Paschke dilations

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Stinespring dilation

$$\mathscr{A} \xrightarrow[\text{normal linear completely positive contractive}]{\varphi} B(\mathscr{H})$$

$\mathscr{A}$ von Neumann algebra, $\mathscr{H}$ Hilbert space

Stinespring dilation

$$\mathscr{A} \xrightarrow{\varphi \text{ process}} B(\mathscr{H})$$

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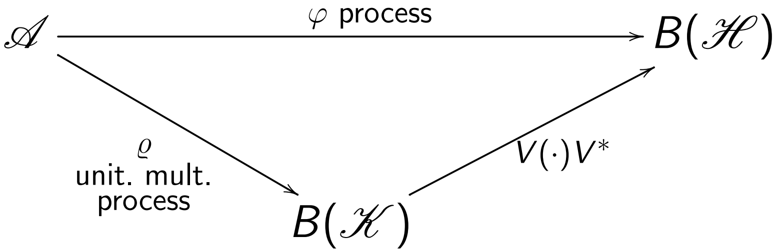
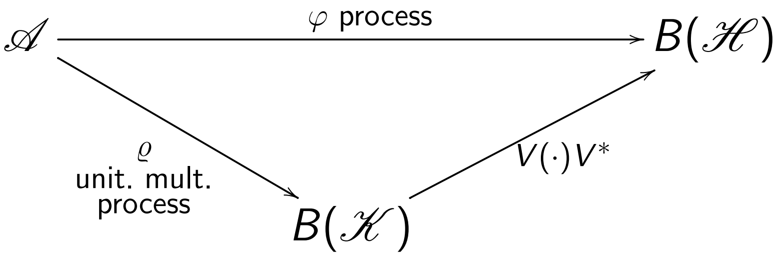


$\mathscr{K}$ Hilbert space, $V : \mathscr{H} \to \mathscr{K}$ bounded linear

Minimal Stinespring dilation



minimal $\equiv$ ($\mathrm{span}\,\varrho(\mathscr{A})V\mathscr{H}$ dense in $\mathscr{K}$)

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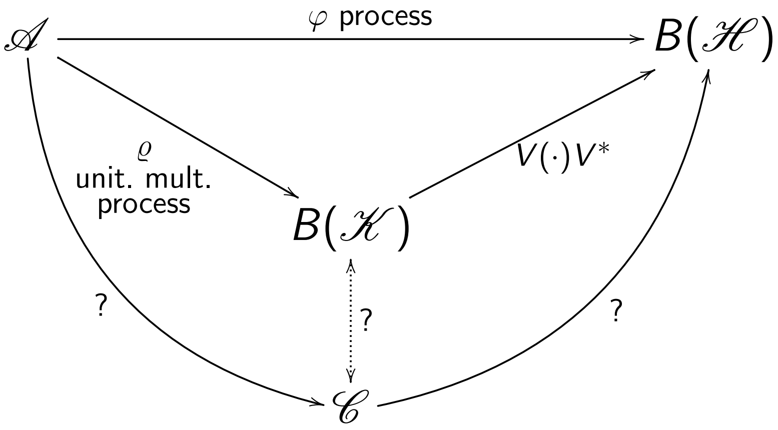

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5.3 **Corollary.** *Let A and B be as above, and $\phi: A \to B$ a completely positive map such that $\phi(1) = 1$. There is a B^*-algebra $\mathfrak{A}$ containing B, a projection $p \in \mathfrak{A}$ such that $B = p\mathfrak{A}p$, and a *-homomorphism $\pi: A \to \mathfrak{A}$ such that $\phi(a) = p\pi(a)p \;\; \forall a \in A$.*

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by WL Paschke - 1973 - Cited by 523 - Related articles

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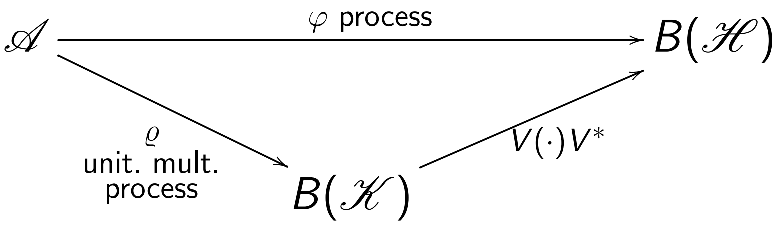
3. Thus (surprisingly): Paschke is a generalization of Stinespring.

Chris Heunen's contribution



$(\varrho, \mathscr{K}, V)$ minimal Stinespring

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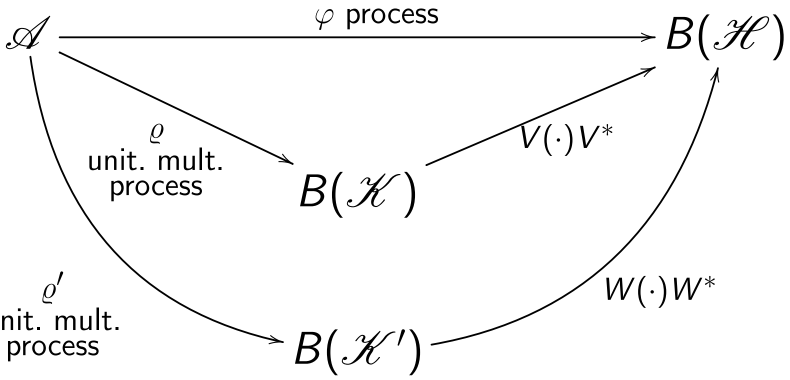

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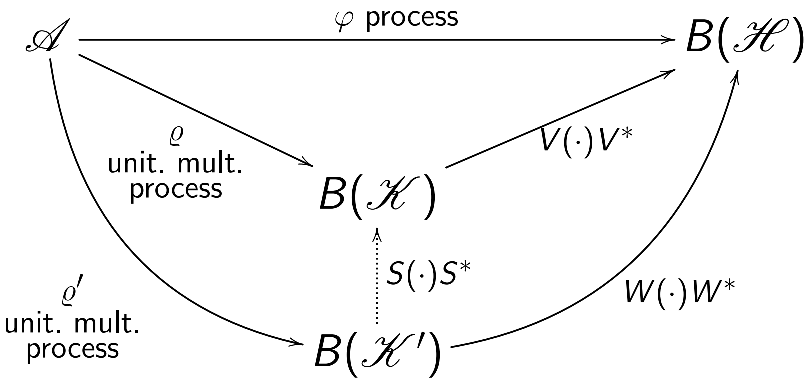

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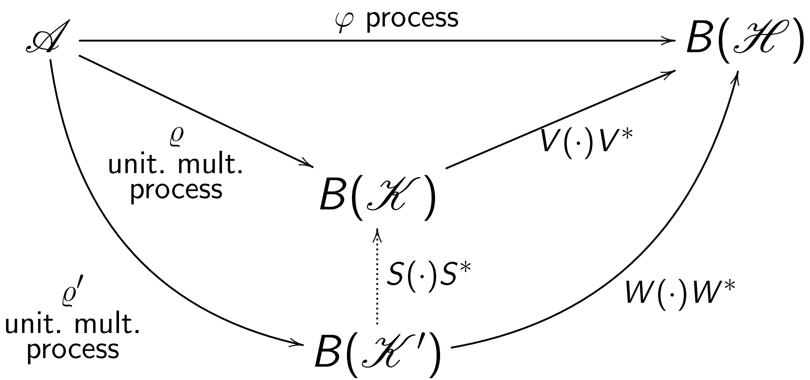

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(WW filled a gap in the proof.)

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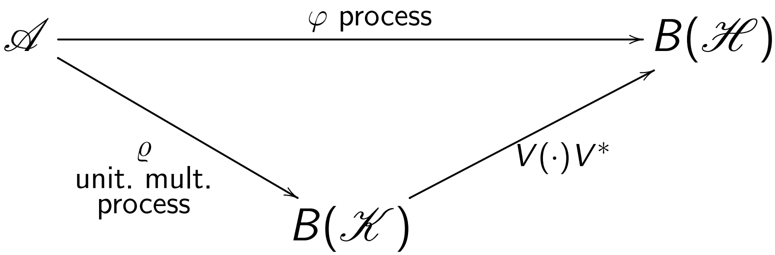

$(\varrho, \mathscr{K}, V)$ minimal Stinespring dilation of φ

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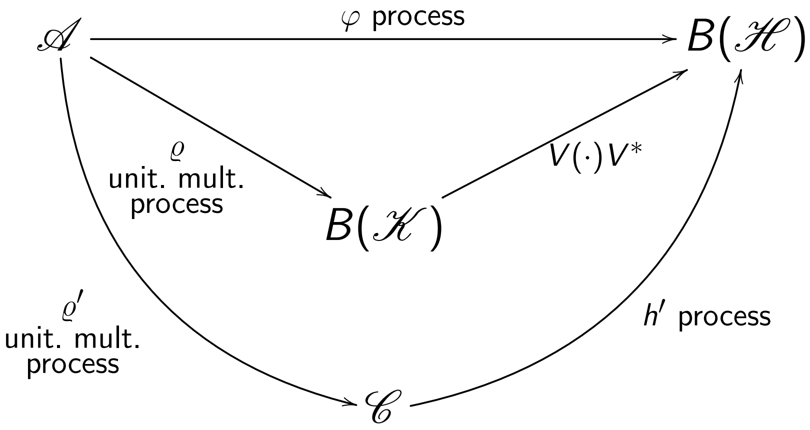

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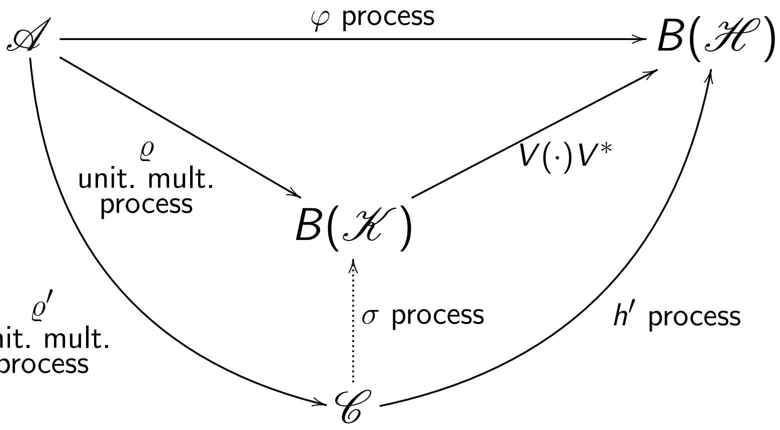

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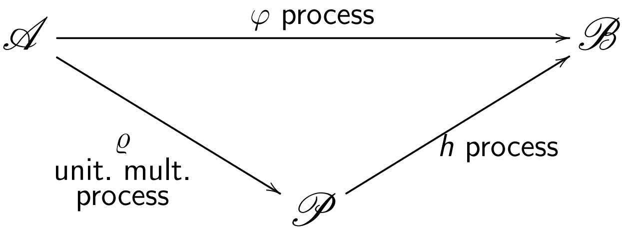

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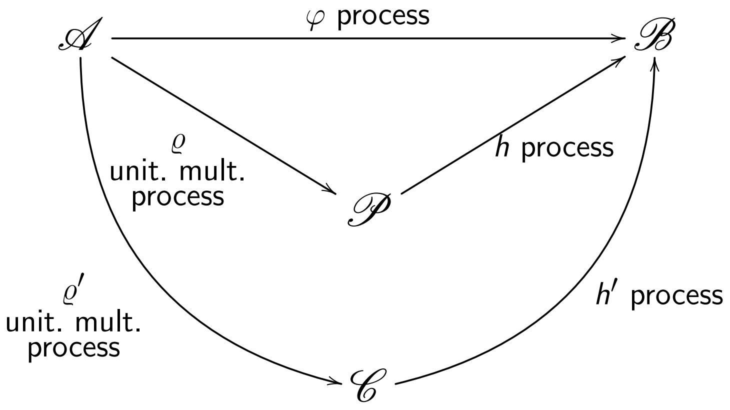

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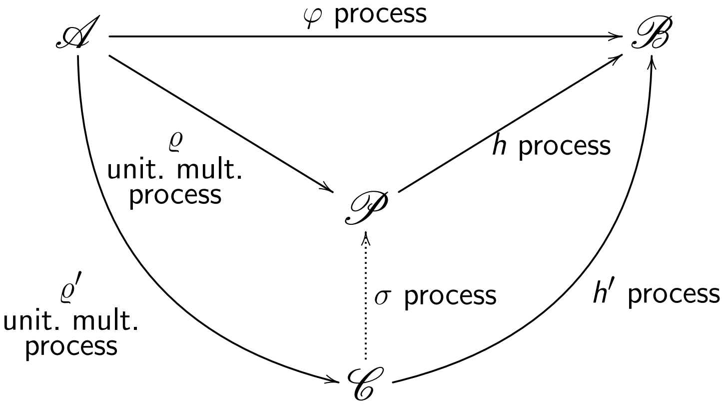

Remainder talk

1. Sketch construction $\mathscr{P}$
2. Examples of dilations
3. Pure maps
4. Future research

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On algebraic tensor $\mathscr{A} \odot \mathscr{B}$, define $[a \otimes b, \alpha \otimes \beta]_\varphi := b^*\varphi(a^*\alpha)\beta$.

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$\varrho(\alpha) a \otimes b = (\alpha a) \otimes b$ and

$h(T) = \langle T1 \otimes 1, 1 \otimes 1 \rangle_\varphi$

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(h corner if $h(x) = \vartheta(pxp)$ for some projection $p \in \mathscr{P}$ and isomorphism $\vartheta\colon p\mathscr{P}p \to \mathscr{B}$.)

Examples 2/3

- $\langle \varphi_1, \varphi_2 \rangle : \mathscr{A} \to \mathscr{B}_1 \oplus \mathscr{B}_2$ has P-dill.
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  with $\mathscr{A} \xrightarrow{\varrho_i} \mathscr{P}_i \xrightarrow{h_i} \mathscr{B}_i$ Paschke dilation of φ_i.
- Thus in the finite dimensional case, the Paschke dilation is componentwise minimal Stinespring.

Examples 3/3

- $\mathscr{A} \xrightarrow{C_p(\cdot)C_p} C_p\mathscr{A}C_p \xrightarrow{p(\cdot)p} p\mathscr{A}p$ is the Paschke dilation of the corner $h\colon \mathscr{A} \to p\mathscr{A}p, x \mapsto pxp$

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Clearly $\mathrm{Ad}_V\colon B(\mathscr{H}) \to B(\mathscr{K})$ should be pure with $\mathrm{Ad}_V^\dagger = \mathrm{Ad}_{V^*}$

Our proposal

$$\mathscr{A} \xrightarrow{\varphi \text{ process}} \mathscr{B}$$

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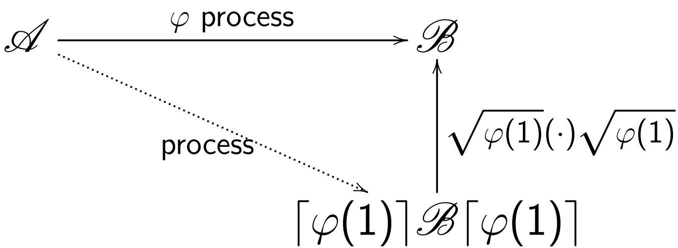

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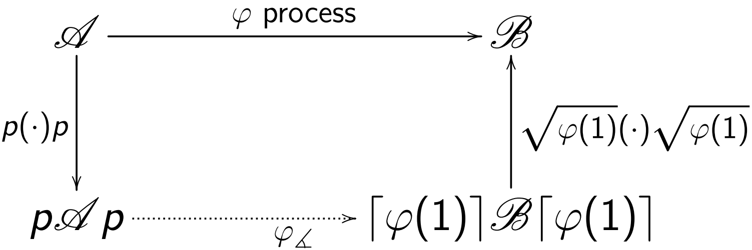
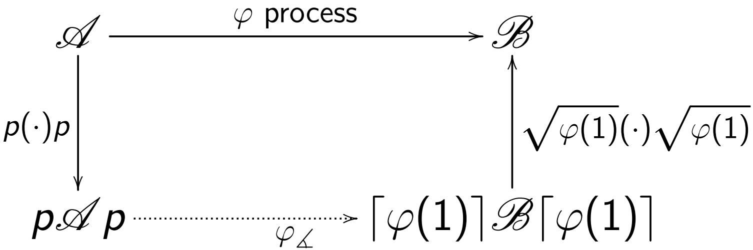


p carrier projection of φ

Our proposal



φ pure $:=$ $\varphi_{\measuredangle}$ isomorphism

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- h is pure
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- Pure processes are extreme among processes with the same value on 1
- (To be published: there is a unique* dagger on pure maps.)

Future work

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Questions?