1) \otimes ((p \otimes q)^{\perp} * q_2)$
$= (q * q_1) \otimes (p \otimes (p \otimes q)^{\perp} * q_2)$
$= (q * q_1) \otimes (q^{\perp} * q_2)$
$= (\mathrm{char}_q)^*(q_1, q_2)$

(h) Suppose $b\colon n \to 3n$; $[[\kappa_1, \kappa_2], \kappa_2] \circ b = \mathrm{char}_p$ and $[[\kappa_2, \kappa_1], \kappa_2] \circ b = \mathrm{char}_q$. Proving inductively, we may assume $([[\kappa_1, \kappa_2], \kappa_2] \circ b)^* = (\mathrm{char}_p)^*$ and $([[\kappa_2, \kappa_1], \kappa_2] \circ b)^* = (\mathrm{char}_q)^*$. That is:

$$\begin{aligned}(b)^*(q_1, q_2, q_2) &= (p * q_1) \otimes (p^{\perp} * q_2)\\ (b)^*(q_2, q_1, q_2) &= (q * q_1) \otimes (q^{\perp} * q_2)\end{aligned}$$

and thus

$$\begin{aligned}(b)^*(x, 0, 0) &= p * x\\ (b)^*(0, x, 0) &= q * x\\ (b)^*(x, x, 0) &= (b)^*(x, 0) \otimes (b)^*(0, x)\\ &= (p * x) \otimes (q * x)\\ (b)^*(0, x, x) &= p^{\perp} * x\\ (b)^*(0, 0, x) &= (b)^*(0, x, x) \ominus (b)^*(0, x, 0)\\ &= (p^{\perp} * x) \ominus (q * x)\\ &= (p^{\perp} \ominus q) * x\\ &= (p \otimes q)^{\perp} * x\end{aligned}$$

hence

$$\begin{aligned}(b)^*(1, 1, 0) &= (p * 1) \otimes (q * 1)\\ &= p \otimes q\end{aligned}$$

consequently $p \perp q$ and thus $b_{p,q}$ exists and

$$\begin{aligned}(b)^*(q_1, q_2, q_3) &= (b)^*(q_1, 0, 0) \otimes (b)^*(0, q_2, 0) \otimes (b)^*(0, 0, q_3)\\ &= (p * q_1) \otimes (q * q_2) \otimes ((p \otimes q)^{\perp}) * q_3\\ &= (b_{p,q})^*(q_1, q_2, q_3).\end{aligned}$$

(i) $([\mathrm{id},\mathrm{id}] \circ \mathrm{char}_p)^*(q) = (\mathrm{char}_p)^* \circ \langle \mathrm{id},\mathrm{id}\rangle (q)$

$$\begin{aligned} &= (\mathrm{char}_p)^*(q,q) \\ &= (p * q) \otimes (p^\perp * q) \\ &= 1 * q \\ &= q \\ &= (\mathrm{id})^*(q) \end{aligned}$$

(j) $([(\kappa_1+\kappa_1)\circ \mathrm{char}_p, (\kappa_2+\kappa_2)\circ \mathrm{char}_q])^*((q_1,q_1'),(q_2,q_2'))$

$$\begin{aligned} &= \langle (\mathrm{char}_p)^* \circ (\pi_1 \times \pi_1), (\mathrm{char}_q)^* \circ (\pi_2 \times \pi_2)\rangle ((q_1,q_1'),(q_2,q_2')) \\ &= ((\mathrm{char}_p)^*(q_1,q_2), (\mathrm{char}_q)^*(q_1',q_2')) \\ &= ((p * q_1) \otimes (p^\perp * q_2), (q * q_1') \otimes (q^\perp * q_2')) \\ &= ((p,q) * (q_1,q_1')) \otimes ((p,q)^\perp * (q_2,q_2')) \\ &= (\mathrm{char}_{(p,q)})^*((q_1,q_1'),(q_2,q_2')) \end{aligned}$$

(k) Suppose $\kappa_1 \circ f = \kappa_1 \circ g$. Reasoning inductively, we may assume $(f)^* \circ \pi_1 = (g)^* \circ \pi_1$. But then $(f)^* = (g)^*$, since projections are epimorphisms in EMod_M. The argument for the case with κ_2 is the same.

(l) $(\mathrm{char}_{p^\perp})^*(q_1,q_2) = (p^\perp * q_1) \otimes (p^{\perp\perp} * q_2)$

$$\begin{aligned} &= (p^\perp * q_1) \otimes (p * q_2) \\ &= (\mathrm{char}_p)^*(q_2,q_1) \\ &= (\mathrm{char}_p)^* \circ \langle \pi_2,\pi_1\rangle (q_1,q_2) \\ &= ([\kappa_2,\kappa_1] \circ \mathrm{char}_p)^*(q_1,q_2) \end{aligned}$$

3. With the same argument as in the proof of Theorem 93, we see $\mathscr{C}$ has coproducts.

4. Now we check the effect logic axioms. Define

$$\textstyle\coprod_{\kappa_1} p = (p,0) \text{ and } \prod_{\kappa_1} p = (p,1).$$

We identify an element $p \in E^n$ with the map $\mathrm{char}_p \colon n \to 2n$.

(EL1) For any $\mathrm{char}_p \in \mathrm{Pred}(n)$, we have $[\mathrm{id},\mathrm{id}]\circ\mathrm{char}_p = \mathrm{id}$ by construction.

(EL2) i. For any $\mathrm{char}_p, \mathrm{char}_q \in \mathrm{Pred}(n)$. If $p \perp q$, then by construction, $b_{p,q}$ is the appropriate bound.
Conversely, suppose there is a map b such that $[[\kappa_1,\kappa_2],\kappa_2]\circ b = p$ and $[[\kappa_2,\kappa_1],\kappa_2] \circ b = q$. Then, by construction $b = b_{p,q}$ and $p \perp q$.

ii. By construction $[\kappa_2,\kappa_1] \circ p = [\kappa_2,\kappa_1] \circ \mathrm{char}_p = \mathrm{char}_{p^\perp} = p^\perp$.

iii. By construction $1 = \mathrm{char}_1 = \kappa_1$.

(EL3) Clearly $\coprod_{\kappa_1} \dashv (\kappa_1)^* = \pi_1 \dashv \prod_{\kappa_1}$. Furthermore:

$$\begin{aligned} \textstyle\coprod_{\kappa_1} p &= (p,0) \\ &= \mathrm{char}_{(p,0)} \\ &= [(\kappa_1+\kappa_1)\circ \mathrm{char}_p, (\kappa_2+\kappa_2)\circ \mathrm{char}_0] \\ &= [(\kappa_1+\kappa_1)\circ \mathrm{char}_p, (\kappa_2+\kappa_2)\circ \kappa_2] \\ &= [(\kappa_1+\kappa_1)\circ \mathrm{char}_p, \kappa_2\circ \kappa_2] \\ &= [(\kappa_1+\kappa_1)\circ p, \kappa_2\circ \kappa_2]. \end{aligned}$$

and similarly $\coprod_{\kappa_1} p = [(\kappa_1 + \kappa_1) \circ p, \kappa_1 \circ \kappa_2]$.

(EL4) By construction, coprojections are monic.

(EL5) By construction, $\mathrm{char}_1 = \kappa_1$ and $\mathrm{char}_0 = \kappa_2$.

(EL6)
$$\begin{aligned}(\mathrm{char}_p)^* \textstyle\coprod_{\kappa_1} 1 &= (\mathrm{char}_p)^*(1,0)\\ &= (p * 1) \ovee (p^\perp * 0)\\ &= p\end{aligned}$$

5. Finally, note $\mathrm{Pred}(1) = E$ and for $p, q \in E$:

$$\langle p?\rangle (q) = (\mathrm{char}_p)^* \textstyle\coprod_{\kappa_1} q = (\mathrm{char}_p)^*(q,0) = (p * q) \ovee (p^\perp * 0) = p * q. \quad \square$$

Remark 115. One might hope the effect logic constructed in the previous Theorem is internal. This is, in general, not the case. Suppose it is internal. Note that $[\mathrm{char}_b, \kappa_2] \circ \mathrm{char}_a$ is always an internal predicate:

$$\begin{aligned}[\mathrm{id}, \mathrm{id}] \circ [\mathrm{char}_b, \kappa_2] \circ \mathrm{char}_a &= [[\mathrm{id}, \mathrm{id}] \circ \mathrm{char}_b, [\mathrm{id}, \mathrm{id}] \circ \kappa_2] \circ \mathrm{char}_a\\ &= [\mathrm{id}, \mathrm{id}] \circ \mathrm{char}_a\\ &= \mathrm{id}.\end{aligned}$$

Because our effect logic is internal, there is a $z \in E$ such that

$$[\mathrm{char}_b, \kappa_2] \circ \mathrm{char}_a = \mathrm{char}_z$$

and consequently

$$\begin{aligned}(z * c) \ovee (z^\perp * d) &= (\mathrm{char}_z)^*(c, d)\\ &= (\mathrm{char}_a)^* \circ \langle (\mathrm{char}_b)^*, \pi_2\rangle (c, d)\\ &= (\mathrm{char}_a)^*((b * c) \ovee (b^\perp * d), d)\\ &= (a * (b * c)) \ovee (a * (b^\perp * d)) \ovee (a^\perp * d).\end{aligned}$$

And thus, if we set $c = 1$ and $d = 0$, we see: $z = a * b$. Hence, with $d = 0$:

$$(a * b) * c = a * (b * c).$$

Our left-additive weak sequential effect module E must be associative. This is not always the case.

Theorem 116. *Any effect monoid is represented in a full internal effect logic.*

Proof. Given any effect monoid M. By Proposition 107, there is a full and internal effect logic $\mathrm{Pred}\colon \mathscr{K}\ell(\mathcal{D}_M) \to \mathsf{EMod}_M^{\mathrm{op}}$ such that

- $\mathrm{Pred}(X) = M^X$ the set of maps from X to M, which is an M-effect module and effect monoid with pointwise operations.
- For any X and $p, q \in \mathrm{Pred}(X)$, we have $\langle p?\rangle (q) = p \odot q$.

Thus in particular M is represented in this effect logic as the image of the one-element-set. $\square$

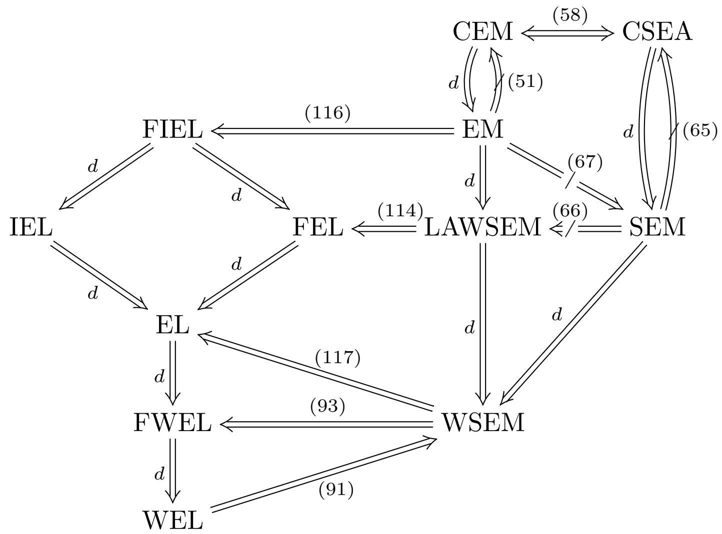
Theorem 117. *Any weak sequential effect module is represented in an effect logic.*

Proof. Recall Theorem 114: any left-additive weak sequential effect module is represented in a full effect logic. In the proof of that theorem, we required the left-additivity of the weak effect module only to show the preservation of equalities by Pred related to $b_{p,q}$. If we do not require the resulting effect logic to be full, we can leave the arrows $b_{p,q}$ out of $\mathscr{D}$ and do not need the left-additivity to define the functor. □

5 Conclusions

5.1 Summary

We can summarize our main results as follows. Given an effect algebra E with a binary operation $*$. Then we have the following (non)implications.



Abbr.	Property	See
CEM	Commutative Effect Monoid	39
CSEA	Commutative Sequential Effect Algebra	57
FIEL	Represented in Full Internal Effect Logic	92
EM	Effect Monoid	36
IEL	Represented in Internal Effect Logic	96 and 92
FEL	Represented in Full Effect Logic	96 and 92
LAWSEM	Left-Additive Weak Sequential Effect Module	90
SEM	Sequential Effect Module	56
EL	Represented in Effect Logic	96 and 92
FWEL	Represented in Full Weak Effect Logic	84 and 92
WSEM	Weak Sequential Effect Module	90
WEL	Represented in Weak Effect Logic	84 and 92

Recall the starting point of this thesis: are there categorical axioms, which the examples Set, $\mathscr{K}\ell(\mathcal{D})$ and $\mathsf{Hilb}_{\mathrm{isom}}$ obey, such that the *andthen* forms a sequential effect algebra. To this question, we have not found an answer. Our candidate axioms all fall short. The three strongest sets of axioms, the full and/or internal effect logics, are not obeyed by $\mathsf{Hilb}_{\mathrm{isom}}$. The other candidates, the (full) (weak) effect logics imply only weak properties of *andthen*.

The proof (known to the author) that *andthen* in $\mathsf{Hilb}_{\mathrm{isom}}$ is a sequential effect algebra (Theorem 59) requires non-elementary functional analysis. It would be surprising if relatively simple axioms, such as those investigated, would entail this same result.

Even though we were not able to decide our initial question, we have found some results along the way. For instance:

- A convex effect monoid is an interval of an ordered vector space with a certain kind of product (Theorem 46) and vice versa (Proposition 44).
- Finite effect monoids are simple, Proposition 40.

Furthermore, we can present a future candidate for axioms of effect logics with two tests: does the pathalogical model $\mathscr{H}\ell(\mathcal{D}_M)$ obey them and does the syntactic construction used in the proof of Theorems 93, 116 and 117 apply?

5.2 Further investigation

We will discuss some possible future investigation. First, the following strengthenings of the effect logic axioms can be considered.

- In the axioms of an effect logic, we require the effect algebra operations to correspond to the natural operations on the internal predicates, see axiom (EL2). One can also define the scalars and scalar multiplication on predicates internally. See Definition 6 of [9]. Would it make a difference, to require the scalar multiplication to correspond to the internal scalar multiplication?
- Some of the predicate functors are full and faithful, such as $\mathsf{CStar}_{\mathrm{PU}} \to \mathsf{EMod}^{\mathrm{op}}_{[0,1]}$. The predicate functor $\mathsf{Set} \to \mathsf{EA}$, depending on the set theory[1], is not. However, one could investigate, full and faithfull predicate functors to $\mathsf{EMod}^{\mathrm{op}}_{\mathrm{dc}}$, the directed complete effect modules.

Also, the current syntactic construction might be improved.

- In the current construction we chose $\mathrm{Pred}(n) = E^n$. Can we prove more with a different choice? Could we represent any weak sequential effect module in a full and internal effect logic?
- In the current construction we ensured coprojections to be monic by forcing the required equality. It seems that without this forced equality, one can prove with a syntactic analysis of the arrows, that coprojections are monic. Such methods will also likely be helpful to prove that a syntactic model is internal.

5.3 Acknowledgments

With several people I have had pleasant and fruitful discussions. For this thesis, I am particularly grateful to Arnoud van Rooij for his help with functional analysis and ordered vector spaces; Robert Furber and Jorik Mandemaker for answering many silly questions; Bram Westerbaan for scrutinizing the early proofs and some cunning suggestions; Mathys Renella for some fresh ideas and my supervisor, Bart Jacobs.

[1] If there is a non principal ultrafilter $\mathscr{G}$ on $\mathcal{P}(\mathbb{N})$, then we have a counterexample:

$$\mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N} \cup \{\infty\}) \qquad U \mapsto \begin{cases} U & U \notin \mathscr{G} \\ U \cup \{\infty\} & U \in \mathscr{G} \end{cases}$$

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