Bankruptcy: Is It Enough to Forgive or Must We Also Forget?
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Online Appendix

Derivation of Beliefs — $p_0(s_0, q)$ and $p^S(p)$

We first discuss how equation (5), the probability that an agent with a credit history of zero length is the safe type, is derived.

Recall that in each period a unit mass of entrepreneurs is born; a fraction s_0 of them are of the safe type and $1 - s_0$ risky. In addition, there are entrepreneurs who were born in previous generations, but likewise have no credit history — these are risky agents who failed in all projects since they were born, and had each of these failures forgotten (and also did not die). The total mass of such entrepreneurs is $(1-s_0)(1-\pi_{e^r(p_0)})\delta q + (1-s_0)(1-\pi_{e^r(p_0)})^2\delta^2 q^2 + \cdots$, where recall that $\pi_{e^r(p_0)} = e^r(p_0)\pi_h + (1 - e^r(p_0))\pi_l$ denotes the probability of success with equilibrium effort strategy $e^r(p_0)$ when $p = p_0$. Thus, at any point in time, the total mass of risky entrepreneurs with no credit history is $(1 - s_0)\frac{1}{1-(1-\pi_{e^r(p_0)})\delta q}$, and so:

$$p_0(s_0, q) = \frac{s_0}{s_0 + (1 - s_0)\frac{1}{(1-(1-\pi_{e^r(p_0)})\delta q)}} = s_0\left[\frac{1 - (1 - \pi_{e^r(p_0)})\delta q}{1 - s_0(1 - \pi_{e^r(p_0)})\delta q}\right],$$

which is the expression in (5).

The derivation of $p^S(p)$ is analogous. For a unit mass of entrepreneurs with credit score p at the beginning of some period, there is a mass p of safe entrepreneurs, a fraction δ of them survive into the next period and (if financed) will have credit score $p^S(p)$. Similarly, there is a mass $1 - p$ of risky entrepreneurs, a fraction $\delta\pi_{e^r(p)}$ succeed at their projects and survive into the next period, also ending with the same score $p^S(p)$. But in addition there are again risky entrepreneurs from previous generations, who at the beginning of the period had a credit score $p^S(p)$, their project failed but the failure was forgotten. The total mass of these additional risky entrepreneurs, relative to the unit mass of entrepreneurs with score

p, is $(1-p)\delta\pi_{e^r(p)}\left[(1-\pi_{e^r(p^S(p))})\delta q + (1-\pi_{e^r(p^S(p))})^2\delta^2 q^2 + \cdots\right]$.[56] Therefore:

$$p^S(p) = \frac{\delta p}{\delta p + (1-p)\delta\pi_{e^r(p)} + (1-p)\delta\pi_{e^r(p)}\left[(1-\pi_{e^r(p^S(p))})\delta q + (1-\pi_{e^r(p^S(p))})^2\delta^2 q^2 + \cdots\right]}$$

$$= \frac{p}{p + (1-p)\frac{\pi_{e^r(p)}}{1-\left(1-\pi_{e^r(p^S(p))}\right)\delta q}},$$

as in (6).

Proofs of Claims 1-5

Claim 1: *A solution $p_h \in (0,1)$ to (A.6) always exists.*

Since $\hat{p}^S(p)$ and $r_{zp}(p,1)$ are both continuous for all $p \in (0,1)$, $\hat{v}^r(p)$ is also continuous. As $p \to 1^-$, $r_{zp}(p,1) \to 1$ and $\hat{p}^S(p) \to 1$, and so $\hat{v}^r(p) \to \frac{\pi_h(R-1)-c}{1-\tilde{\beta}(\pi_h+(1-\pi_h)q)}$. And since $c < \frac{(R-1)(1-\tilde{\beta}q)}{1-\tilde{\beta}(\pi_l+(1-\pi_l)q)}$ in this region, for p close to 1 we have

$$\frac{c\pi_l}{\pi_h - \pi_l} < \hat{v}^r(p)(1-\tilde{\beta}q)$$

Conversely, as $p \to 0^+$ it is immediate to see that (since $\hat{p}^S(p) \to 0$ and $r_{zp}(p,1) \to 1/\pi_h$), $\hat{v}^r(p) \to \frac{\pi_h(R-1/\pi_h)-c}{1-\tilde{\beta}(\pi_h+(1-\pi_h)q)}$. Then since $c > \frac{(R-1/\pi_h)(1-\tilde{\beta}q)}{1-\tilde{\beta}(\pi_l+(1-\pi_l)q)}$ in this region, for p close to 0 we have

$$\frac{c\pi_l}{\pi_h - \pi_l} > \hat{v}^r(p)(1-\tilde{\beta}q)$$

Thus by the continuity $\hat{v}^r(\cdot)$, there must be a solution $p_h \in (0,1)$ to (A.6); moreover, by the monotonicity of $\hat{v}^r(\cdot)$ this solution is unique.

Claim 2: *There exists a lowest value $p_m \le p_h$ for which there is a solution $e^r(p)$ to (A.9) for all $p \in [p_m, p_h]$, with $e^r(p)$ increasing in p. Moreover, there is only a single period of mixing along the equilibrium path.*

Recall that we defined $\tilde{p}^S(p,e)$ to be the posterior following a success, when the risky entrepreneurs exert effort e at p, and follow the equilibrium path for $p' > p$. We now denote the inverse of this map by $\left(\tilde{p}^S\right)^{-1}(p',e)$; that is, if $p = \left(\tilde{p}^S\right)^{-1}(p',e)$, then $\tilde{p}^S(p,e) = p'$.

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[56]The expression follows from the fact that every entrepreneur with credit score $p^S(p)$ must previously have had a credit score p, succeeded once, and then failed one or more times (having this failure forgotten each time).

Let $p^{**} = p_h$ and $p^* \equiv (\tilde{p}^S)^{-1}(p^{**}, e^r(p^{**}))$ be the inverse of $p^{**} = p_h$ when the equilibrium effort $e^r(p^{**}) = 1$ is exerted at both p^* and p^{**}. It is clear from (6) that this implies that $p^* < p^{**} = p_h$. Also, recall that we have already demonstrated that $v^r(p) = \hat{v}^r(p)$ for $p \geq p_h$, and hence is monotonic for $p \geq p^{**}$. Note finally that $(p^{**}, e^r(p^{**}))$ solves (A.9).

For each $p \in [p^*, p^{**})$, define $e^r(p)$ to be the maximal value of e that solves (A.9), subject to the constraint that $e \leq e^r(p^{**})$;[57] If there is no solution to (A.9) (subject to this constraint) for some $p \in [p^*, p^{**}]$, take p_m to be the lowest value of p for which a solution exists. (We know that $e = e^r(p^{**}) = e^r(p_h) = 1$ solves (A.9) at $p^{**} = p_h$, so this is well-defined.) Let then $v^r(p) = \tilde{v}^r(p, e^r(p))$.

We now show that $e^r(p)$ is monotonic in p. In particular, whenever (p, e) solves (A.9) with $e \leq e^r(p^{**})$, for all $p' \in (p, p^{**})$ there exists a solution (p', e') to (A.9), with $e' \in (e, e^r(p^{**})]$. To see this, first note that $\tilde{v}^r(p', e)(1 - \tilde{\beta}q) > \frac{c\pi_l}{\pi_h - \pi_l}$; the reason is that $r_{zp}(p', e) < r_{zp}(p, e)$ and also $v^r(\tilde{p}^S(p', e)) \geq v^r(\tilde{p}^S(p, e))$ (the latter follows from the monotonicity of $v^r(\cdot)$ for $p \geq p^{**}$). Conversely, we have $\tilde{v}^r(p', e^r(p^{**}))(1 - \tilde{\beta}q) < \tilde{v}^r(p^{**}, e^r(p^{**}))(1 - \tilde{\beta}q) = \frac{c\pi_l}{\pi_h - \pi_l}$. The result then follows by the continuity of $\tilde{v}^r(p', \cdot)$ when we take $e = e^r(p)$. Also note that $v^r(\cdot)$ is constant in the mixing region, and so (weakly) monotonic above p^* (or above p_m if there was such a minimal value).

If a minimal value $p_m > p^*$ exists, then we are done. If not, we iterate the argument. That is, set p^{**} equal to the value of p^* at the previous iteration and $p^* = (\tilde{p}^S)^{-1}(p^{**}, e^r(p^{**}))$. We then apply the same argument as above to establish that for any $p \in [p^*, p^{**})$ for which there is a solution e to (A.9) with $e \leq e^r(p^{**})$, for all $p' \in (p, p^{**}]$ there exists a solution $e' \in (e, e^r(p^{**}))$.

Finally, we show that there is only a single period of mixing along the equilibrium path, and so we must have $p_m \geq (\tilde{p}^S)^{-1}(p_h, 0)$.

Suppose there were more than one period of mixing, beginning at some $p < p_h$. That is, we mix at both p and $p^S(p)$, with $p^S(p) \leq p_h$ and $p^S(p^S(p)) \geq p_h$. Recall that (A.9) implies that $v^r(p) = v^r(p^S(p)) = v^r(p_h)$.

Now, $v^r(p) \equiv \pi_{e^r(p)}(R - r_{zp}(p)) - c \cdot e^r(p) + \pi_{e^r(p)}\tilde{\beta}v^r(p^S(p)) + (1 - \pi_{e^r(p)})\tilde{\beta}qv^r(p)$.

Also, $v^r(p^S(p)) \equiv \pi_{e^r(p^S(p))}(R - r_{zp}(p^S(p))) - c \cdot e^r(p^S(p)) + \pi_{e^r(p^S(p))}\tilde{\beta}v^r(p^S(p^S(p))) + (1 - \pi_{e^r(p^S(p))})\tilde{\beta}qv^r(p^S(p))$.

Now, we have $v^r(p^S(p^S(p))) \geq v^r(p_h)$. Also, $r_{zp}(p) > r_{zp}(p^S(p))$ by the monotonicity of effort in the mixing region, established above. This implies that $v^r(p^S(p)) > v^r(p)$, a contradiction. Thus there can be only a single period of mixing along the equilibrium path.

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[57]This constraint does not bind in the first step of the induction, since we have $p^{**} = p_h$ and so $e^r(p^{**}) = 1$.

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So we can take p_m to be the lowest value of p for which we can solve (A.9) subject to this constraint.

Claim 3: *The contract $r_{zp}(p, 0)$ satisfies the low-effort IC constraint for $p \in [p_{NF}, p_m)$.*
We proceed by induction.

- First consider $p \in \left[\max[p_{\mathrm{NF}}, (\tilde{p}^S)^{-1}(p_m, e^r(p_m))], p_m \right)$, where recall that $(\tilde{p}^S)^{-1}(p', e)$ denotes the preimage of p' under the map $\tilde{p}^S(p, e)$. Also recall that in the proof of part b-ii. of Proposition 1 $\tilde{v}^r(p, e)$ was defined to be the utility a risky agent would receive if the effort exerted at p were e, and the same as on the equilibrium path for all $p' > p$.

  Intuitively, if low effort were not incentive compatible in this region, that would contradict the construction of p_m as minimal. To see this more formally, suppose it were not true, i.e., $\tilde{v}^r(p, 0)(1 - \tilde{\beta}q) > \frac{c\pi_l}{\pi_h - \pi_l}$. Now, by the monotonicity of $\tilde{v}^r(p, e)$ and $\tilde{p}^S(p, e)$ in p, we also know that we have $\tilde{v}^r(p, e^r(p_m))(1 - \tilde{\beta}q) < \frac{c\pi_l}{\pi_h - \pi_l}$ (since this held with equality at p_m). In addition, we have demonstrated in proving Claim 2 that $v(p')$ is monotonic and continuous for $p' \geq p_m$. Thus $\tilde{v}^r(p, e)$ must be continuous in e, and so this would imply that there must be a solution $e' \in (0, e^r(p_m))$ to (A.9) at p, which contradicts the construction of p_m as minimal.

  So we conclude that $e^r(p) = 0$ for $p \in \left[p_{\mathrm{NF}}, (\tilde{p}^S)^{-1}(p_m, e^r(p_m))], p_m \right)$, and thus $v^r(p) = \tilde{v}^r(p, 0)$ in this interval. By the monotonicity of $r_{zp}(p, 0)$ and $\tilde{v}^r(\cdot, 0)$, we can also conclude that $v(p)$ is monotonic for $p \geq \max[p_{\mathrm{NF}}, (\tilde{p}^S)^{-1}(p_m, 0)]$.

- If $\max[p_{\mathrm{NF}}, (\tilde{p}^S)^{-1}(p_m, e^r(p_m))] = p_{\mathrm{NF}}$, then we are done. Otherwise, we need to iterate the argument. Recall that we have already demonstrated that low effort is incentive compatible at $p' = \max[p_{\mathrm{NF}}, (\tilde{p}^S)^{-1}(p_m, e^r(p_m))]$ and that $v^r(p)$ is monotonic for $p \geq p'$.

  We conclude the proof by showing that if low effort is incentive compatible at some p' and that $v^r(p)$ is monotonic for $p \geq p'$, then low effort is also incentive compatible for $p \geq (\tilde{p}^S)^{-1}(p', 0)$ and also $v^r(p)$ is monotonic for $p \geq (\tilde{p}^S)^{-1}(p', 0)$.

  Now, $p^S(p) = \tilde{p}^S(p, 0)$ is increasing in p, and also $p^S(p) \geq p'$ when $p \geq (\tilde{p}^S)^{-1}(p', 0)$, thus from the monotonicity of $v^r(\cdot)$ we have $v^r(p^S(p)) < v^r(p^S(p'))$. Also, $r_{zp}(p, 0)$ is monotonic in p. These then imply that $\tilde{v}^r(p, 0) < v^r(p')$ and so low effort must also be incentive compatible at p; thus $v^r(p) = \tilde{v}^r(p, 0)$. The desired monotonicity of $v^r(p)$ for $p \geq (\tilde{p}^S)^{-1}(p', 0)$ follows immediately.

  We can then iterate the same argument as above, and continue doing so until reaching p_{NF}.

Claim 4: *When $\delta q < \frac{1 - \pi_h}{1 - \pi_l}$, the equilibrium constructed in Proposition 1 is such that the risky entrepreneurs' effort $e^r(p)$ is higher, at any p, than at any other symmetric sequential MPE.*

First note that this is immediate for region c., since the equilibrium of Proposition 1 implements high effort for all $p > 0$. As far as region a., for the values of c in this region it is not hard to show that only low effort can be incentive compatible. So we can restrict attention to region b.

Let $v, p^S(p), e^r$ etc. denote the value function, updating, effort, etc. for the equilibrium of Proposition 1, and let $\bar{v}, \bar{p}^S(p), \bar{e}^r$ etc. denote the same for another MPE.

Note that it is immediate that $e^r(p) \geq \bar{e}^r(p)$ for all $p \geq p_h$, as the equilibrium of Proposition 1 implements high effort in this region. Also, since the equilibrium of Proposition 1 is characterized by (the minimal) break-even interest rates, we have $v^r(p) \geq \bar{v}^r(p)$ in this region as well.

Now, suppose the result does not hold, and we can implement higher effort at the other MPE, at some $\tilde{p}$, i.e., $\bar{e}^r(\tilde{p}) > e^r(\tilde{p})$. We will show that this contradicts the construction of the equilibrium of Proposition 1. Now, if there is more than one such value of $\tilde{p}$ for which higher equilibrium is supported in the alternative equilibrium, we choose $\tilde{p}$ such that, for all successor nodes of $\tilde{p}$ along the equilibrium path of the other MPE, effort is (weakly) higher in the equilibrium of Proposition 1: $\bar{e}^r(\bar{p}^S(\tilde{p})) \leq e^r(\bar{p}^S(\tilde{p}))$ and so on for all successor nodes of $\tilde{p}$ in the other equilibrium. This can be done because $\bar{p}^S(\tilde{p}) > \tilde{p}$ by the condition on δq.

We show now that if higher effort can be supported at $\tilde{p}$ in the other equilibrium, this implies that effort level $e' \geq \bar{e}^r(\tilde{p}) > e^r(\tilde{p})$ is incentive compatible at p when we follow the equilibrium path of Proposition 1 at all successor nodes. To see why, note that if we adopted the effort level $\bar{e}^r(\tilde{p})$ at $\tilde{p}$ and then followed the equilibrium of Proposition 1 at all successor nodes, we would obtain utility $\tilde{v} \geq \bar{v}^r(\tilde{p})$ at $\tilde{p}$. The reason is, first, that the equilibrium of Proposition 1 is characterized by interest rates satisfying the zero-profit condition, and second, that we have taken $\tilde{p}$ to be the highest point along the equilibrium path of the other equilibrium for which effort is higher.

To see why this implies that higher effort e' is incentive compatible, observe that if $\bar{e}^r(\tilde{p}) = 1$ then it is immediate that high effort is also incentive compatible at $\tilde{p}$ when we follow the equilibrium path of Proposition 1 at all successor nodes. Alternatively, if $\bar{e}^r(\tilde{p}) \in (0, 1)$, while the incentive compatibility condition (A.9) may not hold with equality when effort level e' is exerted and we then follow the equilibrium path of Proposition 1 (since $\tilde{v}$ might be too high), a continuity argument similar to that in Claim 2 can be used to show

that it is possible to find some $e' > \bar{e}^r(\tilde{p})$ which is incentive compatible. In either case, we can find $e' > e^r(\tilde{p})$ which is incentive compatible at $\tilde{p}$.

To demonstrate that this contradicts the construction of the equilibrium of Proposition 1, and thereby conclude the proof, we must also verify whether this higher effort level e' satisfies the constraint $e''^r(p^{**})$, for $p^{**} \geq \tilde{p}$ constructed as in the proof of Claim 2.[58]

Suppose it does. Then we have contradicted the construction of the equilibrium of Proposition 1 and, in particular, the construction of $e^r(\tilde{p})$ as maximizing effort subject to the constraint that $e^r(\tilde{p}) \leq e^r(p^{**})$ (see Claim 2).

If, in contrast, $e' > e^r(p^{**})$ (i.e., the constraint binds), we show in what follows that we can apply a similar argument to that given above to establish that higher effort can be implemented at p^{**} (following the equilibrium of Proposition 1 at all successor nodes). This can be done because the monotonicity of $v^r(p)$ and the fact that interest rates are decreasing in p imply that incentives are (weakly) stronger for higher values of p. In particular, if effort $e'^r(p^{**})$ is incentive compatible at $\tilde{p}$, then a (weakly) higher effort is also incentive compatible at $p^{**} > \tilde{p}$. Demonstrating that a higher level of effort can be implemented at p^{**} would again contradict the construction of the equilibrium of Proposition 1. And if the constraint also binds at p^{**}, we iterate forwards again and apply then the same argument as above to the upper bound associated with the new value of p^{**}. We continue iterating forwards as needed; this iteration must eventually stop (at the latest when we reach $(p^S)^{-1}(p_h, 1)$, as there the effort choice was unconstrained).

Claim 5: $W(s_0, q) \geq \overline{\mathcal{W}}(s_0, q)$.

First consider the case $p_0(s_0, q) < p_l$. From (5) it is clear that since the other equilibrium implements lower effort at any p, we must have $\bar{p}_0(s_0, q) \leq p_0(s_0, q)$. Thus from Corollary 2 there can be no financing in either equilibrium and so $W(s_0, q) = \overline{\mathcal{W}}(s_0, q)$.

Next, when $p_0(s_0, q) \geq p_l$, total surplus can be defined as follows:

$$\mathcal{W}(s_0, q) = \mathcal{W}^s(s_0) + \mathcal{W}^r(s_0),$$

where $W^s(s_0) = \frac{s_0(R-1)}{1-\tilde{\beta}}$ and $W^r(s_0) = (1 - s_0)w^r(p_0(s_0, q))$, for $w^r(\cdot)$ defined recursively: $w^r(p) \equiv \left[\pi_{e(p)}R - 1 - c \cdot e^r(p)\right] + \pi_{e(p)}\tilde{\beta}w^r(p^S(p)) + (1 - \pi_{e(p)})q\tilde{\beta}w^r(p)$.[59] We can similarly define $\overline{\mathcal{W}}(s_0), \overline{\mathcal{W}}^s(s_0), \overline{\mathcal{W}}^r(s_0)$, and $\overline{w}^r(p)$ for the other equilibrium.

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[58]While Claim 2 focuses on the mixing region, this condition also holds in the high-effort and low-effort regions.

[59]Here $\pi_{e(p)} \equiv \pi_h e^r(p) + \pi_l(1 - e^r(p))$ is the risky entrepreneurs' success probability given the equilibrium effort level at p, and similarly $\pi_{\bar{e}(p)}$ for the other equilibrium.

It is immediate that $W^s(s_0) \geq \overline{\mathcal{W}}^s(s_0)$. So we will restrict attention in what follows to establishing the same result for $W^r(s_0)$.

Suppose first that there is financing for all $p > p_0(s_0, q)$ in both equilibria. Now, it follows that since the equilibrium of Proposition 1 implements high effort for $p \geq p_h$, we have $w^r(p) \geq \bar{w}^r(p)$. Also note that $w^r(p)$ is constant above p_h. So if $p_0(s_0, q) \geq p_h$, it is then immediate that $W^r(s_0) \geq \overline{\mathcal{W}}^r(s_0)$, since from (5) we know that $p_0(s, 0) \geq \bar{p}_0(s_0, q)$.[60] Otherwise, we proceed by induction.

Consider next $p \in [p_m, p_h)$. We know from Claim 4 that $e^r(p) \geq \bar{e}^r(p)$, which also implies that $\bar{p}^S(p) \geq p^S(p) \geq p_h$, and thus that $w^r(\bar{p}^S(p)) \geq \bar{w}^r(\bar{p}^S(p))$. Also note that $w^r(p') > 0$ for $p' \geq p_h$. So

$$
\begin{aligned}
\overline{w}^r(p) &= \frac{\left(\pi_{\bar{e}(p)} R - 1 - c\bar{e}^r(p)\right) + \pi_{\bar{e}(p)}\tilde{\beta}\overline{w}^r(\bar{p}^S(p))}{1 - (1 - \pi_{\bar{e}(p)})\tilde{\beta}q} \\
&\leq \frac{\left(\pi_{\bar{e}(p)} R - 1 - c\bar{e}^r(p)\right) + \pi_{\bar{e}(p)}\tilde{\beta}w^r(\bar{p}^S(p))}{1 - (1 - \pi_{\bar{e}(p)})\tilde{\beta}q} \\
&\leq \frac{\left(\pi_{e(p)} R - 1 - ce^r(p)\right) + \pi_{e(p)}\tilde{\beta}w^r(\bar{p}^S(p))}{1 - (1 - \pi_{e(p)})\tilde{\beta}q},
\end{aligned}
$$

where the final inequality follows because $\bar{e}^r(p) \leq e^r(p) \leq 1$.[61] If we now replace $w^r(\bar{p}^S(p))$ with $w^r(p^S(p))$ in the right-hand side of the inequality, this cannot decrease its value, since we showed that $w^r(p')$ is constant for $p' \geq p_h$ (and $\bar{p}^S(p) \geq p^S(p)$). This then demonstrates that $w^r(p) \geq \bar{w}^r(p)$ for $p \in [p_m, p_h)$. Another consequence of this is that $w^r(p)$ is weakly increasing for $p \geq p_m$.

Next, observe that for lower values of p; $p \in [p_l, p_m)$, from Claim 4 we know that both equilibria implement low effort in this region, and so $w^r(p) \geq \bar{w}^r(p)$. This again also implies that $w^r(p)$ is weakly increasing. Then since $p_0(s_0, q) \geq \bar{p}(s_0, q)$, we can conclude that $W^r(s_0) \geq \overline{\mathcal{W}}^r(s_0)$.

If the other equilibrium implements exclusion for $p \geq p_l$, it is not hard to extend the argument above, once we note that financing generates positive social surplus in every period (since the lenders make zero profits, on average).

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[60]Since $p_0(s, 0)$ is decreasing in the effort exerted in the initial period, and this effort cannot be higher in the other equilibrium.

[61]The reason is that increasing from $\bar{e}^r(p)$ to $e^r(p)$ raises the probability of success (and hence continuing rather than staying at the same score). Since the agent exerts high effort at $\bar{p}^S(p)$, this then increases welfare generated by the risky agents.